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RECOVERING THE SHAPE OF AN EQUILATERAL QUANTUM TREE WITH THE DIRICHLET CONDITIONS AT THE PENDANT VERTICES

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Abstract. We consider two spectral problems on an equilateral rooted tree with the standard (continuity and Kirchhoff's type) conditions at the interior vertices (except of the root if it is interior) and Dirichlet conditions at the pendant vertices (except of the root if it is pendant). For the first (Neumann) problem we impose the standard conditions (if the root is an interior vertex) or Neumann condition (if the root is a pendant vertex) at the root, while for the second (Dirichlet) problem we impose the Dirichlet condition at the root. We show that for caterpillar trees the spectra of the Neumann problem and of the Dirichlet problem uniquely determine the shape of the tree. Also, we present an example of co-spectral snowflake graphs.

Keywords: Sturm–Liouville equation, eigenvalue, spectrum, equilateral tree, caterpillar tree, snowflake graph, root, standard conditions, Dirichlet boundary condition, Neumann boundary condition.

Mathematics Subject Classification: 34B45, 34B24, 34L20.

1. INTRODUCTION

A review on the classical problem of recovering the shape of a combinatorial graph using the eigenvalues of its adjacency matrix is described in [6, Chapter 6], where several examples of co-spectral graphs are given. However, if the number of vertices in a graph is sufficiently small the spectrum of the graphs adjacency matrix uniquely determines the shape of the graph.

The situation in quantum graph theory is similar. It was shown in [10] that if the lengths of the edges are non-commensurate then the spectrum of the spectral Sturm–Liouville problem on a graph with standard conditions (Neumann conditions at the pendant vertices and continuity+ Kirchhoff's at the interior vertices) uniquely determines the shape of this graph. In [1], it was shown that in case of commensurate lengths of the edges there exist co-spectral quantum graphs.

While generalizations of Ambarzumian's theorem for graphs are known (see, e.g. [7, 12, 18]), the authors of [3, 13] proved the "geometric" Ambarzumian's theorem: it was shown that the spectrum of the Neumann problem with zero potential on P_2 (on a tree of two vertices, i.e. on a segment) uniquely determines the shape of the graph. This result was generalized in [9] where it was shown that if the graph is simple connected equilateral with the number of vertices less or equal 5 and the potentials on the edges are real L_2 functions then the spectrum of the Sturm–Liouville problem with standard conditions at the vertices uniquely determines the shape of the graph. The authors of [9] used the result of [8] which states that the L_2-potentials do not influence the first and the second terms of the eigenvalue asymptotics and the result of [2] relating the spectrum of the Sturm–Liouville problem with the standard conditions on a graph to the spectrum of the normalized Laplacian of this graph.

By co-spectral quantum graphs we mean two (or more) non-isomorphic metric graphs for which the spectra of the Sturm–Liouville problems on which coincide. The vertex conditions for these problems must be the same (e.g. standard conditions, i.e. continuity and Kirchhoff's conditions) at the interior vertices and the Neumann conditions at the pendant vertices or standard conditions at the interior vertices and the Dirichlet conditions at the pendant vertices)

In the case of standard conditions the minimal number of vertices in a co-spectral pair of trees is 9 (see [17]). If the number of vertices doesn't exceed 8 then to find the shape of a tree we need just to find in [9] the characteristic polynomial corresponding to the given spectrum. For the problem on a tree with the Dirichlet conditions at the pendant vertices similar to those in [9] results were obtained in [5], namely it was shown that if the number of vertices in a tree does not exceed 8, then the spectrum of the Sturm–Liouville problem uniquely determines the shape of a tree.

In this paper we consider two spectral problems on an equilateral rooted tree with the standard conditions at the interior vertices (except of the root if it is interior) and Dirichlet conditions at the pendant vertices (except of the root if it is pendant). For the first (Neumann) problem we impose the standard conditions (if the root is an interior vertex) or Neumann condition (if the root is an interior vertex) at the root while for the second (Dirichlet) problem we impose the Dirichlet condition at the root. The idea to use two spectra comes from the inverse problem of recovering the potential of the Sturm–Liouville equation on an interval by two such spectra solved in classical works ([4], see also [14]).

We show how to find the shape of a tree using two spectra: the spectra of the Dirichlet and of the Neumann problems described above. This method works even in case of large number of vertices. If the solution is not unique we can find all the solutions. We give an example of double-co-spectral trees, i.e. of a pair of trees with the same spectrum of the Dirichlet problem and the same spectrum of the Neumann problem. We prove that the solution of such inverse problem is unique in case of caterpillar trees. The inverse problem of recovering the shape of a tree with the Neumann conditions at the pendant vertices using two spectra is considered in [19], where the technics of expanding into a branched continued fraction was introduced. The inverse problem on a caterpillar tree with the Neumann conditions at the pendant vertices (vertices of degree 1) is solved in [11].

