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MULTIPLICITY RESULTS FOR PERTURBED FOURTH-ORDER KIRCHHOFF-TYPE PROBLEMS

Mohamad Reza Heidari Tavani, Ghasem Alizadeh Afrouzi, and Shapour Heidarkhani

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Abstract. In this paper, we investigate the existence of three generalized solutions for fourth-order Kirchhoff-type problems with a perturbed nonlinear term depending on two real parameters. Our approach is based on variational methods.

Keywords: multiplicity results, multiple solutions, fourth-order Kirchhoff-type equation, variational methods, critical point theory.

Mathematics Subject Classification: 34B15, 58E05.

1. INTRODUCTION

In this paper, we consider the following perturbed fourth-order Kirchhoff-type problem

$$\begin{cases} u^{iv} + K\left(\int\limits_0^1 (-A|u'(x)|^2 + B|u(x)|^2)dx\right)(Au'' + Bu) \\ \quad = \lambda f(x,u) + \mu g(x,u) + h(u), \qquad x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1.1}$$

where A and B are real constants, λ is a positive parameter, μ is a non-negative parameter, $K : [0,+\infty[\to \mathbb{R}$ is a continuous function such that there exist positive numbers m_0 and m_1 with $m_0 \le K(t) \le m_1$ for all $t \ge 0$, $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are two L^2-Carathéodory functions and $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e.,

$$|h(t_1) - h(t_2)| \le L|t_1 - t_2|$$

for every $t_1, t_2 \in \mathbb{R}$, and $h(0) = 0$.

The problem (1.1) is related to the stationary problem

$$\rho \frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

for $0 < x < L$, $t \geq 0$, where $u = u(x,t)$ is the lateral displacement at the space coordinate x and the time t, E the Young modulus, ρ the mass density, h the cross-section area, L the length and ρ_0 the initial axial tension, proposed by Kirchhoff [16] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. The Kirchhoff's model takes into account the length changes of the string produced by transverse vibrations. Some interesting results can be found, for example in [3, 31]. On the other hand, nonlocal boundary value problems model several physical and biological systems where u describes a process which depend on the average of itself, as for example, the population density. We refer the reader to [2, 5, 12, 13, 19–24, 29, 30] for some related works.

We refer to the recent monograph by Molica Bisci, Rădulescu and Servadei [25] for related problems concerning the variational analysis of solutions of some classes of nonlocal problems.

It is well known that the static form change of beam or the support of rigid body can be described by a fourth-order equation, and specially a model to study travelling waves in suspension bridges can be furnished by the fourth-order equation of nonlinearity, so studying fourth-order boundary value problems is important to Physics. In [17], Lazer and McKenna have pointed out that this type of nonlinearity furnishes a model to study travelling waves in suspension bridges. Due to this, many researchers have studied the existence and multiplicity of solutions for fourth-order two-point boundary value problems, we refer the reader to [1, 4, 8, 26]. In [32], Wang and An using the mountain pass theorem established the existence and multiplicity of solutions for a fourth-order nonlocal elliptic problem, and in [33] the authors by using the mountain pass techniques and the truncation method studied the existence of nontrivial solutions for a class of fourth order elliptic equations of Kirchhoff-type. In particular, in [11], using variational methods and critical point theory, multiplicity results of nontrivial and nonnegative solutions for a fourth-order Kirchhoff type elliptic problem, by combining an algebraic condition on the nonlinear term with the classical Ambrosetti-Rabinowitz condition was established, while in [14], using variational methods and critical point theory, the existence of one, two and three solutions for the problem (1.1), in the case $\mu = 0$ were discussed. In [18] Ma studied the existence of solutions of a nonlinear fourth order equation of Kirchhoff type, under nonlinear boundary conditions modeling the deformations of beams on elastic supports.

In the present paper, using two kinds of three critical points theorems obtained in [6, 9], the first one due to Bonanno and Marano, and the second one due to Bonanno and Candito which we recall in the next section (Theorems 2.1 and 2.2), we establish the existence of least three generalized solutions for the problem (1.1). The treatment is variational and basic tools are three critical point theorems recently established by Bonanno at al. and which goes back to the pioneering contributions of Pucci and

Serrin [27,28]. These theorems have been successfully used to ensure the existence of at least three solutions for perturbed boundary value problems in the papers [7,10,15].

2. PRELIMINARIES

Our main tools are the following three critical points theorems, the first one due to Bonanno and Marano, and the second one due to Bonanno and Candito. In the first one the coercivity of the functional $\Phi - \lambda\Psi$ is required, in the second one a suitable sign hypothesis is assumed.

