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ON 4-DIMENSIONAL CUBE AND SUDOKU

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Abstract. The number puzzle SUDOKU (Number Place in the U.S.) has recently gained great popularity. We point out a relationship between SUDOKU and 4-dimensional Latin cubes. Namely, we assign the cells 4-tuples of numbers – coordinates in 4-dimensional space and then consider the game plan as a 4-dimensional cube. We also mention some variants of SUDOKU.

Recently the SUDOKU puzzle became popular in Europe. This game came into existence about 30 years ago in the U.S. The name of the game – a number puzzle – is derived from Japanese words SU (number) and DOKU (single), and the game became popular in Japan in the year 1986. SUDOKU was introduced in Europe in 2004, in the well know British journal The Times. As the signs for numbers are common to many languages, the game spread very quickly and became popular in many other countries. In the U.S. the SUDOKU game is usually called *Number Place.* For more information we advise to consult the Internet.

The *game plan* of the SUDOKU puzzle is a grid consisting of 9×9 squares, called *cells*. The grid is further divided into 3×3 *subgrids* or *regions*. Some cells contain numbers ranging from 1 to 9, known as *givens*. They are usually inscribed into cells which are symmetrical with respect to the center of the game plan (see Figure 1). The aim of the game is to inscribe a number from 1 to 9 into each of the empty cells, so that each row, column and region contains only one instance of each numeral. To simplify our formulas, we shall use numbers from the set $M = \{0, 1, 2, \ldots, 8\}$ instead of $\{1, 2, 3, \ldots, 9\}$.

From the point of view of mathematics, there arise many questions concerning SUDOKU. Several papers have considered the questions connected with solving the game. These are question of the type: For which configurations of the givens does the game possess a unique solution? How many solutions does the game have? What is the best time complexity of an algorithm solving SUDOKU? Here, however, we shall consider a different viewpoint. We shall label the individual cells by quadruples of numbers – coordinates, and represent the game plan by a 4-dimensional cube (also called a *hypercube*). The goal of the paper is to show some connections between the SUDOKU puzzle, Latin squares and their 4-dimensional analogies. The reader can get himself acquainted with 4-dimensional Latin cubes (Latin hypercubes) which have been getting an ever greater attention of mathematicians recently. Hence the text offers a small excursion into the 4-dimensional space and it can also inspire scientific research for both teachers and students. The involved reader can design different variations of the SUDOKU puzzle. These modifications can be adjusted with respect to the age and capabilities of the solver.

Figure 1 shows a SUDOKU puzzle (the numbers inscribed are from the set $M = \{0, 1, 2, \ldots, 8\}$) and Figure 2 shows its solution in the ternary, i.e. base-3 numeral system. The ternary numbers range from 00 (decimal 0) to 22 (decimal 8) and we consider them as ordered pairs of numerals $[i, j]$, $0 \leq i, j \leq 2$. Each ordered pair occurs exactly once in each row, column and region. We recommend the reader to divide the solution from Figure 2 into two tables – one containing the first numerals and the other one containing the second numerals of the ternary numbers. In both tables, each row, column and region will contain exactly three occurrences of the numbers $0, 1, 2$. This property might give the reader a new perspective on the SUDOKU puzzle or even help him solving it.

4			6	7				0
	8	0					3	
6		7			3	1		2
			7		5	3		
8	3			0			2	7
		2	1		8			
3		8	2			6		5
	1					4	0	
7				6	4			1

FIGURE 1

11	02	10	20	21	01	22	12	00
01	22	00	12	11	02	21	10	20
20	12	21	00	22	10	01	11	02
00	20	11	21	02	12	10	01	22
22	10	01	11	00	20	12	02	21
12	21	02	01	10	22	00	20	11
10	11	22	02	01	00	20	21	12
02	01	20	22	12	21	11	00	10
21	00	12	10	20	11	02	22	01

FIGURE 2

Because we assume that most of the readers have never before encountered the notion of a hypercube (i.e. a 4-dimensional cube), which we shall use later in this text, we shall first simplify the SUDOKU puzzle into its "3-dimensional version" and demonstrate its relation to the usual 3-dimensional cube. Figure 3 shows a cube consisting of $3 \times 3 \times 3$ cells. Letters A, B, C denote three of its nine layers. Each layer consists of 3×3 cells. To obtain the second and the third triple of layers, we have to "cut" the cube using two planes parallel to the base and the side face of the cube, respectively. The numbers from the set $M = \{0, 1, 2, \ldots, 8\}$ are inscribed into the cells in such a way that the numbers in each of the nine layers are pairwise different.