It should be mentioned that the spectra of the Neumann and Dirichlet problems can be obtained from the S-function of the scattering problem obtained by attaching a lead at the root of the graph (see [16]).

In Section 2 we describe the Neumann and the Dirichlet spectral problems. Also we expose known results which we use in the sequel.

In Section 3 we prove a theorem where the fraction of the characteristic polynomial of the normalized Laplacian of the corresponding combinatorial tree and the modified characteristic polynomial of its certain subgraph (a tree or a forest), obtained by deleting the root and the incident edges is presented as a branched continued fraction of a special form. This continued fraction contains information about the shape of the tree.

In Section 4 we consider particular cases of snowflake trees and caterpillar trees where the above mentioned fraction together with the total number of edges is used to determine the shape of the combinatorial tree.

In Section 5 we describe asymptotics of eigenvalues of the Neumann and of the Dirichlet problems on an equilateral metric tree and show the relation of these asymptotics to the eigenvalues of the corresponding normalized Laplacians and of the modified characteristic polynomial of a subgraph of the tree.

In Section 6 we show the procedure of recovering the shape of a tree using asymptotics of the spectra of the Neumann and Dirichlet problems. We present an example of double-co-spectral snowflake graphs. We prove that there are no double-co-spectral caterpillar trees.

2. STATEMENT OF THE PROBLEMS AND AUXILIARY RESULTS

Let T be an equilateral tree with p vertices and $q = p - 1$ edges each of the length l. We choose an arbitrary vertex v_0 as the root and direct all the edges away from the root. Let us describe the *Neumann* spectral problem on this tree.

2.1. NEUMANN PROBLEM

We consider the Sturm–Liouville equations on the edges

$$-y_j'' + q_j(x)y_j = \lambda y_j, \quad j = 1, 2, \ldots, g, \tag{2.1}$$

where $q_j \in L_2(0, l)$ are real.

For each edge e_j incident with a pendant vertex which is not the root we impose the Dirichlet condition

$$y_j(l) = 0. \tag{2.2}$$

At each interior vertex which is not the root we impose the continuity conditions

$$y_j(l) = y_k(0) \tag{2.3}$$

for the edge e_j incoming into v_i and for all edges e_k outgoing from v_i and the Kirchhoff's conditions

$$y_j'(l) = \sum_k y_k'(0) \tag{2.4}$$

where the sum is taken over all edges e_k outgoing from v_i.

If the root is an interior vertex then the conditions at v_0 are

$$y_i(0) = y_j(0) \tag{2.5}$$

for all indices i and j of the edges incident with the root and

$$\sum_i y_i'(0) = 0. \tag{2.6}$$

If the root is pendant and its incident edge is e_1 then

$$y_1'(0) = 0. \tag{2.7}$$

In the sequel, if the potentials are the same on all the edges we omit the lower index in q_j, y_j, s_j, c_j. The following theorem is Theorem 6.4.2 of [15] adopted for a tree with the standard condition at the root and at all the interior vertices and the Dirichlet conditions at all the pendant vertices (except of the root if it is pendant vertex).

If we delete the pendant vertices (except of the root if it is pendant) with all the incident edges from our tree T then we obtain a tree which we denote by T'.

Theorem 2.1. *Let T be a tree with $p \geq 2$ vertices and p_{pen} pendant vertices. Let r be the number of pendant vertices with the Dirichlet conditions ($r = p_{pen}$) if the root is an interior vertex and $r = p_{pen} - 1$ if the root is a pendant vertex). Assume that all edges have the same length l and the same potentials symetric with respect to the midpoints of the edges ($q(l-x) = q(x)$). Then the spectrum of problem* (2.1)–(2.6) *or* (2.1)–(2.4), (2.7) *coincides with the set of zeros of the function*

$$\phi_N(\lambda) = s^{r-1}(\sqrt{\lambda}, l)\psi(c(\sqrt{\lambda}, l)), \tag{2.8}$$

where

$$\psi(z) = \det(-zD' + A'). \tag{2.9}$$

Here A' is the adjacency matrix of T', v_0, $v_1, \ldots,$ v_{p-1-r} are the vertices of T',

$$D' = \mathrm{diag}(d(v_0), d(v_1), \ldots, d(v_{p-1-r})),$$

$d(v_i)$ is the degree of the vertex v_i in T, $s(\sqrt{\lambda}, x)$ and $c(\sqrt{\lambda}, x)$ are the solutions of the Sturm–Liouville equation on the edges satisfying the conditions

$$s(\sqrt{\lambda}, 0) = s'(\sqrt{\lambda}, 0) - 1 = 0 \quad \textit{and} \quad c(\sqrt{\lambda}, 0) - 1 = c'(\sqrt{\lambda}, 0) = 0.$$

Here $(-zD' + A')$ is the submatrix of the normalized Laplacian of T obtained by deleting r rows and r columns corresponding to the pendant vertices with the Dirichlet condition.