Theorem 2.1 ([9, Theorem 3.6])**.** *Let X be a reflexive real Banach space, $\Phi : X \longrightarrow \mathbb{R}$ be a coercive continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\Phi(0) = \Psi(0) = 0$.*
Assume that there exist $r > 0$ and $\overline{v} \in X$, with $r < \Phi(\overline{v})$ such that

(a_1) $\frac{\sup_{\Phi(u)\le r}\Psi(u)}{r} < \frac{\Psi(\overline{v})}{\Phi(\overline{v})}$,

(a_2) *for each* $\lambda \in \Lambda_r := \left]\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\right[$ *the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Theorem 2.2 ([6, Corollary 3.1])**.** *Let X be a reflexive real Banach space, $\Phi : X \longrightarrow \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*, $\Psi : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

1. $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;
2. *for each $\lambda > 0$ and for every $u_1, u_2 \in X$ which are local minima for the functional $\Phi - \lambda\Psi$ and such that $\Psi(u_1) \ge 0$ and $\Psi(u_2) \ge 0$, one has*

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \ge 0.$$

Assume that there are two positive constants r_1, r_2 and $\overline{v} \in X$, with $2r_1 < \Phi(\overline{v}) < \frac{r_2}{2}$, such that

$$(b_1)\quad \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1} < \frac{2}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})};$$

$$(b_2)\quad \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} < \frac{1}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})}.$$

Then, for each

$$\lambda \in \left]\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})},\ \min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)},\ \frac{\frac{r_2}{2}}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}\right\}\right[,$$

the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which lie in $\Phi^{-1}(]-\infty, r_2[)$.

Suppose that

$$\max\left\{\frac{A}{\pi^2}, -\frac{B}{\pi^4}, \frac{A}{\pi^2}-\frac{B}{\pi^4}\right\} < 1.$$

Set

$$\sigma := \max\left\{\frac{A}{\pi^2}, -\frac{B}{\pi^4}, \frac{A}{\pi^2}-\frac{B}{\pi^4}, 0\right\}$$

and

$$\delta := \sqrt{1-\sigma}.$$

Let $X := H^2([0,1]) \cap H_0^1([0,1])$ be the Sobolev space endowed with the norm

$$\|u\| = \left(\int_0^1 (|u''(x)|^2 - A|u'(x)|^2 + B|u(x)|^2)dx\right)^{1/2}$$

which is equivalent to the usual one and, in particular, for each $u \in X$ one has

$$\|u\|_\infty \leq \frac{1}{2\pi\delta}\|u\|, \tag{2.1}$$

(see [8, Proposition 2.1]).

We suppose that the Lipschitz constant $L > 0$ of the function h satisfies $\min\{1, m_0\} > \frac{L}{4\pi^2\delta^2}$.

A function $u : [0,1] \to \mathbb{R}$ is a generalized solution to the problem (1.1) if $u \in C^3([0,1])$, $u''' \in AC([0,1])$, $u(0) = u(1) = 0$, $u''(0) = u''(1) = 0$, and

$$\begin{aligned} &u^{iv} + K\left(\int_0^1 (-A|u'(x)|^2 + B|u(x)|^2)dx\right)(Au'' + Bu) \\ &= \lambda f(x, u(x)) + \mu g(x, u(x)) + h(u(x)) \end{aligned}$$

for almost every $x \in [0,1]$, and it is a weak solution to the problem (1.1) if $u \in X$ and

$$\begin{aligned} &\int_0^1 u''(x)v''(x)dx \\ &\quad + K\left(\int_0^1 (-A|u'(x)|^2 + B|u(x)|^2)dx\right)\int_0^1 (-Au'(x)v'(x) + Bu(x)v(x))dx \\ &\quad - \lambda\int_0^1 f(x,u(x))v(x)dx - \mu\int_0^1 g(x,u(x))v(x)dx - \int_0^1 h(u(x))v(x)dx = 0 \end{aligned}$$

for every $v \in X$. Each weak solution to the problem (1.1) is a generalized one (see [8, Proposition 2.2]). If f, g are continuous, then each generalized solution u of the problem (1.1) is a classical solution.

Put

$$F(x,t) = \int_0^t f(x,\xi)d\xi \quad \text{for all } (x,t) \in [0,1] \times \mathbb{R},$$

$$G(x,t) = \int_0^t g(x,\xi)d\xi \quad \text{for all } (x,t) \in [0,1] \times \mathbb{R},$$

$$\widetilde{K}(t) = \int_0^t K(\xi)d\xi \quad \text{for all } t > 0$$

and

$$H(t) = \int_0^t h(\xi)d\xi \quad \text{for all } t \in \mathbb{R}.$$

Moreover, set

$$G^\theta := \int_0^1 \sup_{|t|\le\theta} G(x,t)dx$$

for every $\theta > 0$ and

$$G_\eta := \inf_{[0,1]\times[0,\eta]} G(x,t)$$

for every $\eta > 0$. If g is sign-changing, then $G^\theta \ge 0$ and $G_\eta \le 0$. Put

$$k = 2\delta^2\pi^2\Big(\frac{2048}{27} - \frac{32}{9}A + \frac{13}{40}B\Big)^{-1}.$$

Then, $0 < k < 1/2$ (see [8, p. 1168]).

