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A MODEL FOR OPTIMIZING ENTERPRISE'S INVENTORY COSTS. A FUZZY APPROACH

Applicability of a fuzzy approach to a problem originating from administrative accounting, namely to determine an economic order quantity (EOQ) in a variable competitive environment with imprecise and vague data, has been presented. For this purpose, the model of ordered fuzzy numbers developed by the first author and his two co-workers is used. The present approach generalizes the one developed within the framework of convex fuzzy numbers and stays outside the probabilistic one.

Keywords: *economic order quantity, optimization problem, ordered fuzzy numbers*

1. Introduction

We encounter inventory management issues in commercial and industrial enterprises. According to the accounting act, inventories include: a) finished products produced or processed by the entity, b) products in the process of production, c) semi-finished products – products that have undergone stages of closed-tech manufacturing processes, d) goods purchased for resale without further processing. Cost-effective inventory management can indirectly affect the value of the company measured as the value of expected cashflows. The level of cash flow results from the volumes of the company's incomes and expenditure. The volume of sales (revenue) depends on the availability of finished products and changes in working capital, as growth in these is

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associated with a drop in sales, while a reduction is associated with an increase in sales. Maintaining large inventories also increases fixed and variable costs.

The aim of this paper is to show the applicability of a fuzzy approach to a problem originating from administrative accounting, namely to determine an economic order quantity (EOQ) in a variable competitive environment with imprecise and vague data. For this purpose, the model of ordered fuzzy numbers developed by the first author [4] and his two co-workers was used. The present approach generalizes the one developed within the framework of convex fuzzy numbers and stays outside the probabilistic one.

For many years the only tool representing imprecise and vague notions was probability theory. Hence, any suggestion of substituting this tool by an approach related to fuzzy logic and fuzzy sets leads to the question: is it worth doing this, and if yes, then why? In this paper, we focus our fuzzy approach on applications to problems in economics, for which modelling the influence of imprecise quantities and preferences on a decision maker's opinions is important. With the help of a fuzzy number it is possible to express incomplete knowledge about a quantity by giving the range in which its realization can appear, and writing it in the form of a (subjective) function of the information, representing the capability degree of any realization. In this case, only one condition appears, namely capability degrees may take values from the interval [0, 1], where complete impossibility is expressesed by 0, while 1 expresses full capability. There are no more restrictions on the form of such a function. On the other hand, one may use a random variable but in order to describe and to model this situation with imprecise quantities, the random variable has to supply information about a probability, rather than the capability degrees of the possible realizations (values of the quantity).

This means that in this probabilistic case, one is forced here to give the probability distribution of the quantity (even it is subjective). Then we have to fulfil some constraint which follows from the definition of a probability distribution. In the case of a fuzzy number approach, on the other hand, the decision maker is completely free as far as the form of the capability degree function of the imprecise quantity is concerned.

The notion of an ordered fuzzy number (OFN) arrived a decade ago, proposed by the first author, together with two co-workers: P. Prokopowicz and D. Ślęzak, to eliminate several drawbacks of classical convex fuzzy numbers (CFN), such as the unbounded growth in fuzziness which results from a large number of arithmetic operations on fuzzy numbers and the lack of solutions to simple linear equations using fuzzy numbers [4]. This new model is not based on the classical notion of the membership function introduced by Zadeh. A less restrictive concept is used, namely a membership relation. Each convex fuzzy number possesses two representatives in the form of ordered fuzzy numbers which differ by their orientation. This means that each element of an OFN has an additional attribute, namely its orientation, which is not present in a CFN [1, 3, 8]. In this way, we are able to reflect information about the trend of changes in such imprecise quantities in the modeling. This information is not present whenever the CFN approach or the probabilistic one are applied.

In this paper, wel propose a solution to a problem originating from administrative accounting, i.e. inventory management using the model of an OFN, namely to determine an economic order quantity (EOQ) in a variable competitive environment with imprecise and vague data. If the decision maker chooses his tool for making crisp numbers from fuzzy ones, i.e. a defuzzification functional, the method proposed here gives a tool for determining the optimal order quantity that minimizes the total inventory cost. In this way, a decision support tool is constructed for when data are fuzzy. The final result of this paper manifests the applicability of ordered fuzzy numbers as a tool based on which investors can make investment decisions.

