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FUZZIFIED PROBABILITY: FROM KOLMOGOROV TO ZADEH AND BEYOND

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ABSTRACT

We discuss the fuzzification of classical probability theory. In particular, we point out similarities and differences between the so-called fuzzy probability theory and the so-called operational probability theory.

1. INTRODUCTION

Clearly, the classical (Kolmogorovian) probability theory ([12]), CPT for short, based on Boolean logic and set theory, has its limitations when modelling real life situations and when uncertainty is to be taken into account. Accordingly, L. A. Zadeh ([19]) proposed to extend random events to fuzzy random events. In the literature, there are various approaches to uncertainty based on fuzzy sets and fuzzy logic. Also, there are several generalizations of CPT and we will deal with two of them, both deserving to be called fuzzified probability theory; just to avoid misunderstanding, we call them *fuzzy probability theory*, FPT for short, and *operational probability theory*, OPT for short. While FPT is more oriented to applications and engineering and its basic ideas and constructions are outlined in [1], OPT originated in modelling quantum phenomena in physics, but its mathematical results can be used, e.g., in social sciences, and the interested reader is referred to [11], [2], [3], [7], and references therein.

We aim at a better understanding of the fuzzification of CPT. Since we will work with more than one probability theory, we need to put them into

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a perspective. To do so, we first analyse and discuss some basic notions and constructions of probability theory.

Our paper is written as a series of questions and answers. This way we hope to provide more information (stress is on "why" and "what") within a limited space and make the presentation more readable.

2. Basic notions of CPT and their roles

Probability theory has been axiomatized (mathematized) by A. N. Kolmogorov. The influence of his "Grundbegriffe" ([12]) on the education in the area of probability and its impact on research in stochastics cannot be overestimated. In this section we present few remarks about basic mathematical notions of CPT: *random events, probability measures, and random variables.*

At the beginning we have a *probability space* $(\Omega, \mathbf{A}, P)$, where Ω is the set of all outcomes of a random experiment, $\mathbf{A}$ is a σ-field of subsets of Ω, each $A \in \mathbf{A}$ is called an *event*, events of the form $A = \{\omega\}$, $\omega \in \Omega$, are called elementary events, and $P: \mathbf{A} \to [0,1]$ is a normed σ-additive measure called *probability*; $P(A)$ measures how "big" is $A \in \mathbf{A}$ in comparison to Ω. The most important example is $(R, \mathbf{B}_R, p)$, where R are the real numbers, $\mathbf{B}_R$ is the real Borel σ-field, and p is a probability on $\mathbf{B}_R$. Let f be a measurable map of Ω into R, i.e., $f^{\leftarrow}(B) = \{\omega \in \Omega \mid f(\omega) \in B\} \in \mathbf{A}$ for all $B \in \mathbf{B}_R$. If $p(B) = P(f^{\leftarrow}(B))$ for all $B \in \mathbf{B}_R$, then f is said to be a *random variable* and p is said to be the distribution of f. More generally, if $(\Xi, \mathbf{B})$ is a measurable space then a measurable map $g : \Omega \longrightarrow \Xi$ is said to be a $(\Xi, \mathbf{B})$-valued random variable. If points of Ω and Ξ are viewed as the degenerated probability point-measures, then g gives rise to a map T_g on the set $\mathcal{P}(\mathbf{A})$ of all probability measures on $\mathbf{A}$ into the set $\mathcal{P}(\mathbf{B})$ of all probability measures on $\mathbf{B}$, $\Omega \subset \mathcal{P}(\mathbf{A})$, T_g and g coincide on Ω.