3. MAIN RESULTS

In order to introduce our first result, fixing two positive constants θ and η such that

$$\frac{(\max\{1,m_1\} + \frac{L}{4\pi^2\delta^2})\eta^2}{k\int_{\frac{3}{8}}^{\frac{5}{8}} F(x,\eta)dx} < \frac{(\min\{1,m_0\} - \frac{L}{4\pi^2\delta^2})\theta^2}{\int_0^1 \sup_{|t|\le\theta} F(x,t)dx},$$

and taking

$$\lambda \in \Lambda := \left]\frac{\frac{2\pi^2\delta^2}{k}(\max\{1,m_1\} + \frac{L}{4\pi^2\delta^2})\eta^2}{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x,\eta)dx},\ \frac{2\pi^2\delta^2(\min\{1,m_0\} - \frac{L}{4\pi^2\delta^2})\theta^2}{\int_0^1 \sup_{|t|\le\theta} F(x,t)dx}\right[,$$

set $\delta_{\lambda,g}$ given by

$$\min\left\{\frac{2\pi^2\delta^2(\min\{1,m_0\}-\frac{L}{4\pi^2\delta^2})\theta^2-\lambda\int\limits_0^1\sup_{|t|\le\theta}F(x,t)dx}{G^\theta},\right.$$
$$\left.\frac{2\pi^2\delta^2(\max\{1,m_1\}+\frac{L}{4\pi^2\delta^2})\eta^2-k\lambda\int\limits_{\frac{3}{8}}^{\frac{5}{8}}F(x,\eta)dx}{kG_\eta}\right\} \tag{3.1}$$

and

$$\overline{\delta}_{\lambda,g}:=\min\left\{\delta_{\lambda,g},\ \frac{1}{\max\left\{0,\frac{1}{2\pi^2\delta^2(\min\{1,m_0\}-\frac{L}{4\pi^2\delta^2})}\limsup_{|t|\to+\infty}\frac{\sup_{x\in[0,1]}G(x,t)}{t^2}\right\}}\right\} \tag{3.2}$$

where we read $\rho/0=+\infty$, so that, for instance, $\overline{\delta}_{\lambda,g}=+\infty$ when

$$\limsup_{|t|\to+\infty}\frac{\sup_{x\in[0,1]}G(x,t)}{t^2}\le 0,$$

and $G_\eta=G^\theta=0$.

Now we formulate our main results as follows.

Theorem 3.1. *Assume that there exist two positive constants θ and η with $\theta<\frac{\eta}{\sqrt{k}}$ such that*

(A_1) *$F(x,t)\ge 0$, for each $(x,t)\in[0,\frac{3}{8}]\cup]\frac{5}{8},1]\times[0,\eta]$;*

(A_2) $$\frac{\int\limits_0^1\sup_{|t|\le\theta}F(x,t)dx}{\theta^2}<\frac{k(\min\{1,m_0\}-\frac{L}{4\pi^2\delta^2})}{\max\{1,m_1\}+\frac{L}{4\pi^2\delta^2}}\frac{\int\limits_{\frac{3}{8}}^{\frac{5}{8}}F(x,\eta)dx}{\eta^2};$$

(A_3) $\limsup\limits_{|t|\to+\infty}\frac{\sup_{x\in[0,1]}F(x,t)}{t^2}\le 0.$

Then, for each $\lambda\in\Lambda$ and for every L^2-Carathéodory function $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ satisfying the condition

$$\limsup_{|t|\to+\infty}\frac{\sup_{x\in[0,1]}G(x,t)}{t^2}<+\infty, \tag{3.3}$$

there exists $\overline{\delta}_{\lambda,g}>0$ given by (3.2) such that, for each $\mu\in[0,\overline{\delta}_{\lambda,g}[$, the problem (1.1) admits at least three distinct generalized solutions in X.

Proof. Let $\Phi,\ \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) = \frac{1}{2}\int_0^1 |u''(x)|^2 dx + \frac{1}{2}\widetilde{K}\left(\int_0^1 (-A|u'(x)|^2 + B|u(x)|^2)dx\right) - \int_0^1 H(u(x))dx \quad (3.4)$$

and

$$\Psi(u) = \int_0^1 F(x,u(x))dx + \frac{\mu}{\lambda}\int_0^1 G(x,u(x))dx \quad (3.5)$$

for every $u \in X$. It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_0^1 f(x,u(x))v(x)dx + \frac{\mu}{\lambda}\int_0^1 g(x,u(x))v(x)dx,$$

as well as is sequentially weakly upper semicontinuous. Furthermore, $\Psi' : X \to X^*$ is a compact operator. Moreover, Φ is continuously differentiable whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = \int_0^1 u''(x)v''(x)dx + K\left(\int_0^1 (-A|u'(x)|^2 + B|u(x)|^2)dx\right)$$
$$\times \int_0^1 (-Au'(x)v'(x) + Bu(x)v(x))dx - \int_0^1 h(u(x))v(x)dx$$

for every $v \in X$, while [14, Proposition 2.4.] gives that Φ' admits a continuous inverse on X^*. Furthermore, Φ is sequentially weakly lower semicontinuous. Put

$$r := 2\pi^2\delta^2(\min\{1,m_0\} - \frac{L}{4\pi^2\delta^2})\theta^2$$

and

$$w(x) := \begin{cases} -\frac{64\eta}{9}(x^2 - \frac{3}{4}x) & \text{if } x \in [0, \frac{3}{8}], \\ \eta & \text{if } x \in]\frac{3}{8}, \frac{5}{8}], \\ -\frac{64\eta}{9}(x^2 - \frac{5}{4}x + \frac{1}{4}) & \text{if } x \in]\frac{5}{8}, 1]. \end{cases} \quad (3.6)$$