The organization of the paper is as follows. In Section 2 we present the problem of inventory management. We quote the basic indicators of the quality of such management. Then we considerthreats usingrisk analysis. Here, we specify risk categories in tabular form, together with a description of their sources and levels of impact on costs. This section ends with a list of 3 options that can be used to select an optimal strategy for inventory management.

In Section 3, we formulate the model for determining the economic order quantity (EOQ) in a deterministic setup. Then we pass on to its fuzzy formulation. Section 4 brings some conclusions. In the Appendix, the main concepts and definitions involved in the model of an ordered fuzzy number are briefly presented.

2. The issue of inventory management

Effective inventory management improves the liquidity of a company, lowers the cost of storage and results in an uninterrupted production process. Maintaining either excessive or too low inventory levels leads to a reduction in the efficiency of the company. Too much inventory generates costs associated with the maintenance of warehouse space, inventory storage costs and costs of frozen capital. Too low inventory levels can interfere with the production process and sales through a lack of materials or finished products. Here you can find many sources of emerging risks. One of the indicators summarizing the level of inventories in a company is the inventory turnover ratio

$$W_{\mathrm{rot}} = \frac{P}{\frac{Z_p + Z_k}{2}} \tag{1}$$

where P – revenue from the sale of goods, materials, finished products, Z_p – inventory at the beginning of the analyzed period, Z_k – inventory at end of the analyzed period.

Another indicator evaluating the correctness of inventory management is the inventory turnover period, given by

$$O_{\mathrm{rot}} = \frac{365}{W_{\mathrm{rot}}} \tag{2}$$

The turnover period indicates how long on average current supplies will last to conduct operations. An increase in the turnover ratio means that stocks will last for a shorter period of sales or production. The inventory turnover period indicates the number of days between renewals of the company's inventory. A long inventory turnover period means a slow turnover of stocks, a short period – rapid turnover. A too long period for holding stocks results in an increase in the costs of production and worsens the company's financial situation. The size of stocks should result from the current demand for materials and goods. At the same time, when analyzing the efficiency of warehouse management, one must take into account the adverse effects of increasing inventory turnover ratios such as: ordering costs increase due to more frequent purchases, the risk of inventory shortages, disrupting of the rhythm of production, loss of customers.

Costs associated with inventory management can be divided into three groups:

1. The costs of maintaining an inventory:
- frozen capital costs,
- storage costs,
- costs of inventory maintenance,
- risk associated with the loss of physical inventories and random events.

2. The costs of creating stocks:
- procurement costs,
- transportation costs,
- costs of receiving inventory.

3. The costs of inventory shortages:
- costs of production downtime,
- costs of lost sales,
- costs of lost reputation.

2.1. Risk analysis

In a company, an increase in operational risk associated with inventory management results in a decrease in the value of the company measured as a sum of cash flows. Too low value of inventory increases the likelihood of interference to the rhythm of production. Reducing operational risk by increasing the value of stock increases fixed and variable costs which, in turn, leads to a reduction in the value of cash flows and therefore the value of the company. Table 1 describes categories of operational risk.

Table 1. Categories of operational risk

Risk category	Description of sources of risk	Degree of impact on the cost
Increase in stock prices	Uncertainty about the price of the materials used in the company caused by: the geopolitical situation, inflation, changes in exchange rates. These risks are minimized by maintaining larger stocks.	The increase in costs as a result of price increases – for example: prices of materials used in production .
Providing steady production	Lack of materials in stock results in increased costs associated with downtime such as loss of customers, contractual penalties. Disappointed customers can switch to the competition to satisfy their needs.	The increase in costs incurred by the company and the loss of revenue from sales.
Uncertainty in supply and delays	Difficulties in ensuring steady production which, in turn, may lead to loss of customers and profit from sales.	
Degradation and aging of inventory	Some stocks require the provision of specific conditions such as: appropriate humidity, optimal temperature. Not providing such conditions results in loss of their utility value.	The increase in costs associated with maintaining optimal conditions for storing stocks and the costs of inventory liquidation.