Remark 2.1. *Each random event $A \in \mathbf{A}$ can be viewed as the indicator (characteristic) function χ_A of A, or as a propositional function "$\omega \in A$" and operations on events correspond to (Boolean) logical operations on propositional functions. Observe that P can be viewed as a fuzzy subset of $\mathbf{A}$.*

Remark 2.2. *A random variable is neither a variable, nor random. Indeed, f is a function and the assignment $\omega \mapsto f(\omega)$ is not random. Instead, for example, if $F : R \to [0,1]$ is a distribution function corresponding to p, then the pair (R, F) looks like "randomized real variable". In probability theory and mathematical statistics, in fact, laws of probability on R, R^n, or R^R (in terms of distribution functions, characteristic functions, density functions) are the main objects of study and the random variables, as measurable functions, play only an auxiliary role.*

Remark 2.3. *In some sense, more important than a random variable f is its dual preimage map $f^{\leftarrow}$, mapping $\mathbf{B}_R$ into $\mathbf{A}$. Indeed, $f^{\leftarrow}$ "preserves" the Boolean structure of $\mathbf{B}_R$, i.e. it is a Boolean homomorphism mapping real events $\mathbf{B}_R$ into the events of the original (sample) space $(\Omega, \mathbf{A}, P)$, and p is the composition $P \circ f^{\leftarrow}$ of $f^{\leftarrow}$ and P. The stochastic information about an "observed" event $B \in \mathbf{B}$ is obtained by finding the corresponding "theoretical" event $f^{\leftarrow}(B) \in \mathbf{A}$ and $P(f^{\leftarrow}(B))$ is the needed stochastic information about B. In CPT, strange enough, $f^{\leftarrow}$ does not have its name.*

Remark 2.4. *Let f be a measurable map of Ω into R. Then f induces a map D_f of the set $\mathcal{P}(\mathbf{A})$ of all probability measures on $\mathbf{A}$ into the set $\mathcal{P}(\mathbf{B}_R)$ of all probability measures on $\mathbf{B}_R$: for $Q \in \mathcal{P}(\mathbf{A})$ we define*

$$D_f(Q) = Q \circ f^{\leftarrow};$$

we say that f pushes forward Q to $D_f(Q)$. If we identify each $\omega \in \Omega$ and the Dirac point-probability δ_ω and, similarly, each $r \in R$ and δ_r, then a straightforward calculation shows that $D_f(\delta_\omega) = \delta_{f(\omega)}$. Consequently, the distribution map D_f can be considered as an extension of f, mapping $\Omega \subseteq \mathcal{P}(\mathbf{A})$ into $R \subseteq \mathcal{P}(\mathbf{B}_R)$, to D_f mapping $\mathcal{P}(\mathbf{A})$ into $\mathcal{P}(\mathbf{B}_R)$.

QUESTION 1. *What is the role of a random variable in* CPT?

ANSWER 1. *It is a channel through which stochastic information is transported (the dual preimage map transports the real random events $\mathbf{B}_R$ into the sample random events $\mathbf{A}$ and each probability measure P on $\mathbf{A}$ is transported via the composition of the preimage map and P to become a probability measure on $\mathbf{B}_R$).*

3. Basic notions of FPT and their roles

In order to understand the transition from CPT to FPT, we first recollect some facts about FPT. The next lines are borrowed from [1]:

> "The development of fuzzy probability theory was initiated by H. Kwakernaak ([13]) with the introduction of fuzzy random variables in 1978... Fuzzy probability theory is an extension of probability theory to dealing with mixed probabilistic/non-probabilistic uncertainty... If a set of uncertain perceptions of a physical quantity is present in the form of a random sample, then the overall uncertainty possesses a mixed probabilistic/non-probabilistic character. Whilst the scatter of the realizations of the physical quantity possesses a probabilistic character (frequentative or subjective), each particular realization from the population may, additionally, exhibit non-probabilistic uncertainty. Consequently,

> a realistic modelling in those cases must involve both probabilistic and non-probabilistic uncertainty. This modelling without distorting or ignoring information is the mission of fuzzy probability theory. A pure probabilistic modelling would introduce unwarranted information in the form of a distribution function that cannot be justified and Fuzzy Probability Theory would thus diminish the trustworthiness of the probabilistic results."