We clearly observe that $w \in X$ and, in particular,

$$\frac{2\pi^2\delta^2}{k}\Big(\min\{1,m_0\} - \frac{L}{4\pi^2\delta^2}\Big)\eta^2 \le \Phi(w) \le \frac{2\pi^2\delta^2}{k}\Big(\max\{1,m_1\} + \frac{L}{4\pi^2\delta^2}\Big)\eta^2.$$

Taking into account $\theta < \frac{\eta}{\sqrt{k}}$, we observe that

$$0 < r < \Phi(w).$$

The inequality

$$\frac{1}{2}\Big(\min\{1,m_0\}-\frac{L}{4\pi^2\delta^2}\Big)\|u\|^2\leq \Phi(u)$$

for each $u\in X$ in conjunction with (2.1) yields

$$\begin{aligned}\Phi^{-1}(]-\infty,r]) &= \{u\in X;\ \Phi(u)\leq r\}\\ &= \Big\{u\in X;\ \frac{1}{2}\Big(\min\{1,m_0\}-\frac{L}{4\pi^2\delta^2}\Big)\|u\|^2\leq r\Big\}\\ &\subseteq \{u\in X;\ |u(x)|\leq\theta \text{ for each } x\in[0,1]\}\,,\end{aligned}$$

which follows

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u) &= \sup_{u\in\Phi^{-1}(]-\infty,r])}\int_0^1[F(x,u(x))+\frac{\mu}{\lambda}G(x,u(x))]dx\\ &\leq \int_0^1\sup_{|t|\leq\theta}F(x,t)dx+\frac{\mu}{\lambda}G^\theta.\end{aligned}$$

On the other hand, in view of (A_1), since $0\leq w(x)\leq\eta$ for each $x\in[0,1]$, we have

$$\begin{aligned}\Psi(w) &\geq \int_{\frac{3}{8}}^{\frac{5}{8}}F(x,\eta)dx+\frac{\mu}{\lambda}\int_0^1 G(x,w(x))dx\\ &\geq \int_{\frac{3}{8}}^{\frac{5}{8}}F(x,\eta)dx+\frac{\mu}{\lambda}\inf_{[0,1]\times[0,\eta]}G(x,t)\\ &= \int_{\frac{3}{8}}^{\frac{5}{8}}F(x,\eta)dx+\frac{\mu}{\lambda}G_\eta.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} &= \frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\int_0^1[F(x,u(x))+\frac{\mu}{\lambda}G(x,u(x))]dx}{r}\\ &\leq \frac{\int_0^1\sup_{|t|\leq\theta}F(x,t)dx+\frac{\mu}{\lambda}G^\theta}{2\pi^2\delta^2(\min\{1,m_0\}-\frac{L}{4\pi^2\delta^2})\theta^2},\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}\frac{\Psi(w)}{\Phi(w)} &\geq \frac{\int\limits_{\frac{3}{8}}^{\frac{5}{8}} F(x,\eta)dx + \frac{\mu}{\lambda}\int\limits_0^1 G(x,w(x))dx}{\frac{2\pi^2\delta^2}{k}(\max\{1,m_1\} + \frac{L}{4\pi^2\delta^2})\eta^2} \\ &\geq \frac{\int\limits_{\frac{3}{8}}^{\frac{5}{8}} F(x,\eta)dx + \frac{\mu}{\lambda}G_\eta}{\frac{2\pi^2\delta^2}{k}(\max\{1,m_1\} + \frac{L}{4\pi^2\delta^2})\eta^2}.\end{aligned} \tag{3.8}$$

Since $\mu < \delta_{\lambda,g}$, one has

$$\mu < \frac{2\pi^2\delta^2(\min\{1,m_0\} - \frac{L}{4\pi^2\delta^2})\theta^2 - \lambda\int\limits_0^1 \sup_{|t|\leq\theta} F(x,t)dx}{G^\theta},$$

this means

$$\frac{\int\limits_0^1 \sup_{|t|\leq\theta} F(x,t)dx + \frac{\mu}{\lambda}G^\theta}{2\pi^2\delta^2(\min\{1,m_0\} - \frac{L}{4\pi^2\delta^2})\theta^2} < \frac{1}{\lambda}.$$

Furthermore,

$$\mu < \frac{2\pi^2\delta^2(\max\{1,m_1\} + \frac{L}{4\pi^2\delta^2})\eta^2 - k\lambda\int\limits_{\frac{3}{8}}^{\frac{5}{8}} F(x,\eta)dx}{kG_\eta},$$

this means

$$\frac{\int\limits_{\frac{3}{8}}^{\frac{5}{8}} F(x,\eta)dx + \frac{\mu}{\lambda}G_\eta}{\frac{2\pi^2\delta^2}{k}(\max\{1,m_1\} + \frac{L}{4\pi^2\delta^2})\eta^2} > \frac{1}{\lambda}.$$

Then,

$$\frac{\int\limits_0^1 \sup_{|t|\leq\theta} F(x,t)dx + \frac{\mu}{\lambda}G^\theta}{2\pi^2\delta^2(\min\{1,m_0\} - \frac{L}{4\pi^2\delta^2})\theta^2} < \frac{1}{\lambda} < \frac{\int\limits_{\frac{3}{8}}^{\frac{5}{8}} F(x,\eta)dx + \frac{\mu}{\lambda}G_\eta}{\frac{2\pi^2\delta^2}{k}(\max\{1,m_1\} + \frac{L}{4\pi^2\delta^2})\eta^2}. \tag{3.9}$$