Corollary 2.2. *If the spectra of problems* (2.1)–(2.6) (*or* (2.1)–(2.4), (2.7)) *on two trees coincide then these trees have the same number of edges.*

2.2. DIRICHLET PROBLEM

Now we consider the Dirichlet problem. We change the standard condition at the root v_0 for the Dirichlet condition:

$$y_i(0) = 0 \tag{2.10}$$

for all edges incident with v_0, and consider the Dirichlet problem which consists of equations (2.1)–(2.4) and (2.10).

Then we can consider T as a union of $d(v_0)$ subtrees T_1, T_2, ..., $T_{d(v_0)}$. Of course, if v_0 is pendant the we have one tree. These trees have the common vertex v_0. We denote by $\hat{T}$ the tree obtained by deleting all the pendant vertices and the root together with the edges incident with them and by $\hat{A}$ the adjacency matrix of $\hat{T}$.

Let us consider spectral problems on the subtrees T_js of T meaning that the Dirichlet conditions are imposed at v_0 and at all the pendant vertices while at the interior vertices we keep the standard conditions. Thus, we obtain $d(v_0)$ problems on the subtrees. The spectrum of our Dirichlet problem is the union of the spectra of the Dirichlet problems on the subtrees T_j.

If now we delete all the pendant vertices and the root from T with all incident edges, we obtain a forest (if the root v_0 is an interior vertex) or a tree (if the root is a pendant vertex). We denote this forest or tree by $F = \hat{T}_1 \cup \hat{T}_2 \cup \ldots \cup \hat{T}_{d(v_0)}$.

Let $\hat{A}_i$ be the adjacency matrix of $\hat{T}_i$, let

$$\hat{D}_{T,i} = \mathrm{diag}\{d(v_{i,1}), d(v_{i,2}), \ldots, d(v_{i,p_i})\},$$

where $d(v_{i,j})$ is the degree of the vertex $v_{i,j}$ in T_i (we underline that in T_i, not in $\hat{T}_i$!) and p_i is the number of vertices $\{v_{i,1}, \ldots, v_{i,p_i}\}$ in $\hat{T}_i$.

We consider the polynomials defined by

$$\hat{\psi}_i(z) := \det(-z\hat{D}_{T,i} + \hat{A}_i). \tag{2.11}$$

Theorem 2.3. *Let T_i be a subtree of the tree T rooted at a vertex v_0. Let the Dirichlet condition be imposed at all pendant vertices of T_i and the standard conditions at all its interior vertices. Assume that all edges have the same length l and the same potentials symmetric with respect to the midpoints of the edges ($q(l-x) = q(x)$). Then the spectrum of problem* (2.1)–(2.4), (2.10) *on T_i coincides with the set of zeros of the characteristic function*

$$\phi_{D,i}(\lambda) = s^{p_{pen,i}-1}(\sqrt{\lambda}, l)\hat{\psi}_i(c(\sqrt{\lambda}, l)), \tag{2.12}$$

where $p_{pen,i}$ is the number of pendant vertices in $\hat{T}_i$.

Proof. Theorem 2.1 is nothing but Theorem 6.4.2 of [15] adapted to the case of a tree with the Dirichlet conditions at all the pendant vertices. □

It is clear that the characteristic function of the Dirichlet problem (2.1)–(2.4), (2.10) on F' is

$$\phi_D(\lambda) = \prod_{i=1}^{d(v_0)} \phi_{D,i}(\lambda) = \prod_{i=1}^{d(v_0)} \hat{\psi}_i(c\sqrt{\lambda}, l) = \prod_{i=1}^{d(v_0)} \det(-c(\sqrt{\lambda}, l)\hat{D}_{T,i} + \hat{A}_i). \tag{2.13}$$

Denote by

$$\hat{\psi}(z) := \prod_{i=1}^{d(v_0)} \hat{\psi}_i(z). \tag{2.14}$$

It is clear that

$$\hat{\psi}(z) = \det(-z\hat{D} + \hat{A}), \tag{2.15}$$

where $\hat{A}$ be the adjacency matrix of the forest F.

3. NORMALIZED LAPLACIAN, ITS PRINCIPAL SUB-MATRIX AND THE SHAPE OF A TREE

First of all we notice that $-z\hat{D} + \hat{A}$ is the principal sub-matrix of matrix $-zD + A$ obtained by deleting those rows and columns which correspond to v_0 and to the pendant vertices.

Theorem 3.1. *Let T be an equilateral tree. Then the fraction $\frac{\psi(z)}{\hat{\psi}(z)}$ can be expanded in branched continued fraction of the special form* (3.1)–(3.3)*. The coefficients before $+z$ or $-z$ correspond to the degrees of the vertices. The beginning fragment*

$$-m_0 z + \sum_{k=1}^{m_0} \frac{1}{m_k z - \ldots} \tag{3.1}$$

of the expansion means that the vertex v_0 is connected by edges with m_0 vertices, say $v_1, v_2, \ldots, v_{m_0}$. A fragment

$$\ldots \pm \sum_{i=1}^{r} \frac{1}{-m_i z + \sum_{k=1}^{m_i - 1} \frac{1}{+m_{i,k} z + \ldots}} \tag{3.2}$$

means that there are r vertices each have one incoming edge and $m_i - 1$ $(i = 1, \ldots, r)$ outgoing edges. A fragment

$$\ldots \pm \frac{1}{mz} \tag{3.3}$$

at an end of a branch of the continued fraction means m edges ending with pendant vertices with the Dirichlet boundary conditions.