QUESTION 2. *What is the meaning of "fuzzy probability law"?*

ANSWER 2. *A classical distribution function F represents a classical probability law and its fuzzification $\tilde{F}$, called fuzzy distribution function, represents a "fuzzy probability law". The fuzzification is based on a suitable non-probabilistic procedure transforming a classical probability space $(R, \mathbf{B}_R, F)$ into $(R, \mathbf{B}_R, \tilde{F}) \equiv (R, \mathbf{B}_R, \mu, F)$, where μ is a membership function constructed via a map of Ω into the fuzzy real numbers (satisfying certain technical conditions) and $\tilde{F}$ represents the fuzzy set of distribution functions determined by μ.*

Remark 3.1. *Observe that random events in CPT and FPT are crisp sets forming a σ-field of sets. The transition from CPT to FPT is based on the transition from F to $\tilde{F}$.*

QUESTION 3. *What is the role of fuzzy random variables?*

M. R. Puri and D. A. Ralescu ([18]) formalized fuzzy random variables (also called random fuzzy sets) as an extension of random sets as follows. Let B be a separable Banach space. Denote $\mathcal{K}(B)$ the set of all non-empty bounded closed subsets of B and denote $\mathcal{F}(B)$ the class of the normal upper semi-continuous $[0,1]$-valued functions defined on B with bounded closure of the support. Let $(\Omega, \mathbf{A}, P)$ be a probability space. A *fuzzy random variable* is a mapping $X : \Omega \longrightarrow \mathcal{F}(B)$ such that, for all $\alpha \in [0,1]$, the set-valued α-level mapping $X_\alpha : \Omega \longrightarrow \mathcal{K}(B)$, $X_\alpha(\omega) = (X(\omega))_\alpha$ is a compact random set, that is, it is Borel measurable with respect to the Borel σ-field generated by the topology associated with the well-known Hausdorff metric on $\mathcal{K}(B)$.

The usefulness of this somewhat technical notion follows, e.g., from a detailed discussion in [4] explaining how, using the notion of fuzzy random variable as a tool, classical methods of mathematical statistics can be modified to get corresponding methods of "fuzzy statistics".

ANSWER 3. *Fuzzy random variables serve as a (non-random) tool to transform classical probability law (in terms of distribution functions) into fuzzy probability law (in terms of fuzzy distribution functions)*

4. Basic notions of OPT and their roles

In OPT (cf. [11], [2], [3]) random events (called also effects) are measurable fuzzy sets and fuzzified probability measures (also called states) are integrals with respect to probability measures — this is in accordance with Zadeh's proposal ([19]). The essential novelty is in the fuzzification of random variables: *the outcome* ω *of a random experiment can be mapped to a probability measure on* $\mathbf{B}_R$ (or on some other target measurable space)! Accordingly, the measurability of operational random variable (also statistical map, or a fuzzy random variable in the Bugajski-Gudder sense) and the corresponding probability distribution have to be defined in a different way than in CPT. This leads to different interpretation, mathematical apparatus, and applications (cf. [5], [14], [15], [16], [17], [10]).

QUESTION 4. *What is the role of operational random variables*?

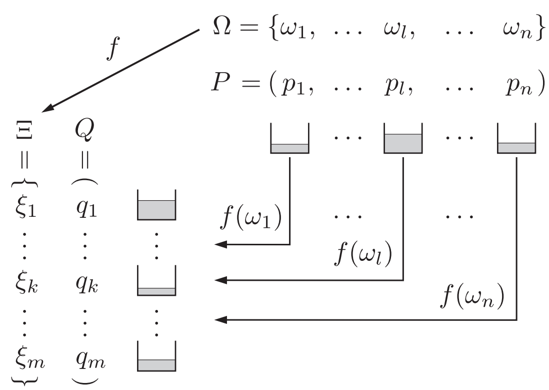

Figure 1

To avoid technicalities, we illustrate the involved notions and constructions on the discrete (finite) case (cf. [9]). So, assume that