To determine optimal management strategies, an entrepreneur should calculate which option generates the highest cash flow. We discuss three such options.

Option 1. Maintain high values of stock while benefiting from the discounts and lower transport costs resulting from less frequent deliveries of large batches of materials.

Option 2. Minimize the value of stock by: more frequent deliveries, shortening the production cycle, shortening the duration of storage of finished goods, which leads to lower costs of:

- frozen capital in the inventory and forsaken opportunities to invest available cash,
- storage and management of stocks, such as rental of warehouse space,
- related to the ageing of stocks.

Option 3. Maintain a high value of stock, in order to reduce operational risk which has the effect of increasing the cost of:

- production downtime,
- lost sales due to a lack of goods or finished products,
- lost reputation due to inadequate customer service.

3. Inventory optimization

Every firm faces the challenge of matching its supply volume to customer demand. How well a firm manages this challenge has a major impact on its profitability. Also, the amount of inventory held has a major impact on the amount of cash available. With working capital at a premium, it is important for companies to keep inventory levels as low as possible and to use or sell inventory as quickly as possible. Inventory is one of the most key factors that analysts take into account when making earnings forecasts or recommendations to buy and sell.

The challenge of managing inventory is becoming ever more complex. Models of inventory optimization can be deterministic – with each set of variable states being uniquely determined by the parameters in the model, or stochastic – with variable states described by probability distributions or fuzzy numbers. In this paper, we propose a fuzzy number approach.

3.1. Deterministic model

Inventory management within an enterprise is an integral part of its operational activities as it affects the liquidity of its financial performance and competitive advantage. The purpose of inventory management is to have stock at a high enough level to operate smoothly, while incurring the lowest possible operating costs.

The present formulation is within the general framework of the model of the economic order quantity (EOQ). We consider an abstract inventory item. To estimate the cost of inventory management, we formulate the main assumptions in the EOQ model:

- the abstract inventory item is split into units,
- we refer to some time unit, say one year,
- demand is constant in time,
- sales are uniform in time and known,
- the next delivery arrives just as the stock level falls to zero.

Let us start with a deterministic formulation in which the following objects appear:

D – annual inventory demand, measured in number of units,
Q – order quantity, measured in number of units,
c – unit purchase cost,
$r(Q)$ – discount function,
K_t – transportation cost of a single delivery,
K_s – unit inventory cost,
K_u – unit cost of loss (aging or extraordinary loss),

θ – fraction of possible unexpected inventory loss,

R – banking interest rate, used to calculate the cost of frozen capital.

The assumption about the arrival of each supply makes it possible to simplify the problem and to state that the average level of inventory is $Q/2$. However, the statement that the annual frequency of delivery D/Q is obvious. Now we can write the general expression for the total cost $K(Q)$, as the sum of the purchase cost K_z and the storage cost K_m, i.e.

$$K(Q) = K_z + K_m$$

We assume that the discount function is a step function of the following form (in line with Options 1 and 2 described above):

$$r(Q) = \begin{cases} r_0, & \text{if} \quad Q_0 \le Q < Q_1 \\ r_1, & \text{if} \quad Q_1 \le Q < Q_2 \\ r_2, & \text{if} \quad Q \ge Q_2 \end{cases} \tag{3}$$

where Q_0, Q_1 and Q_2 are fixed quantities. Here three steps are assumed. However, more steps can also be considered. The purchase cost function $K_p(Q)$ depends on the quantity in a single delivery Q, the frequency of deliveries, the discount $r(Q)$ and the unit price c, and is given by

$$K_p(Q) = c\left(1 - r(Q)\right)Q\frac{D}{Q} = c\left(1 - r(Q)\right)D.$$

From the form of the discount function $r(Q)$, we can see that this is a piecewise constant function.