Proof. Let the first row in matrix $-zD + A$ correspond to v_0 and the next $d(v_0)$ rows correspond to the vertices adjacent with v_0. The first row expansion of the determinant of matrix $-zD + A$ gives

$$\det(-zD + A) = -d(v_0)z\det(-z\hat{D} + \hat{A}) - \sum_{k=1}^{d(v_0)} (-1)^k \det(-z\check{D}_k + \check{A}_k) \tag{3.4}$$

where the principal sub-matrix $(-z\hat{D} + \hat{A})$ is obtained from $(-zD + A)$ by deleting the first row and the first column while $(-z\check{D}_k + \check{A}_k)$ is the one obtained from $(-z\hat{D} + \hat{A})$ by deleting its kth row and its kth column. The corresponding subtrees can be seen at Figures 1–4.

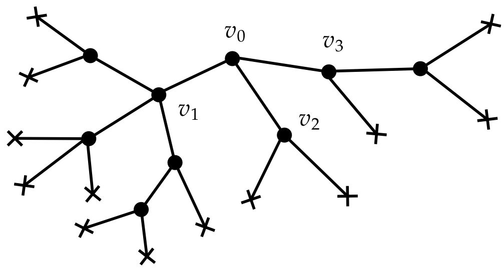

Fig. 1. A tree T

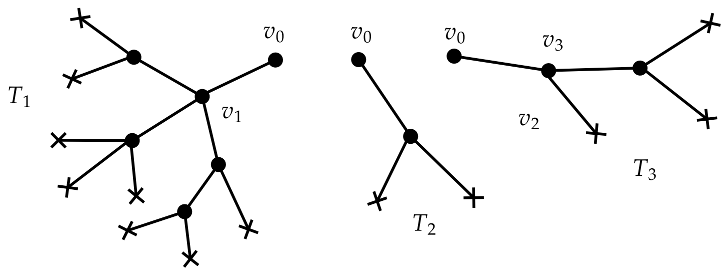

Fig. 2. Subtrees T_1, T_2, T_3 of the tree T

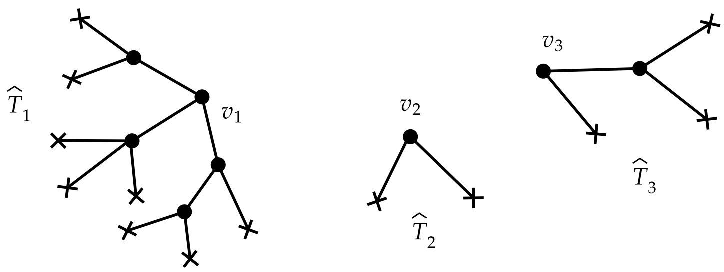

Fig. 3. The subforest $\hat{T}$ consisting of the subtrees $\hat{T}_1$, $\hat{T}_2$, $\hat{T}_3$

Fig. 4. The subforest $\check{T}$

Dividing both parts of (3.4) by $\det(-z\hat{D}+\hat{A})$ we continue expanding into branched fraction and obtain

$$\begin{aligned}\frac{\psi(z)}{\hat{\psi}(z)} &= \frac{\det(-zD+A)}{\det(-z\hat{D}+\hat{A})} = -d(v_0)z - \frac{\sum_{i=1}^{d(v_0)}\det(-z\check{D}_i+\check{A}_i)}{\det(-z\hat{D}+\hat{A})} \\ &= -d(v_0)z + \sum_{i=1}^{d(v_0)} \frac{1}{d(v_i)z - \sum_{i=1}^{d(v_i)-1}\frac{\hat{\psi}_i(z)}{\check{\psi}_i(z)}}.\end{aligned} \tag{3.5}$$

Here $\check{\psi}_i(z)$ is the modified characteristic polynomial of the subtree $\check{T}_i$. To finish the proof we need to continue this procedure. □

Example 3.2. Let $\psi(z) = -144z^5 + 88z^3 - 7z$ and $\hat{\psi}(z) = 72z^4 - 14z^2$. Then

$$\begin{aligned}\frac{\psi(z)}{\hat{\psi}(z)} &= -2z + \frac{60z^3-7z}{72z^4-14z^2} = -2z + \frac{1}{2z} + \frac{24z^3}{72z^4-14z^2} \\ &= -2z + \frac{1}{2z} + \frac{1}{3z - \frac{7}{12z}}.\end{aligned} \tag{3.6}$$

Now we need to present $\frac{7}{12z}$ as a sum $\frac{1}{m_1 z} + \frac{1}{m_2 z}$ where m_1 and m_2 are natural numbers. It can be done in two ways:

$$\frac{\psi(z)}{\hat{\psi}(z)} = -2z + \frac{1}{2z} + \frac{1}{3z - \frac{1}{4z} - \frac{1}{3z}}. \tag{3.7}$$

and

$$\frac{\psi(z)}{\hat{\psi}(z)} = -2z + \frac{1}{2z} + \frac{1}{3z - \frac{1}{2z} - \frac{1}{12z}}. \tag{3.8}$$

However, we underline that these trees have different number of edges. Thus, this branched continued fraction corresponds to the trees of Figure 5.