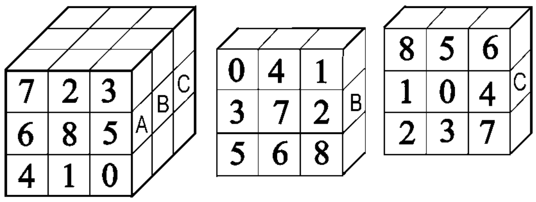

FIGURE 3

Let us assign a triple of coordinates to each cell in the natural way, as shown in Figure 4. (The commas between the coordinates are omitted in the picture.) We can now formalize the notion of a layer: a *layer* is a nine-tuple of cells which have the same coordinate in one of the three positions. Each layer is determined by one of its elements (cells) and two of the three directions given by the edges of the cube.

$a(111)$	$a(112)$	$a(113)$	$a(211)$	$a(212)$	$a(213)$	$a(311)$	$a(312)$	$a(313)$
$a(121)$	$a(122)$	$a(123)$	$a(221)$	$a(222)$	$a(223)$	$a(321)$	$a(322)$	$a(323)$
$a(131)$	$a(132)$	$a(133)$	$a(231)$	$a(232)$	$a(233)$	$a(331)$	$a(332)$	$a(333)$

FIGURE 4

Exercise: Figure 5 shows a cube with numbers inscribed into one third of its cells. Inscribe a number from the set $M = \{0, 1, 2, \ldots, 8\}$ into each cell so that each number occurs in each layer exactly once.

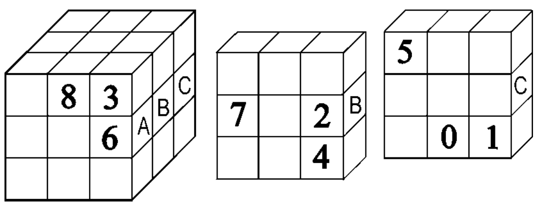

FIGURE 5

After this "3-dimensional trip", let us return to SUDOKU.

When the SUDOKU puzzle is solved, i.e. when the 9×9 grid correctly completed, then each row and column of the grid contains a permutation of the set M. In mathematics, a grid with this property is called a *Latin square*. Formally: a *Latin square* of order n is an $n \times n$ matrix (table) $\mathbf{R}_n = |r(k,l);\ 1 \le k,l \le n|$ comprising n^2 numbers $r(k,l) \in \{0,1,2,\dots,n-1\}$ with the property that each row and column is a permutation of the set $\{0,1,\dots,n-1\}$. The SUDOKU puzzle contains an additional constraint that each region is a permutation of the set M.

We assign to each cell of the game plan a quadruple of numbers – its coordinates – as in Figure 6. The first pair of the quadruple determines the region, and the second pair the position in the particular region.

A *hypercube* of order n is a 4-dimensional matrix

$$\mathbf{A}_n = |a(i,j,k,l);\ 1 \le i,j,k,l \le n|$$

comprising n^4 elements $a(i,j,k,l)$.

$a(1111)$	$a(1112)$	$a(1113)$	$a(1211)$	$a(1212)$	$a(1213)$	$a(1311)$	$a(1312)$	$a(1313)$
$a(1121)$	$a(1122)$	$a(1123)$	$a(1221)$	$a(1222)$	$a(1223)$	$a(1321)$	$a(1322)$	$a(1323)$
$a(1131)$	$a(1132)$	$a(1133)$	$a(1231)$	$a(1232)$	$a(1233)$	$a(1331)$	$a(1332)$	$a(1333)$
$a(2111)$	$a(2112)$	$a(2113)$	$a(2211)$	$a(2212)$	$a(2213)$	$a(2311)$	$a(2312)$	$a(2313)$
$a(2121)$	$a(2122)$	$a(2123)$	$a(2221)$	$a(2222)$	$a(2223)$	$a(2321)$	$a(2322)$	$a(2323)$
$a(2131)$	$a(2132)$	$a(2133)$	$a(2231)$	$a(2232)$	$a(2233)$	$a(2331)$	$a(2332)$	$a(2333)$
$a(3111)$	$a(3112)$	$a(3113)$	$a(3211)$	$a(3212)$	$a(3213)$	$a(3311)$	$a(3312)$	$a(3313)$
$a(3121)$	$a(3122)$	$a(3123)$	$a(3221)$	$a(3222)$	$a(3223)$	$a(3321)$	$a(3322)$	$a(3323)$
$a(3131)$	$a(3132)$	$a(3133)$	$a(3231)$	$a(3232)$	$a(3233)$	$a(3331)$	$a(3332)$	$a(3333)$

Figure 6

A *row* of a hypercube $\mathbf{A}_n$ of order n is an n-tuple of elements whose coordinates differ on exactly one position. A *layer* of a hypercube is an n^2-tuple of elements, whose coordinates differ on exactly two positions. A layer is determined by one of its elements and two directions. There are four *directions* in a hypercube; they are given by its edges. Each pair of directions defines a *lay*. (Note: We use the notions of *direction* and *lay* in a similar sense as in analytic geometry. There, direction (lay) denotes a one-dimensional (two-dimensional) vector space and together with a point it defines a straight line (plane).) Each direction is contained in three lays. Two different layers of a hypercube have the same lay if and only if they are disjoint. An order n hypercube contains $6n^2$ layers belonging to six lays.)