It is obvious that the cost of frozen capital depends on:

- the number of deliveries D/R,
- the money spent on a single delivery,
- the banking interest rate R,
- the annual frequency of deliveries.

The form of the purchase cost K_p leads to the following cost K_f of frozen capital:

$$K_f = \frac{D}{Q}\left(1 - r(Q)\right)Qc\frac{R}{\frac{D}{Q}}$$

which reduces to

$$K_f = \left(1 - r(Q)\right) QcR \tag{4}$$

where the function $r(Q)$ is given by Eq. (3). We can see that the expression K_f represents a step function, which is piecewise linear. Hence, the cost $K_z(Q)$, related to the total costs of frozen capital, together with the purchase, and transportation (delivery) to a warehouse of D/Q deliveries per year, is given by the expression

$$K_z(Q) = c\left(1 - r(Q)\right)D + \left(1 - r(Q)\right)QcR + K_t \frac{D}{Q} \tag{5}$$

The conditions of Option 2 lead to the following total storage cost K_m

$$K_m = \underbrace{K_s \frac{Q}{2}}_{\text{storage cost}} + \underbrace{K_u \theta \frac{Q}{2}}_{\text{unexpected loss}} \tag{6}$$

Hence, the function describing the total cost

$$K(Q) = K_z(Q) + K_m(Q)$$

can be expressed as

$$K(Q) = \underbrace{c\left(1 - r(Q)\right)\left(D + RQ\right)}_{\text{purchase cost + capital cost}} + \underbrace{\left(K_s + K_u\,\theta\right)\frac{Q}{2}}_{\text{storage cost + unexpected loss}} + \underbrace{K_t \frac{D}{Q}}_{\text{transportation cost}} \tag{7}$$

Optimization problem. Hence, the optimization problem of inventory management requires us to find the minimum of the cost function $K(Q)$ in Eq. (7). The argument which gives the minimum is the optimal value of the order quantity.

Notice that in Eq. (5) the first component depends on Q in a piecewise way, and the search for the optimal value should be performed in a piecewise way, i.e. considering each subinterval

$$L_0 := (0, Q_1), \qquad L_1 := [Q_1, Q_2) \qquad \text{and} \qquad L_2 := [Q_2, D]$$

Thus the global optimum is the quantity which gives the minimal cost over the three optimal values calculated from each subinterval. Notice that in each of these subintervals L_i, $i = 0, 1, 2$ the local optimum is attained at

$$\overline{q}_i = \left(\frac{K_t D}{M}\right)^{1/2}, \qquad M = c(1 - r_i)R + \frac{K_s + K_u\theta}{2}, \qquad i = 0, 1, 2 \tag{8}$$

provided a given $\overline{q}_i$ belongs to the corresponding subinterval L_i, $i = 0, 1, 2$. Otherwise, $\overline{q}_i$ is assumed to be the end point of the correspoinding interval which gives the minimum cost. From these three candidates, the optimal value is calculated according to

$$\arg\left\{\min\left[\left(K\left(\overline{q}_i\right):\overline{q}_i \in L_i, i = 0, 1, 2\right), \left(K\left(Q_j\right), j = 0, 1, 2\right)\right]\right\} \tag{9}$$

More complex problems can also be formulated, in which the assumption concerning the arrival of the next delivery will be omitted and the so-called security level of stock appears, and then some dynamics will enter our optimization problem. This more general case will be considered in a future paper.

3.2. Fuzzy optimization problem

This formulation is within the framework of the model of the economic order quantity (EOQ) and similar to the one proposed in the set of CFN by [9] and considered by [7].

In terms of the fuzzy optimization problem, our aim is to give a general solution with the cost function given by (7) when D, K_t, K_s, K_u and θ are fuzzy and represented by ordered fuzzy numbers (OFN).

It will be easy to see that the arithmetic of OFN manifests its superiority over the arithmetic of convex fuzzy numbers (CFN), and the complex calculations performed by the authors of [9] and [7] can be avoided. The only thing we need to do is to choose the defuzzification functional which suits the decision maker the most.