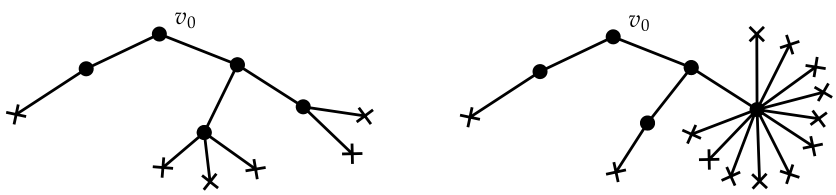

Fig. 5. The trees corresponding to
$\psi(z) = -144z^5 + 88z^3 - 7z$ and $\hat{\psi}(z) = 72z^4 - 14z^2$

Let us show that there are no other trees corresponding to $\psi(z) = -144z^5+88z^3-7z$ and $\hat{\psi}(z) = 72z^4 - 14z^2$. To prove it we notice that

$$\lim_{z\to\infty} = -\frac{1}{z}\frac{\psi(z)}{\hat{\psi}(z)} = 2$$

what means that the degree of the root is 2. Now

$$\lim_{z\to\infty} z\left(\frac{\psi(z)}{\hat{\psi}(z)}+2z\right)=\frac{5}{6}.$$

Therefore,

$$\frac{5}{6}=\frac{1}{d_1}+\frac{1}{d_2}$$

where d_1, d_2 are the degrees of the vertices adjacent with the root. This equation has the only (up to permutations) solution $d_1=2$, $d_2=3$ in natural numbers $d\geq 2$, $d_2\geq 2$. Since the denominator of

$$\frac{60z^2-7}{z(72z^2-14)}$$

contains a factor z we conclude that expansion of this fraction into sum of two summands must contain either $\frac{1}{2z}$ or $\frac{1}{3z}$. In the first case we have (3.6). Since the equation

$$\frac{1}{d_1}+\frac{1}{d_2}=\frac{7}{12}$$

possesses only two (up to permutations) solutions in integer numbers such that $d_1\geq 2$, $d_2\geq 2$, namely $d_1=4$, $d_2=3$ and $d_1=2$, $d_2=12$, what leads to (6.1) and (6.2), respectively.

In the second case we have

$$\frac{\psi(z)}{\hat{\psi}(z)}=-2z+\frac{1}{3z}+\frac{108z^3-7z}{216z^4-42z^2}=-2z+\frac{1}{3z}+\frac{1}{2z-\frac{28z}{108z^2-7}}.$$

The only way to continue expanding into continued fraction as in Theorem 3.1 is

$$\frac{\psi(z)}{\hat{\psi}(z)}=2z+\frac{1}{3z}+\frac{1}{2z-\frac{1}{\frac{9}{7}z-\frac{1}{4z}}}$$

what does not correspond to any tree.

Thus, there are only two trees corresponding to $\psi(z)=-144z^5+88z^3-7z$ and $\hat{\psi}(z)=72z^4-14z^2$.

4. PARTICULAR CASES

4.1. SNOWFLAKE TREES

Definition 4.1. By snowflake graph we mean a tree with the combinatorial distance between the root and any pendant vertex less or equal two (see an example in Figure 6).

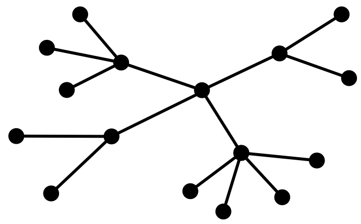

Fig. 6. Snowflake graph

In the case of a snowflake graph equation (3.5) looks as follows:

$$\frac{\psi(z)}{\hat{\psi}(z)} = -d(v_0)z + \sum_{k=1}^{d(v_0)} \frac{1}{d(v_k)z}. \tag{4.1}$$

Thus, we can find the degree of the central vertex:

$$\lim_{z\to\infty} \left(-\frac{1}{z}\frac{\psi(z)}{\hat{\psi}(z)}\right) = d(v_0) \tag{4.2}$$

Thus, we can calculate

$$z\left(\frac{\psi(z)}{\hat{\psi}(z)} + d(v_0)z\right) = \sum_{k=1}^{d(v_0)} \frac{1}{d(v_k)}. \tag{4.3}$$