Hence from (3.7)–(3.9), we observe that the condition (a_1) of Theorem 2.1 is fulfilled. Finally, since $\mu < \overline{\delta}_{\lambda,g}$, we can fix $l > 0$ such that

$$\limsup_{|t|\to\infty} \frac{\sup_{x\in[0,1]} G(x,t)}{t^2} < l,$$

and

$$\mu l < \frac{(4\pi^2\delta^2 \min\{1, m_0\} - L)}{2}.$$

Therefore, there exists a function $\rho \in L^1([0,1])$ such that

$$G(x,t) \leq lt^2 + \rho(x) \tag{3.10}$$

for every $x \in [0,1]$ and $t \in \mathbb{R}$. Now, fix

$$0 < \epsilon < \frac{(4\pi^2\delta^2 \min\{1, m_0\} - L)}{2\lambda} - \frac{\mu l}{\lambda}.$$

From (A_3) there exists a function $\rho_\epsilon \in L^1([0,1])$ such that

$$F(x,t) \leq \epsilon t^2 + \rho_\epsilon(x) \tag{3.11}$$

for every $x \in [0,1]$ and $t \in \mathbb{R}$. Taking (2.1) into account, it follows that, for each $u \in X$,

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) \geq{}& \frac{1}{2}\Big(\min\{1, m_0\} - \frac{L}{4\pi^2\delta^2}\Big)\|u\|^2 - \lambda\epsilon \int_0^1 u^2(x)dx - \lambda\|\rho_\epsilon\|_1 \\ & - \mu l \int_0^1 u^2(x)dx - \mu\|\rho\|_1 \\ \geq{}& \Big(\frac{1}{2}\Big(\min\{1, m_0\} - \frac{L}{4\pi^2\delta^2}\Big) - \frac{\lambda\epsilon}{4\pi^2\delta^2} - \frac{\mu l}{4\pi^2\delta^2}\Big)\|u\|^2 - \lambda\|\rho_\epsilon\|_1 - \mu\|\rho\|_1,\end{aligned}$$

and thus

$$\lim_{\|u\|\to+\infty} (\Phi(u) - \lambda\Psi(u)) = +\infty,$$

which means the functional $\Phi - \lambda\Psi$ is coercive, and the condition (a_2) of Theorem 2.1 is verified. From (3.7)–(3.9) one also has

$$\lambda \in \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(u)\leq r} \Psi(u)}\right[.$$

Finally, since the generalized solutions of the problem (1.1) are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$, Theorem 2.1 (with $\overline{v} = w$) ensures the conclusion. □

Now, we present a variant of Theorem 3.1 in which no asymptotic condition on the nonlinear term is requested.

Fix positive constants θ_1, θ_2 and η such that

$$\begin{aligned}&\frac{3}{2}\frac{(\max\{1, m_1\} + \frac{L}{4\pi^2\delta^2})\eta^2}{k\int_{\frac{3}{8}}^{\frac{5}{8}} F(x,\eta)dx} \\ &< \Big(\min\{1, m_0\} - \frac{L}{4\pi^2\delta^2}\Big) \min\left\{\frac{\theta_1^2}{\int_0^1 \sup_{|t|\leq\theta_1} F(x,t)dx}, \frac{\theta_2^2}{2\int_0^1 \sup_{|t|\leq\theta_2} F(x,t)dx}\right\}\end{aligned}$$

and let

$$\Lambda' := \left]\frac{3\pi^2\delta^2(\max\{1,m_1\}+\frac{L}{4\pi^2\delta^2})\eta^2}{k\int_{\frac{3}{8}}^{\frac{5}{8}}F(x,\eta)dx},\right.$$

$$\left.2\pi^2\delta^2(\min\{1,m_0\}-\frac{L}{4\pi^2\delta^2})\min\left\{\frac{\theta_1^2}{\int_0^1\sup_{|t|\le\theta_1}F(x,t)dx},\ \frac{\theta_2^2}{2\int_0^1\sup_{|t|\le\theta_2}F(x,t)dx}\right\}\right[.$$

Theorem 3.2. *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a nonnegative L^2-Carathéodory function. Assume that there exist three positive constants $\theta_1,\ \theta_2$ and η with*

$$(2k)^{\frac{1}{2}}\ \theta_1<\eta<\Big(\frac{k(\min\{1,m_0\}-\frac{L}{4\pi^2\delta^2})}{2(\max\{1,m_1\}+\frac{L}{4\pi^2\delta^2})}\Big)^{\frac{1}{2}}\ \theta_2$$

such that the assumption (A_1) in Theorem 3.1 holds. Furthermore, suppose that

$$(B_1)\ \frac{\int\limits_0^1\sup_{|t|\le\theta_1}F(x,t)dx}{\theta_1^2}<\frac{2}{3}\frac{k(\min\{1,m_0\}-\frac{L}{4\pi^2\delta^2})}{\max\{1,m_1\}+\frac{L}{4\pi^2\delta^2}}\frac{\int\limits_{\frac{3}{8}}^{\frac{5}{8}}F(x,\eta)dx}{\eta^2};$$

$$(B_2)\ \frac{\int\limits_0^1\sup_{|t|\le\theta_2}F(x,t)dx}{\theta_2^2}<\frac{1}{3}\frac{k(\min\{1,m_0\}-\frac{L}{4\pi^2\delta^2})}{\max\{1,m_1\}+\frac{L}{4\pi^2\delta^2}}\frac{\int\limits_{\frac{3}{8}}^{\frac{5}{8}}F(x,\eta)dx}{\eta^2}.$$