$$\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$$

and

$$\Xi = \{\xi_1, \xi_2, ..., \xi_m\}$$

are two finite sets, $\mathbf{A}$ and $\mathbf{B}$ are the sets of all subsets of Ω and Ξ, and P and Q are probability measures on $\mathbf{A}$ and $\mathbf{B}$, respectively. Then P and Q reduce to probability functions $P = (p_1, p_2, ..., p_n)$, $p_l = P(\{\omega_l\})$, $l = 1, 2, ..., n$, and $P(A) = \sum_{\omega_l \in A} p_l$, $A \subseteq \Omega$, resp. $Q = (q_1, q_2, ..., q_m)$, $q_k = Q(\{\xi_k\})$, $k = 1, 2, ..., m$, and $Q(B) = \sum_{\xi_k \in B} q_k$, $B \subseteq \Xi$. Each map $f : \Omega \longrightarrow \Xi$ is measurable and f is a $((\Xi, \mathbf{B})$-valued, classical) random variable whenever $Q(B) = P(f^{\leftarrow}(B))$, where $q_k = Q(\{\xi_k\}) = \sum_{\omega_l \in f^{\leftarrow}(\{\xi_k\})} p_l$

and $Q(B) = \sum_{\xi_k B} q_k = \sum_{\omega_l \in f^{\leftarrow}(B)} p_l$. Simply, the random variable f can be viewed as a system of pipelines through which the probability measure P is distributed to become Q, see Fig. 1. Observe that each p_l goes exactly to one ξ_k.

An operational random variable can be viewed as a more complex system of fuzzified pipelines, see Fig. 2. Each p_l is distributed along Ξ via the probability function (measure) $(w_{1l}, w_{2l}, ..., w_{ml})$.

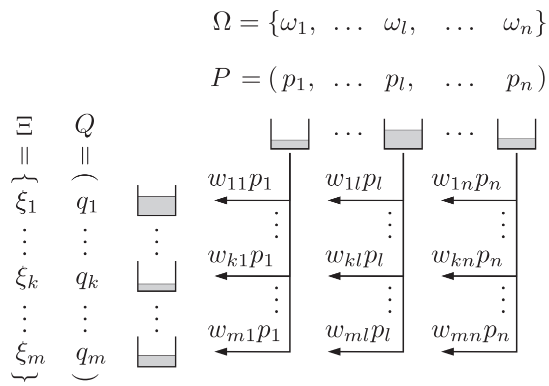

FIGURE 2

For each $l \in \{1, 2, ..., n\}$, p_l is distributed among the elements of Ξ as follows: $w_{kl}p_l$ goes to ξ_k, $k \in \{1, 2, ..., m\}$, and the sum $\sum_{l=1}^{n} w_{kl}p_l$ represents the total input q_k of the probability which flows into ξ_k. Further, for $B \subseteq \Xi$, the sum $\sum_{\xi_k \in B} q_k$ represents the probability $Q(B)$ and, explicitly,

$$Q(B) = \sum_{\xi_k \in B} \sum_{l=1}^{n} w_{kl}p_l = \sum_{l=1}^{n} p_l \sum_{\xi_k \in B} w_{kl}.$$

This points to a fuzzy event $u_B \in [0,1]^{\Omega}$ defined by $u_B(\omega_l) = \sum_{\xi_k \in B} w_{kl}$, $l \in \{1, 2, ..., n\}$, and to a canonical map h_c (defined by $h_c(\chi_B) = u_B$) sending crisp events $\mathbf{B}$ to fuzzy events $[0,1]^{\Omega}$.

The importance of h_c comes from the fact, that $Q(B)$ is "the integral of u_B with respect to P"!

Finally, h_c can be canonically extended to a map h on fuzzy events $[0,1]^{\Xi}$ into fuzzy events $[0,1]^{\Omega}$ so that "the integral with respect to Q" is the composition of h and "the integral with respect to P"!!

Next, we briefly outline how the discrete case of OPT can be extended to the general case.