The number of edges $q = \sum\limits_{k=1}^{d(v_0)} d(v_k)$. So, to have double-co-spectral graphs, i.e. graphs with the same spectrum of the Neumann problem and the same spectrum of the Dirichlet problem we need the system of equations

$$\sum_{k=1}^{d(v_0)} \frac{1}{d(v_k)} = r, \tag{4.4}$$

$$\sum_{k=1}^{d(v_0)} d(v_k) = m \tag{4.5}$$

to have more than one solution in positive natural numbers. Here we mean the solutions are the same if they can be identified after permutations. The system (4.4), (4.5) has two solutions in case of $d(v_0) = 3$, $m = 26$, $r = \frac{13}{30}$. These solutions are $d(v_1) = 12$, $d(v_2) = 10$, $d(v_3) = 4$ and $d(v_1) = 15$, $d(v_2) = 6$, $d(v_3) = 5$. This example was proposed by Sergey Saprikin.

4.2. CATERPILLAR TREES

Now we consider a caterpillar tree rooted at the end of its stalk with the Dirichlet conditions at some of its pendant vertices except of the root. We consider two problems: the first with the Neumann condition at the root and the second with the Dirichlet condition at it (see Figure 7).

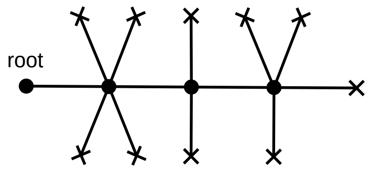

Fig. 7. Caterpillar tree

In this case expansion (3.1)–(3.3) has the following form

$$\frac{\psi(z)}{\hat{\psi}(z)} = -d(v_0)z + \frac{1}{d(v_1)z - \frac{1}{d(v_2)z - \ldots - \frac{1}{d(v_{r-1})z}}}, \tag{4.6}$$

where v_0, v_1,..., v_{r-1} are the vertices on the stalk. Then the degrees $d(v_i)$ can be found by the equations

$$d(v_0) = -\lim_{z\to\infty} \frac{\psi(z)}{z\hat{\psi}(z)}, \tag{4.7}$$

$$d(v_1) = \lim_{z\to\infty} \left(\frac{1}{z}\frac{\psi(z)}{\hat{\psi}(z)} + d(v_0)z \right)^{-1}. \tag{4.8}$$

Thus, we see that in this case the functions $\psi(z)$ and $\hat{\psi}(z)$ uniquely determine the shape of the tree.

It should be mentioned that the caterpillar graphs with the Neumann conditions at the pendant vertices were considered in [11].

5. THE SPECTRA ASYMPTOTICS

Here we describe asymptotics of the Neumann and the Dirichlet problems specctra and show their connection with the functions $\psi(z)$ and $\hat{\psi}(z)$.

Using the asymptotics of the spectrum of the Neumann problem we can find the function $\psi(\lambda)$ (up to a constant factor). Let us show it.

Theorem 5.1. *Let T be an equilateral tree. The eigenvalues of problem* (2.1)–(2.6) *or* (2.1)–(2.4), (2.7) *can be presented as the union of subsequences*

$$\{\lambda_k\}_{k=1}^{\infty} = \bigcup_{i=1}^{p-1} \{\lambda_k^{(i)}\}_{k=1}^{\infty}$$

with the following asymptotics:

$$\sqrt{\lambda_k^{(i)}} \underset{k\to\infty}{=} \frac{2\pi(k-1)}{l} \pm \frac{1}{l}\arccos\alpha_i + O\left(\frac{1}{k}\right) \quad \textit{for } i=1,2,\dots,p-r,\ k=1,2,\dots, \tag{5.1}$$

$$\sqrt{\lambda_k^{(i)}} \underset{k\to\infty}{=} \frac{\pi k}{l} + O\left(\frac{1}{k}\right) \quad \textit{for} \ \ i=p-r+1,\dots,p-1, \quad k=1,2,\dots \tag{5.2}$$

Here $\alpha_1, \alpha_2, \dots, \alpha_{p-r}$ are the zeros of the polynomial $\psi(z)$.

Proof. Let $\{\tilde\lambda_k\}_{k=1}^{\infty} = \bigcup_{i=1}^{p-1} \{\tilde\lambda_k^{(i)}\}_{k=1}^{\infty}$ be the spectrum in the case of $q_j(x) \equiv 0$ for all j. By Theorem 2.1, we obtain the following for the case of zero potentials:

$$\sqrt{\tilde\lambda_k^{(i)}} = \frac{2\pi(k-1)}{l} \pm \frac{1}{l}\arccos\alpha_i \quad \text{for } i=1,2,\dots,p-r,\ k=1,2,\dots \tag{5.3}$$

$$\sqrt{\tilde\lambda_k^{(i)}} = \frac{\pi k}{l} \quad \text{for } i=p-r+1,\dots,p-1, \quad k=1,2,\dots \tag{5.4}$$

Here branch (5.1) is generated by the solutions of the equations $\cos(\sqrt{\lambda}l) = \alpha_i$ while branch (5.2) by the solutions of equation $(\frac{\sin(\sqrt{\lambda}l)}{\sqrt{\lambda}})^{-1+r} = 0$.