Then, for each $\lambda\in\Lambda'$ and for every nonnegative L^2-Carathéodory function $g:[0,1]\times\mathbb{R}\to\mathbb{R}$, there exists $\delta^*_{\lambda,g}>0$ given by

$$\min\left\{\frac{2\pi^2\delta^2(\min\{1,m_0\}-\frac{L}{4\pi^2\delta^2})\theta_1^2-\lambda\int\limits_0^1\sup_{|t|\le\theta_1}F(x,t)dx}{G^{\theta_1}},\right.$$
$$\left.\frac{\pi^2\delta^2(\min\{1,m_0\}-\frac{L}{4\pi^2\delta^2})\theta_2^2-\lambda\int\limits_0^1\sup_{|t|\le\theta_2}F(x,t)dx}{G^{\theta_2}}\right\}$$

such that, for each $\mu\in[0,\delta^*_{\lambda,g}[$, the problem (1.1) *admits at least three distinct generalized solutions u_i for $i=1,2,3$, such that*

$$0\le u_i(x)<\theta_2 \text{ for all } x\in[0,1],\ \ (i=1,2,3).$$

Proof. Fix λ, g and μ as in the conclusion and take Φ and Ψ as in the proof of Theorem 3.1. We observe that the regularity assumptions of Theorem 2.2 on Φ and Ψ are satisfied. Hence, our aim is to verify (b_1) and (b_2). To this end, choose w as given in (3.6),

$$r_1:=2\pi^2\delta^2\Big(\min\{1,m_0\}-\frac{L}{4\pi^2\delta^2}\Big)\theta_1^2$$

and

$$r_2 := 2\pi^2\delta^2\Big(\min\{1, m_0\} - \frac{L}{4\pi^2\delta^2}\Big)\theta_2^2.$$

Therefore, using the condition

$$\big(2k\big)^{\frac{1}{2}}\ \theta_1 < \eta < \Big(\frac{k(\min\{1, m_0\} - \frac{L}{4\pi^2\delta^2})}{2(\max\{1, m_1\} + \frac{L}{4\pi^2\delta^2})}\Big)^{\frac{1}{2}}\ \theta_2$$

one has $2r_1 < \Phi(w) < \frac{r_2}{2}$. Since $\mu < \delta^*_{\lambda,g}$ and $G_\eta = 0$, one has

$$\begin{aligned}
\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1} &= \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\int_0^1[F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))]dx}{r_1} \\
&\le \frac{\int_0^1 \sup_{|t|\le\theta_1} F(x,t)dx + \frac{\mu}{\lambda}G^{\theta_1}}{2\pi^2\delta^2(\min\{1,m_0\} - \frac{L}{4\pi^2\delta^2})\theta_1^2} \\
&< \frac{1}{\lambda} < \frac{2}{3}\ \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x,\eta)dx + \frac{\mu}{\lambda}G\eta}{\frac{2\pi^2\delta^2}{k}(\max\{1,m_1\} + \frac{L}{4\pi^2\delta^2})\eta^2} \le \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{2\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} &= \frac{2\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\int_0^1[F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))]dx}{r_2} \\
&\le \frac{\int_0^1 \sup_{|t|\le\theta_2} F(x,t)dx + \frac{\mu}{\lambda}G^{\theta_2}}{\pi^2\delta^2(\min\{1,m_0\} - \frac{L}{4\pi^2\delta^2})\theta_2^2} \\
&< \frac{1}{\lambda} < \frac{2}{3}\ \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x,\eta)dx + \frac{\mu}{\lambda}G\eta}{\frac{2\pi^2\delta^2}{k}(\max\{1,m_1\} + \frac{L}{4\pi^2\delta^2})\eta^2} \le \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.
\end{aligned}$$

Hence, (b_1) and (b_2) of Theorem 2.2 are verified. Finally, We will prove that $\Phi - \lambda\Psi$ satisfies the assumption *2.* of Theorem 2.2. Let u_1 and u_2 be two local minima for $\Phi - \lambda\Psi$. Then u_1 and u_2 are critical points for $\Phi - \lambda\Psi$, and so, they are weak solutions for the problem (1.1). Arguing as given in the proof of [14, Lemma 3.4.] one has $u_1(x) \ge 0$ and $u_2(x) \ge 0$ for every $x \in [0,1]$. Hence, it follows that $su_1 + (1-s)u_2 \ge 0$ for all $s \in [0,1]$, and that

$$(\lambda f + \mu g)(x, su_1 + (1-s)u_2) \ge 0,$$

and consequently, $\Psi(su_1 + (1-s)u_2) \geq 0$, for every $s \in [0,1]$. From Theorem 2.2, for every

$$\lambda \in \left]\frac{3}{2}\frac{\Phi(w)}{\Psi(w)},\ \min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)},\ \frac{r_2/2}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}\right\}\right[,$$

the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which are the generalized solutions of the problem (1.1). □

Now, we point out the following existence result, as a consequence of Theorem 3.1.