Let $\phi(\cdot)$ be the defuzzification functional chosen by the decision maker. Then the problem of minimizing the fuzzy cost $K(Q)$ gives us the economic order quantity. Writting this problem explicitly

$$\text{find } \arg\left\{\min \phi(K(Q)) : Q \in R\right\}$$

the following question arises: how can we find the minimum of this functional? The answer is rather obvious and comes from physics, and is formulated according to the stationary action principle: the minimum of the functional appears at the argument Q where its first variation (the Gâteaux derivative) vanishes. Calculating the first variation of $\phi(K(Q))$ with respect to Q under given D, K_t, K_s, K_u and θ, we get

$$\delta\phi(K(Q)) = \partial_K \phi(K) \partial_Q K(Q) \delta Q \tag{10}$$

Here $\partial_K \phi(K)$ and $\partial_Q K(Q)$ denote functional derivatives. Due to the arbitrariness of δQ, the condition $\delta\phi(K(Q)) = 0$ implies that

$$\partial_K \phi(K) \partial_Q K(Q) = 0$$

and the argument Q^*, at which the product of these derivatives vanishes, gives us the solution to our optimization problem.

To illustrate this, let us consider a class of linear functionals given by Eq. (20). Let us denote the branches of the fuzzy number $K(Q)$ by (f_K, g_K), and for the remaining quantities we adapt the previous notation by using the appropriate subscripts, i.e.

$$\begin{gathered} Q = \left(f_Q, g_Q\right), \quad D = \left(f_D, g_D\right), \quad K_t = \left(f_t, g_t\right) \\ K_u = \left(f_u, g_u\right), \quad \theta = \left(f_\theta, g_\theta\right) \end{gathered} \tag{11}$$

Then the linear functional describing the fuzzy cost $K(Q)$ has the form

$$\phi\big(K(Q)\big) = \phi\left(f_K, g_K\right) = \int_0^1 f_K(s) dh_1(s) + \int_0^1 g_K(s) dh_2(s) \tag{12}$$

where, see Eq. (7),

$$\begin{aligned} f_{K_i}(s) &= f_c(s)(1 - r_i)(f_D(s) + R\, f_Q(s)) \\ &+ \frac{f_t(s)\, f_D(s)}{f_Q(s)} + \frac{f_s(s) + f_u f_\theta(s)}{2} f_Q(s), \qquad i = 0,\, 1,\, 2 \end{aligned} \tag{13}$$

and the form of $g_{K_i}(s)$ is analogous. Now we differentiate Eq. (10) where the functional is given by Eq. (12), to get

$$\begin{aligned} \delta\varphi(K(Q)) &= \int_0^1 \left(f_M(s) - f_t(s) \frac{f_D(s)}{(f_Q(s))^2} \right) \delta f_Q(s) dh_1(s) \\ &+ \int_0^1 \left(g_M(s) - g_t(s) \frac{g_D(s)}{(g_Q(s))^2} \right) \delta g_Q(s) dh_2(s) \end{aligned} \tag{14}$$

where $\left(f_{M_i}(s), g_{M_i}(s)\right), i = 0, 1, 2$ represent 3 fuzzy numbers $\tilde{M}_i$ as OFN, i.e.

$$f_{M_i}(s) = \frac{f_s(s) + f_u f_\theta(s)}{2} + f_c(s)(1 - r_i)R \tag{15}$$

and $g_{M_i}(s)$ has an analogous form.

We can consider two cases:

Case A. The functions h_1 and h_2 are absolutely continuous, and

Case B. The functions h_1 and h_2 are singular, i.e. $h_1'(s)$ and $h_2'(s)$ are equal to zero almost everywhere.