Remark 4.1. *Let* $(\Omega, \mathbf{A})$ *be a measurable space (remember, we consider subsets as crisp fuzzy sets). Denote* $\mathcal{M}(\mathbf{A})$ *the set of all measurable functions* *of Ω into the unit interval $[0,1]$. In OPT the operations on fuzzy random events $\mathcal{M}(\mathbf{A})$ (generalizations of Boolean operations on classical random events) follow the Łukasiewicz logic: $x \oplus y = \min\{1, x+y\}$, $x^c = 1 - x$, $x \odot y = \max\{0, x+y-1\}$ for the unit interval $[0,1]$, and coordinate-wise for fuzzy sets. Observe that if $\mathbf{T} = \{\emptyset, \{a\}\}$ is a two-element field of sets, then $[0,1] \equiv \mathcal{M}(\mathbf{T})$.*

Remark 4.2. *Let X be a set and let $[0,1]^X$ be the set of all fuzzy subsets of X carrying the coordinate-wise partial ordered ($v \leq u$ whenever $v(x) \leq u(x)$ for all coordinates x) and the partial binary operation of difference "$\ominus$" defined coordinate-wise: $(u \ominus v)(x) = u(x) - v(x)$ whenever $v(x) \leq u(x)$ for all coordinates x. In OPT, an important role is played by* D-posets of fuzzy sets*, i.e., subsets $\mathcal{X} \subseteq [0,1]^X$ carrying the inherited coordinate-wise partial ordered, containing the top element 1_X, the bottom element 0_X, and closed with respect to the inherited partial binary operation of difference. Both $\mathbf{A}$ and $\mathcal{M}(\mathbf{A})$, $\mathbf{A} \subset \mathcal{M}(\mathbf{A})$, are distinguished D-posets of fuzzy sets: they model random events in CPT and in OPT, respectively. Let h be a map on a D-poset of fuzzy sets $\mathcal{Y} \subseteq [0,1]^Y$ into a D-poset of fuzzy sets $\mathcal{X} \subseteq [0,1]^X$. If h preserves the order, the top and bottom elements, and the difference, then it is called a D-*homomorphism. *Sequentially continuous (with respect to the coordinate-wise convergence of sequences) D-homomorphisms play a crucial role in OPT (cf. Lemma 4.1 and Corollary 4.2 in* [6]*).*

Theorem 4.3. *Let $(\Omega, \mathbf{A})$ and $(\Xi, \mathbf{B})$ be measurable spaces.*

(i) *Each sequentially continuous D-homomorphism on $\mathbf{B}$ into $\mathcal{M}(\mathbf{A})$ can be uniquely extended to a sequentially continuous D-homomorphism on $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$;*

(ii) *Integrals on $\mathcal{M}(\mathbf{A})$ with respect to probability measures on $\mathbf{A}$ are exactly sequentially continuous D-homomorphism on $\mathcal{M}(\mathbf{A})$ into $[0,1]$.*

Remark 4.4. *Observe that, according to the Lebesgue Dominated Convergence Theorem, every integral with respect to a probability measure is sequentially continuous. It is easy to verify that a composition of two sequentially continuous D-homomorphisms is a sequentially continuous D-homomorphism. Consequently, a composition of a sequentially continuous D-homomorphism on $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$ and a sequentially continuous D-homomorphism on $\mathcal{M}(\mathbf{A})$ into $[0,1]$ is an integral on $\mathcal{M}(\mathbf{B})$ with respect to a probability measure on $\mathbf{B}$.*

Definition 4.5. *Let $(\Omega, \mathbf{A})$, $(\Xi, \mathbf{B})$ be measurable spaces. Let T be a map on $\mathcal{P}(\mathbf{A})$ into $\mathcal{P}(\mathbf{B})$ such that, for each $B \in \mathbf{B}$, the assignment $\omega \mapsto \big(T(\delta_\omega)\big)(B)$ yields a measurable map on Ω into $[0,1]$ and*

$$\text{(BG)} \qquad \big(T(P)\big)(B) = \int \big(T(\delta_\omega)\big)(B)\,\mathrm{d}P$$

for all $P \in \mathcal{P}(\mathbf{A})$ and all $B \in \mathbf{B}$. Then T is said to be a operational random variable *(also a statistical map, or a fuzzy random variable in the sense of Bugajski and Gudder).*