By [8, Theorem 5.4], we conclude that $|\lambda_k^{(j)} - \tilde\lambda_k^{(j)}| \le C < \infty$, where $\tilde\lambda_k^{(j)}$ are the eigenvalues of problem (2.1)–(2.6) or (2.1)–(2.4), (2.7) on the same tree with $q_j \equiv 0$ for all j and therefore, the presence of the $L_2(0,l)$-potentials does not influence the first and the second terms of the asymptotics (5.1)–(5.2). $\square$

Problem (2.1)–(2.4), (2.10) can be considered as $d(v_0)$ independent Dirichlet problems on the subtrees T_j, $j=1,2,\dots,d(v_0)$ (problems with the Dirichlet conditions at all their pendant vertices). Let p_i be the number of the vertices in T_i and r_i the number of pendant vertices in it. By Theorem 2.3 and equation (2.14), we see that the spectrum $\{\nu_k\}_{k=1}^{\infty}$ of the Dirichlet problem (2.1)–(2.4), (2.10) on T is the union

$$\{\nu_k\}_{k=1}^{\infty} = \bigcup_{i=1}^{d(v_0)} \bigcup_{j=1}^{p_i-1} \{\nu_{k,j}^{(i)}\}_{k=1}^{\infty}$$

of the spectra of the Dirichlet problems on the subtrees T_i ($i=1,\dots,d(v_0)$). According to Theorem 2.3, the spectrum of the Dirichlet problem on T_i is the set of zeros of $\hat\psi_i(z)$. Thus we arrive at the following theorem:

Theorem 5.2. *Let T be an equilateral tree. The eigenvalues of the problem* (2.1)–(2.4), (2.10) *can be presented as the union of subsequences $\{\nu_k\}_{k=1}^{\infty} = \bigcup_{i=1}^{d(v_0)} \{\nu_k^{(i)}\}_{k=1}^{\infty}$ with the following asymptotics:*

$$\sqrt{\nu_{k,j}^{(i)}} = \lim_{k\to\infty} \frac{2\pi(k-1)}{l} \pm \frac{1}{l}\arccos\beta_{i,j} + O\left(\frac{1}{k}\right) \tag{5.5}$$
for $j = 1, 2, \ldots, p_i - r_i, \quad k = 1, 2, \ldots,$

$$\sqrt{\nu_{k,j}^{(i)}} = \lim_{k\to\infty} \frac{\pi k}{l} + O\left(\frac{1}{k}\right) \tag{5.6}$$
for $j = p_i - r_i + 1, \ldots, p_i - 1, \quad k = 1, 2, \ldots,$

where $\beta_{i,j}$ are the zeros of $\hat{\psi}_i(z)$ which is the determinant in (2.11).

Theorem 5.3. *Let $\{\lambda_k\}_{k=1}^{\infty}$ be the spectrum of the Neumann problem* (2.1)–(2.6) *or* (2.1)–(2.4), (2.7) *and*

$$\{\nu_k\}_{k=1}^{\infty} = \bigcup_{i=1}^{d(v_0)} \bigcup_{j=1}^{p_i-1} \{\nu_{k,j}^{(i)}\}_{k=1}^{\infty}$$

be the spectrum of the Dirichlet problem (2.1)–(2.4), (2.10) *where the Dirichlet condition is imposed at a vertex v_0 of degree $d(v_0)$. Let $\{\alpha_k\}_{k=1}^{p}$ be the constants in* (5.1), (5.2) *and $\{\beta_{i,j}\}_{i=1,j=1}^{d(v_0),p_i}$ the constants in* (5.5). *Then*

$$\frac{\psi(z)}{\hat{\psi}(z)} = d(v_0)\frac{\prod_{i=1}^{p}(-z+\alpha_i)}{\prod_{i=1}^{d(v_0)}\prod_{j=1}^{p_i-1}(-z+\beta_{i,j})}. \tag{5.7}$$

Proof. By Theorem 2.1, we know that $\{\alpha_k\}_{k=1}^{p}$ is the set of zeros of $\psi(z)$ and by Theorem 2.3 that $\{\beta_{i,j}\}_{i=1,j=1}^{d(v_0),p_i}$ is the set of zeros of $\hat{\psi}(z)$. Thus, we conclude that

$$\frac{\psi(z)}{\hat{\psi}(z)} = C\frac{\prod_{i=1}^{p}(-z+\alpha_i)}{\prod_{i=1}^{d(v_0)}\prod_{j=1}^{p_i-1}(-z+\beta_{i,j})},$$

where C is a nonzero constant. By (3.5), we obtain $C = d(v_0)$. □

6. RECOVERING THE SHAPE OF A QUANTUM TREE BY TWO SPECTRA

To recover the shape of a tree we need to construct $\frac{\psi(z)}{\hat{\psi}(z)}$ according to (5.7).