In [2] we discussed a less complex case and proved that in both cases the forms of h_1 and h_2 in Eq. (12) have no effect on the optimal value of *Q*. In the present case, however, we formulate the following theorem:

Theorem. If the total inventory cost *K*(*Q*) arising from the fuzzy unit costs of purchase *c*, of inventory K_s, of transportation K_t, of loss K_u and its fraction θ, together with the annual demand *D*, the discount function *r*(*Q*), and banking interest rate *R*, are given by Eq. (7) and the decision maker chooses the defuzzification functional ϕ given in Eq. (12), then in the case A the economic order quantity is given by a two phase optimization procedure:

- First phase: the optimal values on each subinterval L_0, L_1, L_2 are found. We have

$$q_i^* = \phi\left(K(Q_i^*)\right), \qquad \text{where} \qquad Q_i^* = \left(f_{Q_i}^*, g_{Q_i}^*\right)$$

if q_i^* belongs to corresponding interval, where $f_{Q_i}^*(s)$ and $g_{Q_i}^*(s)$ are given by

$$f_{Q_i}^*(s) = \left(\frac{f_t(s) f_D(s)}{f_{M_i}}\right)^{1/2}, \qquad g_{Q_i}(s) = \left(\frac{g_t(s) g_D(s)}{g_{M_i}}\right)^{1/2}, \qquad s \in [0,1] \tag{16}$$

where

$$f_{M_i} = f_c(s)\left(1 - r_i\right)R + \frac{f_s(s) + f_u(s) f_\theta(s)}{2}, \qquad i = 0, 1, 2, \qquad s \in [0,1]$$

and the expression for g_{M_i} is analogous. When q_i^* as defined above does not belong to the corresponding interval, then it is defined to be equal to the end point of one of the intervals L_0, L_1, L_2 which gives the smallest value of the defuzzification functional.

• Second phase: from these 6 expressions, the optimal value is calculated according to

$$\arg\left\{\min\left[\left(\phi(f_{K_i}, g_{K_i}) : i = 0, 1, 2\right), \left(\phi(f_{K(Q(j)}, g_{KQ(j)}) : j = 0, 1, 2\right)\right]\right\} \tag{17}$$

using the notation from Eqs. (9), (11), (13), and where the expressions for f_{K_i} and g_{K_i} in Eq. (16) are used to calculate the pair (f_Q, g_Q).

Remark. If the rebate function is just a constant, i.e. $Q_0 = Q_1 = Q_2$, then the economic order quantity is given by

$$q^* = \phi\left(K(Q^*)\right), \qquad \text{where} \qquad Q^* = \left(f_Q^*, g_Q^*\right)$$

with

$$f_Q^*(s) = \left(\frac{f_t(s) f_D(s)}{f_M}\right)^{1/2}, \qquad g_Q(s) = \left(\frac{g_t(s) g_D(s)}{g_M}\right)^{1/2}, \qquad s \in [0, 1]$$

where

$$f_M(s) = f_c(s)\left(1 - r_2\right) R + \frac{f_s(s) + f_u(s) f_\theta(s)}{2}, \qquad s \in [0, 1]$$

and the expression for g_M is analogous. In the particular case $\phi = \phi_{MOM}$, the minimal cost is given by $q^* = \left(K(f_Q^*(1)) + K(g_Q^*(1))\right)/2$, with the function K given by Eq. (7).

4. Conclusions

Here we have solved a problem originating from inventory management, using the model of ordered fuzzy numbers, and we have demonstrated its applicability in modelling the influence of imprecise quantities and preferences of a decision maker.

Thanks to the well-defined arithmetic of OFN, one can construct efficient decision support tools when data are imprecise. In our next paper, we will introduce some dynamics into inventory management and show that OFN can be successfully applied to the presentation of stock prices giving a transparent image of the stock exchange.

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Appendix

The frist author and his two coworkers Prokopowicz and Ślęzak [4–6] recently proposed an extended model of convex fuzzy numbers [3] (CFN), called ordered fuzzy numbers (OFN), which does not require any membership functions. Using this model, we obtain an extension of the CFN model, when one takes a parametric representation of fuzzy numbers, known since 1986 [1].

Definition 1. By an ordered fuzzy number we understand a pair of functions (*f*, *g*) defined on the unit interval [0,1] which are continuous functions (or of bounded variation) [4–6].