Theorem 3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $F(t) := \int_0^t f(\xi)d\xi$ for each $t \in \mathbb{R}$. Assume that $F(\eta) > 0$ for some $\eta > 0$ and $F(\xi) \geq 0$ in $[0,\eta]$ and*

$$\liminf_{\xi\to 0}\frac{F(\xi)}{\xi^2} = \limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^2} = 0.$$

Then, there is $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ and for every L^2-Carathéodory function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfying the asymptotical condition (3.3) there exists $\delta'_{\lambda,g} > 0$ such that, for each $\mu \in [0, \delta'_{\lambda,g}[$, the problem

$$\begin{cases} u^{iv} + K\left(\int\limits_0^1 (-A|u'(x)|^2 + B|u(x)|^2)dx\right)(Au'' + Bu) \\ \quad = \lambda f(u) + \mu g(x,u) + h(u), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases} \tag{3.12}$$

admits at least three generalized solutions.

Proof. Fix

$$\lambda > \lambda^* := \frac{8\pi^2\delta^2}{kF(\eta)}\Big(\max\{1,m_1\} + \frac{L}{4\pi^2\delta^2}\Big)\eta^2$$

for some $\eta > 0$. From the condition

$$\liminf_{\xi\to 0}\frac{F(\xi)}{\xi^2} = 0,$$

there is a sequence $\{\theta_n\} \subset]0,+\infty[$ such that $\lim_{n\to\infty}\theta_n = 0$ and

$$\lim_{n\to\infty}\frac{\sup_{|\xi|\leq\theta_n}F(\xi)}{\theta_n^2} = \lim_{n\to\infty}\frac{F(\xi_{\theta_n})}{\xi_{\theta_n}^2}\frac{\xi_{\theta_n}^2}{\theta_n^2} = 0,$$

where $F(\xi_{\theta_n}) = \sup_{|\xi|\leq\theta_n}F(\xi)$. Therefore, there exists $\overline{\theta} > 0$ such that

$$\frac{\sup_{|\xi|\leq\overline{\theta}}F(\xi)}{\overline{\theta}^2} < \min\left\{\frac{kF(\eta)(\min\{1,m_0\} - \frac{L}{4\pi^2\delta^2})}{4(\max\{1,m_1\} + \frac{L}{4\pi^2\delta^2})\eta^2}, \frac{2\pi^2\delta^2}{\lambda}(\min\{1,m_0\} - \frac{L}{4\pi^2\delta^2})\right\}$$

and $\overline{\theta} < \frac{\eta}{\sqrt{k}}$. Theorem 3.1 follows the result. □

We here present the following example to illustrate Theorem 3.3.

Example 3.4. Let $A = 3,\ B = 2,\ \eta = 2$ and $\mu = 0$, and let

$$f(t) = 50\,t^9 \ln(1 + e^{-0.001\,t}) - \frac{0.005\,t^{10}\,e^{-0.001\,t}}{1 + e^{-0.001\,t}}$$

for all $t \in \mathbb{R}$, $K(t) = \pi + \text{arctan}t$ for all $t \geq 0$ and $h(t) = \tanh$ t for all $t \in \mathbb{R}$. Hence we have, $\delta = \sqrt{1 - \frac{3}{\pi^2}}$, $m_1 = \frac{3\pi}{2}$, $L = 1$ and $F(t) = 5\,t^{10} \ln(1 + e^{-0.001\,t})$. It is clear that

$$\liminf_{\xi \to 0} \frac{F(\xi)}{\xi^2} = \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = 0.$$

So by applying Theorem 3.3, for every

$$\lambda > \frac{\pi^2 - 3}{160\,k \ln(1 + e^{-0.002})} \Big(\frac{3\pi}{2} + \frac{1}{4\pi^2 - 12} \Big)$$

the problem

$$\begin{cases} u^{iv} + \Big(\pi + \arctan \Big(\int\limits_0^1 (-3|u'(x)|^2 + 2|u(x)|^2) dx \Big) \Big)(3u'' + 2u) \\ \quad = \lambda \Big(50\,t^9 \ln(1 + e^{-0.001\,t}) - \frac{0.005\,t^{10}\,e^{-0.001\,t}}{1+e^{-0.001\,t}} \Big) + \tanh \text{ t}, \qquad x \in (0, 1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases}$$

has at least three classical solutions.

As an example, we give the following consequence of Theorem 3.2.