The degree of the root $d(v_0)$ is the number of subsequences of the form (5.5). Then we find $\{\alpha_k\}_{k=1}^{p}$ and $\{\beta_{i,j}\}_{i=1,j=1}^{d(v_0),p_i}$ from asymptotics (5.1), (5.2) and (5.5), (5.6), respectively, and then expanding (5.7) into branched continued fraction we find the shape of the tree. The following example shows that the found tree is not always unique.

Theorem 6.1. *Let the potentials $q_j(x) \equiv 0$ for all j. The graphs shown in Figure 8 are double co-spectral, i.e. have the same spectrum of problem* (2.1)–(2.6) *and the same spectrum of problem* (2.1)–(2.4), (2.10).

Proof. Calculations show that the spectrum of problem (2.1)–(2.6) on each of the trees shown in Figure 8 consists of the subsequences with:

$$\sqrt{\lambda_k^{(4)}} = \frac{2\pi(k-1)}{l} \pm \frac{1}{l}\arccos\sqrt{\frac{13}{90}} \quad \text{for } k = 1, 2, \ldots, \tag{6.1}$$

$$\sqrt{\lambda_k^{(1)}} = \frac{2\pi(k-1)}{l} \pm \frac{1}{l}\arccos\left(-\sqrt{\frac{13}{90}}\right) \quad \text{for } k = 1, 2, \ldots, \tag{6.2}$$

$$\sqrt{\lambda_k^{(i)}} = \frac{\pi(2k-1)}{2l} \quad \text{for } i = 2, 3,\ k = 1, 2, \ldots, \tag{6.3}$$

$$\sqrt{\lambda_k^{(i)}} = \frac{\pi k}{l} \quad \text{for } i = 5, 6, \ldots, 26,\ k = 1, 2, \ldots \tag{6.4}$$

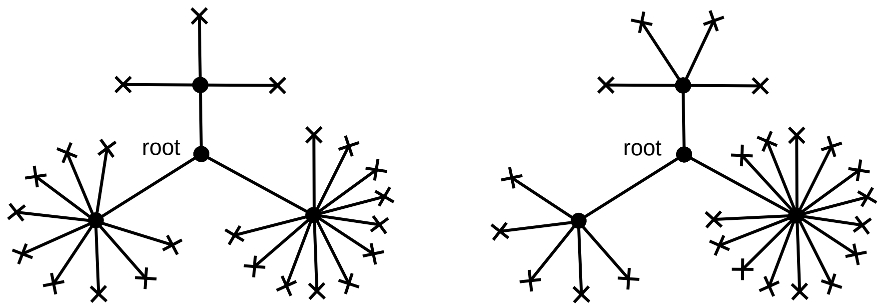

Fig. 8. Trees having the same spectrum in case of the Neumann (standard) conditions at the root and the same spectrum in case of the Dirichlet condition at the root

The spectrum of problem (2.1)–(2.4), (2.10) on each of the trees shown in Figure 8 consists of the subsequences with the following asymptotics:

$$\sqrt{\nu_k^{(i)}} = \frac{\pi(2k-1)}{2l} \quad \text{for } i = 1, 2, 3,\ k = 1, 2, \ldots, \tag{6.5}$$

$$\sqrt{\nu_k^{(i)}} = \frac{\pi k}{l} \quad \text{for } i = 4, 5, \ldots, 26,\ k = 1, 2, \ldots \tag{6.6}$$

□

However, in case of caterpillar graphs the two spectra uniquely determine the shape of the tree.

Theorem 6.2. *Let the spectrum of problem* (2.1)–(2.6) *on a tree consist of subsequences with the asymptotics* (5.1), (5.2) *and the spectrum of problem* (2.1)–(2.4), (2.10) *consist of subsequences* (5.5), (5.6). *Let* $d(v_0)$ *be the number of subsequences* (5.5), (5.6) *and let*

$$\frac{\prod_{i=1}^{p}(-z+\alpha_i)}{\prod_{i=1}^{d(v_0)}\prod_{j=1}^{p_i-1}(-z+\beta_{i,j})} = -z + \frac{1}{a_1 z - \frac{1}{a_2 z - \ldots - \frac{1}{a_{r-1} z}}}. \tag{6.7}$$

Then the tree is a caterpillar graph rooted at one of the ends of its stalk with the degrees of the vertices on the stalk $d(v_i) = a_i$ *for* $i = 1, 2, \ldots, r-1$.

Proof. Asymptotics (5.1), (5.2) and (5.5), (5.6) uniquely determine l, g, $d(v_0) = 1$ and the constants α_i and $\beta_{i,j}$. According to equations of Subsection 4.2 (equations (4.7), (4.8) ..., the constants $d(v_0)$, α_i and $\beta_{i,j}$ uniquely determine the coefficients a_i in (6.7). The coefficients a_i we identify as $d(v_i)$. □

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