Four algebraic operations have been proposed for OFN, denoted by R (or R_{BV}), with the arguments being fuzzy numbers and crisp (real) numbers, in which componentwise operations are present. In particular,

$$f_C(y) = f_A(y) \times f_B(y), \qquad g_C(y) = g_A(y) \times g_B(y), \tag{18}$$

where × can be replaced by $+$, $\cdot$, or $\div$, and where $A \div B$ is defined if the functions and $|g_B|$ are strictly bounded from below by 0. Hence, any fuzzy algebraic equation

$A + X = C$, where A and C are OFN, possesses a solution. A convex fuzzy number corresponds to two OFNs, which differ by their orientation.

A relation of partial ordering in the space of all OFN can be introduced by defining the subset of positive' ordered fuzzy numbers: a number $A = (f, g)$ is not less than zero, and by writing

$$A \geq 0 \qquad \text{iff} \qquad f \geq 0, \qquad g \geq 0 \tag{19}$$

In this way, the set R (or R_{BV}) becomes a partially ordered ring, R using the relations corresponding to operations on reals.

Definition 2. A map ϕ from the space R (or R_{BV}) of all OFN's to reals is called a defuzzification functional if it satisfies: 1) $\phi(c^{\ddagger}) = c$, 2) $\phi(A + c^{\ddagger}) = \phi(A) + c$, 3) $\phi(cA) = c\phi(A)$, 4) $\phi(A) \geq 0$, if $A \geq 0$, for any $c \in R$ and $A \in R$, where $c^{\ddagger}(s) = (c, c), s \in [0,1]$ represents a crisp (real) number $c \in R$.

The linear functionals, MOM (middle of maximum), FOM (first of maximum), LOM (last of maximum) are given by specifying h_1 and h_2 in the following expressions:

$$\phi(f_A, g_A) = \int_0^1 f_A(s) dh_1(s) + \int_0^1 g_A(s) dh_2(s)$$

where h_1, h_2 are non-negative and of bounded variation and $\int_0^1 dh_1(s) + \int_0^1 dh_2(s) = 1.$

Example

In [7], the author considered the problem of minimizing the value of the fuzzy cost $K(Q)$ of a firm in which

$$\bar{K}(Q) = Dc + K_t \frac{D}{Q} + K_s \frac{Q}{2}$$

by neglecting the cost of frozen capital, discount, as well as unexpected loss. This corresponds to the case considered in the optimization problem from Sec. 3.1 and formula (8) for EOQ, but with $M = K_s/2$.

The author of [7] first considered the crisp (deterministic) case with the following data: $D = 1000$, $c = 10$, $K_t = 8$ and $K_s = 7$. According to her calculations, the economic order quantity Q_k is 46 and the total cost $\bar{K}(Q_k)$ corresponding to this order is 10 329. However, using our model, we get Q_k = 47.8 and the corresponding cost is $\bar{K}(Q_k)$ = 10 334.7. These values are different from those of Kuchta in [7].

She also considered the fuzzy case with the same crisp values of D and c but with the fuzzy transportation cost $\tilde{K}_t$ represented by the triangular membership function (7, 8, 9) and the fuzzy storage cost $\tilde{K}_s$ represented by the triangular membership function (1.5, 7, 15). Determination of the economic order quantity in this case is not unique, and is based on some estimation to be done by the decision maker if he/she is supplied with a set of fuzzy cost values determined be a formula in which the fuzzy values $\tilde{K}_t$ and $\tilde{K}_s$ appear, together with $2M + 1$ crisp values of Q from the neighborhood of Q_k, where M is a natural number determined by the decision maker (in Kuchta's paper it was 50). The decision maker then has to choose the most suitable for him/her from those $2M + 1$ fuzzy cost values.