Let us return to the table in Figure 6. We know now that it shows the coordinates of the cells of an order 3 hypercube. The 3×3 blocks bounded by thick lines represent the 9 layers of the hypercube which have the same lay.

For example, the element $a(1,1,1,1)$ is contained in four rows which comprise the following elements:

$$\begin{array}{ll}\text{1-direction:} & (a(1,1,1,1), a(1,1,1,2), a(1,1,1,3)),\\ \text{2-direction:} & (a(1,1,1,1), a(1,1,2,1), a(1,1,3,1)),\\ \text{3-direction:} & (a(1,1,1,1), a(1,2,1,1), a(1,3,1,1)),\\ \text{4-direction:} & (a(1,1,1,1), a(2,1,1,1), a(3,1,1,1)).\end{array}$$

These rows are contained in four different directions which we have denoted by numbers 1,2,3,4. Six layers of the hypercube whose lays are pairwise different and which contain the element $a(1,1,1,1)$ are given by quadruples of corner cells (the symbol *(a-b)-lay* denotes the lay which is given by the directions a and b):

$$\begin{array}{lllll}\text{(1-2)-lay:} & a(1,1,1,1), & a(1,1,1,3), & a(1,1,3,1), & a(1,1,3,3),\\ \text{(1-3)-lay:} & a(1,1,1,1), & a(1,1,1,3), & a(1,3,1,1), & a(1,3,1,3),\\ \text{(1-4)-lay:} & a(1,1,1,1), & a(1,1,1,3), & a(3,1,1,1), & a(3,1,1,3),\\ \text{(2-3)-lay:} & a(1,1,1,1), & a(1,1,3,1), & a(1,3,1,1), & a(1,3,3,1),\\ \text{(2-4)-lay:} & a(1,1,1,1), & a(1,1,3,1), & a(3,1,1,1), & a(3,1,3,1),\\ \text{(3-4)-lay:} & a(1,1,1,1), & a(1,3,1,1), & a(3,1,1,1), & a(3,3,1,1).\end{array}$$

Using this terminology, we can formulate the rules of the SUDOKU puzzle in the following way: Numbers from the set M are inscribed into some cells of an order 3 hypercube. Inscribe a number from M into each empty cell so that each layer with lay (1-2) and (1-3) and (2-4) contains all numbers from the set M.

We would obtain different versions of SUDOKU if we required having all numbers from M in layers with different lays.

Next we show how to fill in such a table in a certain special case. First though, we have to define two notions: orthogonal Latin squares and a Latin hypercube.

Two Latin squares $\mathbf{R}_n = |r(k,l)|$ and $\mathbf{S}_n = |s(k,l)|$ of order n are said to be *orthogonal* if all the ordered pairs $[r(k,l), s(k,l)]$ are pairwise different. It has been known since Leonhard Euler's time that pairs of orthogonal Latin squares can be constructed for all odd n using the following formulas:

$$r(k,l) = (k + l + a) \mod n, \qquad s(k,l) = (k - l + b) \mod n \tag{1}$$

for all $1 \leq k, l \leq n$, where a, b are arbitrary integers. The elements of the table in Figure 7 are pairs $[r(k,l), s(k,l)]$ of elements of two orthogonal Latin squares $\mathbf{R}_9$ a $\mathbf{S}_9$ of order 9 which have been obtained using the above equations with $a = b = 0$.

2, 0	3, 8	4, 7	5, 6	6, 5	7, 4	8, 3	0, 2	1, 1
3, 1	4, 0	5, 8	6, 7	7, 6	8, 5	0, 4	1, 3	2, 2
4, 2	5, 1	6, 0	7, 8	8, 7	0, 6	1, 5	2, 4	3, 3
5, 3	6, 2	7, 1	8, 0	0, 8	1, 7	2, 6	3, 5	4, 4
6, 4	7, 3	8, 2	0, 1	1, 0	2, 8	3, 7	4, 6	5, 5
7, 5	8, 4	0, 3	1, 2	2, 1	3, 0	4, 8	5, 7	6, 6
8, 6	0, 5	1, 4	2, 3	3, 2	4, 1	5, 0	6, 8	7, 7
0, 7	1, 6	2, 5	3, 4	4, 3	5, 2	9, 1	7, 0	8, 8
1, 8	2, 7	3, 6	4, 5	5, 4	6, 3	7, 2	8, 1	0, 0

FIGURE 7

The French De la Hire knew already 300 years ago that a *magic square* can be constructed using a pair of orthogonal Latin squares. (A *magic square* of order n is an $n \times n$ matrix $\mathbf{M}_n = |m(k,l);\ 1 \le k,l \le n|$ consisting of n^2 consecutive positive integers $m(k,l)$ such that the sums of elements in each row, column and on both diagonals are the same.)