Theorem 3.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function such that*

$$\lim_{t \to 0^+} \frac{f(t)}{t} = 0,$$

and

$$\int\limits_0^1 f(\xi) d\xi < \Big(\frac{\min\{1, m_0\} - \frac{L}{4\pi^2}}{\max\{1, m_1\} + \frac{L}{4\pi^2}} \Big) \frac{18\pi^2}{86.111} \int\limits_0^{0.1} f(\xi) d\xi.$$

Assume that

$$\frac{\min\{1, m_0\} - \frac{L}{4\pi^2}}{\max\{1, m_1\} + \frac{L}{4\pi^2}} > \frac{86.111}{108\pi^2}.$$

Then, for every

$$\lambda \in \left] \frac{86.111}{18} \frac{\max\{1, m_1\} + \frac{L}{4\pi^2}}{\int\limits_0^{0.1} f(\xi) d\xi}, \ \frac{(\min\{1, m_0\} - \frac{L}{4\pi^2})\pi^2}{\int\limits_0^1 f(\xi) d\xi} \right[$$

and for every L^2-Carathéodory nonnegative function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ there exists $\delta^*_{\lambda,g} > 0$ such that, for each $\mu \in [0, \delta^*_{\lambda,g}[$, the problem

$$\begin{cases} u^{iv} + K\left(\int\limits_0^1 (|u'(x)|^2 + |u(x)|^2)dx\right)(-u'' + u) & \\ \quad = \lambda f(u) + \mu g(x,u) + h(u), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 & \end{cases}$$

admits at least three generalized solutions.

Proof. Our aim is to employ Theorem 3.2 by choosing $A = -1$, $B = 1$, $\theta_2 = 1$ and $\eta = 0.1$. Since, in this case, $k = \frac{2160\pi^2}{86\,111}$ and $\delta = 1$, we have

$$\frac{3\pi^2\delta^2(\max\{1, m_1\} + \frac{L}{4\pi^2\delta^2})\eta^2}{k \int\limits_{\frac{3}{8}}^{\frac{5}{8}} F(x,\eta)dx} = \frac{86.111}{18} \frac{\max\{1, m_1\} + \frac{L}{4\pi^2}}{\int\limits_0^{0.1} f(\xi)d\xi}$$

and

$$\frac{2\pi^2\delta^2(\min\{1, m_0\} - \frac{L}{4\pi^2\delta^2})\theta_2^2}{2\int\limits_0^1 \sup_{|t|\le\theta_2} F(x,t)dx} = \frac{(\min\{1, m_0\} - \frac{L}{4\pi^2})\pi^2}{\int\limits_0^1 f(\xi)d\xi}.$$

Moreover, since $\lim_{t\to 0^+} \frac{f(t)}{t} = 0$, one has

$$\lim_{t\to 0^+} \frac{\int\limits_0^t f(\xi)d\xi}{t^2} = 0.$$

Then, there exists a positive constant $\theta_1 < \frac{1}{120\pi}\sqrt{\frac{86\,111}{30}}$ such that

$$\frac{\int\limits_0^{\theta_1} f(\xi)d\xi}{\theta_1^2} < \Big(\frac{\min\{1, m_0\} - \frac{L}{4\pi^2}}{\max\{1, m_1\} + \frac{L}{4\pi^2}}\Big)\frac{36\pi^2}{86.111}\int\limits_0^{0.1} f(\xi)d\xi,$$

and

$$\frac{\theta_1^2}{\int\limits_0^{\theta_1} f(\xi)d\xi} > \frac{1}{2\int\limits_0^1 f(\xi)d\xi}.$$

Finally, simple computations show that all the assumptions of the Theorem 3.2 are fulfilled, and Theorem 3.2 follows the conclusion. □

Example 3.6. Choose

$$f(t) := \begin{cases} 18000\,t^2 & \text{if } t \le 0.1, \\ -18000\,t + 1980 & \text{if } 0.1 < t \le 0.11, \\ 0 & \text{if } t > 0.11 \end{cases}$$

and $K(t) = 0.01\, e^{-t} + 1$ for all $t \geq 0$ and $h(t) = \sqrt{t^2+3}$ for all $t \in \mathbb{R}$. We observe $m_0 = 1,\ m_1 = 1.01$ and $L = 1$. By simple calculations we see that all hypothesis of Theorem 3.5 are satisfied.

Remark 3.7. The same statements of the above given results can be written by choosing a particular choice of the function K,

$$K(t) = a_1 t + a_2 \text{ for } t \in [\alpha, \beta],$$

where $a_1,\ a_2,\ \alpha$ and β are positive numbers. In fact, in this special case we have

$$\widetilde{K}(t) = \int_0^t [a_1 s + a_2]\, ds = \frac{(a_1 t + a_2)^2}{2a_1} - \frac{a_2^2}{2a_1} \quad \text{for } t \geq 0,$$

$$m_0 = a_1\alpha + a_2 \text{ and } \ m_1 = a_1\beta + a_2.$$

Remark 3.8. As we mentioned in the proof of Theorem 3.2, if f, g are non-negative functions, the generalized solutions ensured by the previous theorems are non-negative. In addition, if either $f(x,0) \neq 0$ for some $x \in (0,1)$ or $g(x,0) \neq 0$ for some $x \in (0,1)$, or both are true, the solutions are positive.

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Mohamad Reza Heidari Tavani
m.reza.h56@gmail.com

Islamic Azad University
Science and Research Branch
Department of Mathematics
Tehran, Iran

Ghasem Alizadeh Afrouzi
afrouzi@umz.ac.ir

Islamic Azad University
Qaemshahr Branch
Department of Mathematics
Qaemshahr, Iran

Shapour Heidarkhani
s.heidarkhani@razi.ac.ir

Razi University
Faculty of Sciences
Department of Mathematics
67149 Kermanshah, Iran

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