On the other hand, if we apply our method and the linear defuzzification functional (20), then from the theorem for the case A (absolutely continuous h_1 and h_2) in Eq. (12), we get an explicit expression for the fuzzy EOQ. To this end, let us choose the representation of two convex triangular fuzzy numbers $\tilde{K}_t$ and $\tilde{K}_s$ as ordered fuzzy numbers. From the Appendix, we know that each CFN corresponds to two OFNs, which differ in orientation. Hence, for $\tilde{K}_t$ we have $(7 + s, 9 - s)$ and $(9 - s, 7 + s)$, with $s \in [0, 1]$. On the other hand, for $\tilde{K}_s$ we have $(1.5 + 5.5s, 15 - 8s)$ and $(15 - 8s, 1.5 + 5.5s)$. For $\tilde{K}_t$, if we take the first OFN, which has the so-called positive orientation, then this means that our estimate of the future transportation cost is rather pessimistic: the cost is at least around 8. On the other hand, if we take the second OFN, namely $(9 - s, 7 + s)$, then we are rather optimistic: the transportation cost is at most around 8.

For further calculation, we assume the pessimistic viewpoint and define $f_t(s) = 7 + s$ and $g_t(s) = 9 - s$, while $f_s(s) = 1.5 + 5.5s$ and $g_s(s) = 15 - 8s$. Notice that there are three other cases and, consequently, three other solutions for the fuzzy EOQ could follow.

Assuming a deterministic demand of 1000[1], applying the formula for the EOQ with $f_D(s) = 1000$, $f_M(s) = f_s(s)/2$, and $g_D(s) = 1000$, $g_M(s) = g_s(s)/2$, we obtain the fuzzy EOQ as the following ordered fuzzy number

$$f_Q^*(s) = \left(\frac{2000(7+s)}{1.5+5.5s}\right)^{1/2}, \quad g_Q^*(s) = \left(\frac{2000(9-s)}{15-8s}\right)^{1/2}, \quad s \in [0,1]$$

[1]Notice that in our example D is crisp and is represented by the pair of constant functions (1000†, 1000†).

From this expression we could easily calculate the fuzzy minimal inventory cost $\overline{K}(Q^*)$. Notice that neither Q^* nor $\overline{K}(Q^*)$ can be represented in the form of a CFN with a triangular membership function. We could draw figures for them by substituting values of s from the interval [0, 1]. By applying a particular defuzzification functional, we could calculate the crisp values corresponding to Q^* and $\overline{K}(Q^*)$. Finally, the characteristic values of Q^* are

$$f_Q^*(0) = \left(\frac{14\,000}{1.5}\right)^{1/2} = 96.6, \qquad f_Q^*(1) = g_Q^*(1)\left(\frac{16\,000}{7}\right)^{1/2} = 47.8,$$

$$g_Q^*(0) = \left(\frac{18\,000}{15}\right)^{1/2} = 34.6$$

Notice that by applying the defuzzification functional $\phi = \phi_{\mathrm{MOM}}$ to Q^*, we obtain the crisp EOQ $\phi = \phi_{\mathrm{MOM}}(Q^*) = f_Q^*(1) = 47.8$, which is equal to Q_k from the deterministic case.

Corresponding to these values, the characteristic values of the cost are:

$$f_K^*(0) = 10\,000 + f_t(0)\frac{1000}{f_Q^*(0)} + f_s(0)\frac{f_Q^*(0)}{2} = 10\,000 + 7\frac{1000}{96.6} + 1.5\frac{96.6}{2} = 10\,144.9$$

$$f_K^*(1) = g_K^*(1) = 10\,000 + f_t(1)\frac{1000}{f_Q^*(1)} + f_s(1)\frac{f_Q^*(1)}{2} = 10\,3344.6$$

$$g_K^*(0) = 10\,000 + g_t(0)\frac{1000}{g_Q^*(0)} + g_s(0)\frac{g_Q^*}{2} = 10\,000 + 9\frac{1000}{34.6} + 15\frac{34.6}{2} = 10\,519.6$$

Considering the data regarding the values of the fuzzy cost in [7], we can see (Table 7.1, p. 112) that the domains of the triangular membership functions of these values is the interval from 10 138 to 10 513. Moreover, these fuzzy values of cost are related to the range of order quantities from 91 to 36. In our calculations, this range is from 96.6 to 34.6.

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