Setting $a = 3$ and $b = 4$ and using the formula

$$m(k,l) = 9 \cdot r(k,l) + s(k,l) + 1,$$

we obtain a magic square $\mathbf{M}_9 = |m(k,l); 1 \le k,l \le 9|$ of order 9 whose elements are from the set $\{0,1,2,\dots,9^2-1\}$. The choice of the parameters a, b ensures that not only the sums in the rows and columns, but also on the diagonals are the same.

When we generalize the notion of a Latin square to four dimensions, we obtain a Latin hypercube. A *Latin hypercube* of order n is a hypercube

$$\mathbf{T}_n = |t(i,j,k,l);\ 1 \le i,j,k,l \le n|,$$

whose elements are from the set $\{0,1,2,\dots,n-1\}$ and each row and diagonal contains a permutation of this set.

Consider two Latin hypercubes $\mathbf{T}_n = |t(i,j,k,l)|$ and $\mathbf{U}_n = |u(i,j,k,l)|$ of order n given by the following formulas:

$$\begin{aligned} t(i,j,k,l) &= r(i,(r(j,r(k,l))) = (i+j+k+l) \mod n, \\ u(i,j,k,l) &= s(i,(s(j,s(k,l))) = (i-j+k-l) \mod n; \end{aligned}$$

with $\mathbf{R}_n$ and $\mathbf{S}_n$ being a pair of orthogonal Latin squares given by formulas (1). The 9×9 grid from Figure 8, which is a SUDOKU solution, has been constructed using the formula

$$v(i,j,k,l) = t(i,j,k,l) \cdot n + u(i,j,k,l). \tag{2}$$

The careful reader may notice that not only all the layers with the three lays (1-2), (1-3) and (2-4) contain all the elements of M but that the same is true for all the layers with a fourth lay. We have obtained this property thank to a suitable choice of the pair of Latin hypercubes. For a different choice of the pair of Latin squares (obtained for example by a different choice of a and b) we obtain different Latin hypercubes of order n and hence a different hypercube – a different game plan completion. Moreover, the construction based on (2) is such that the sums of the numbers in all the rows of the hypercube are the same for every choice of the parameters a, b. Applying suitable exchanges of rows and columns, we can obtain different grids (SUDOKU solutions). (You can find more details on Latin squares and hypercubes in Internet or [3].)

3	1	8	7	5	0	2	6	4
7	5	0	2	6	4	3	1	8
2	6	4	3	1	8	7	5	0
1	8	3	5	0	7	6	4	2
5	0	7	6	4	2	1	8	3
6	4	2	1	8	3	5	0	7
8	3	1	0	7	5	4	2	6
0	7	5	4	2	6	8	3	1
4	2	6	8	3	1	0	7	5

FIGURE 8

The above text poses more questions than it gives answers. Since the formulas are true for every odd n, the reader may think of game plans of different sizes.

The reader can also design numerous variations of the SUDOKU puzzle, inspired for example by the following notes:

1. Assume that a hypercube of order 3 contains in some of its cells numbers from the set $\{0, 1, 2\}$. The aim of the game is to fill in the missing numbers so that each row contains each of the numbers $0, 1, 2$ exactly once. Hence, the solution of the puzzle is a Latin hypercube of order 3. (A construction of such hypercubes is given in [3].)

2. Instead of the 9×9 grid we can consider a $m^2 \times m^2$ grid consisting of m^2 regions containing $m \times m$ cells. Some cells contain numbers from the set $N = \{1, 2, 3, \ldots, m^2\}$. The aim of the game is to fill in the missing numbers so that each row, column and region contains a permutation of the set N. (The formulas (1) and (2) are valid for all odd n.) In our experience the version with $m = 2$ is suitable for young kids from 6 to 10 years of age.

5	4	3	2	6	1
2	1	6	5	3	4
4	3	5	1	2	6
1	6	2	4	5	3
6	2	1	3	4	5
3	5	4	6	1	2

FIGURE 9

3. We can obtain a different SUDOKU puzzle if we drop the requirement that the regions are squares. For example, the 6×6 grid in Figure 9 is divided into rectangular regions with 3×2 cells.

Acknowledgements

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