Remark 4.6. *The assignment $\omega \mapsto \big(T(\delta_\omega)\big)(B)$ results in a sequentially continuous D-homomorphism h_c on $\mathbf{B}$ into $\mathcal{M}(\mathbf{A})$, hence can be uniquely extended to a sequentially continuous D-homomorphism h on $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$. It is known that this way to each operational random variable T of $\mathcal{P}(\mathbf{A})$ into $\mathcal{P}(\mathbf{B})$ there corresponds a unique sequentially continuous D-homomorphism $h = T^{\triangleleft}$ on $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$ called* observable *and, vice versa (via the composition of a sequentially continuous D-homomorphism h on $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$ and integrals in (BG)), to each sequentially continuous D-homomorphism h on $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$ there corresponds a unique T and $h = T^{\triangleleft}$.*

ANSWER 4. *It is a channel through which a fuzzified stochastic information is transported. Each outcome of a random experiment is mapped to a "local" probability measure (possibly a degenerated point-measure, always in CPT) which fuzzifies the corresponding observation and each probability measure P on the sample random events $\mathbf{A}$, and hence the corresponding integral on $\mathcal{M}(\mathbf{A})$ (taking into the account all "local" probability measures) is then transformed to a fuzzified "global" probability measure Q on the observed crisp random events $\mathbf{B}$, and hence the corresponding integral on the observed fuzzy random events $\mathcal{M}(\mathbf{B})$. This yields an operational random variable $T : \mathcal{P}(\mathbf{A}) \longrightarrow \mathcal{P}(\mathbf{B})$. To T there corresponds a unique observable $T^{\triangleleft} : \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ and the composition of $T^{\triangleleft}$ and the integral on $\mathcal{M}(\mathbf{A})$ with respect to P is the integral on $\mathcal{M}(\mathbf{B})$ with respect to Q.*

5. CPT embedded in OPT

Since to each σ-field $\mathbf{A}$ there corresponds exactly one D-poset of fuzzy sets $\mathcal{M}(\mathbf{A})$ and to each probability measure P on $\mathbf{A}$ there corresponds exactly one integral on $\mathcal{M}(\mathbf{A})$ with respect to P, the extension of CPT to OPT looks "conservative". The real novelty is revealed when comparing the corresponding notions of a random variable.

QUESTION 5. *What is the difference between random variables and observables in CPT and OPT?*

ANSWER 5. *Both a classical random variable and a operational random variable model channels through which probability measures are transported, but the latter has a quantum character: a degenerated probability point-measure can be mapped to a non-degenerate probability measure. Dually, both a classical observable and a fuzzy observable map events (represented* *by special D-posets of fuzzy sets) to events, but the latter can map a crisp event to a genuine fuzzy one.*

QUESTION 6. *How are CPT and OPT related?*

ANSWER 6. *CPT can be embedded into OPT in a natural way: classical random evens form a σ-field $\mathbf{A}$ and are embedded into the fuzzy random events $\mathcal{M}(\mathbf{A})$, every generalized probability measure is an integral on $\mathcal{M}(\mathbf{A})$ with respect to a classical probability measure on $\mathbf{A}$, each classical probability measure is the restriction of such integral to $\mathbf{A}$, and there is a one-to-one correspondence between random events (σ-fields of sets) and fuzzy random events (measurable maps into $[0,1]$). Since every sequentially continuous D-homomorphism on $\mathbf{A}$ into $[0,1] \equiv \mathcal{M}(\mathbf{T})$ can be uniquely extended to an observable on $\mathcal{M}(\mathbf{A})$ to $\mathcal{M}(\mathbf{T})$, to every probability measure P on crisp events $\mathbf{A}$ there corresponds a unique observable (integral with respect to P) on fuzzy events $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{T})$. Consequently, generalized probability measures in OPT become observables. Hence probability measures in CPT are "shadows" of fuzzy morphisms.*

Additional information on categorical approach to probability theory can be found in [7], [8].

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