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NONNEGATIVE SOLUTIONS FOR A CLASS OF SEMIPOSITONE NONLINEAR ELLIPTIC EQUATIONS IN BOUNDED DOMAINS OF $\mathbb{R}^n$

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Abstract. In this paper, we obtain sufficient conditions for the existence of a unique nonnegative continuous solution of semipositone semilinear elliptic problem in bounded domains of $\mathbb{R}^n$ $(n \geq 2)$. The global behavior of this solution is also given.

Keywords: nonnegative solution, semipositone, Kato class, fixed point theorem.

Mathematics Subject Classification: 35J25, 35A01, 35B09, 35J08.

1. INTRODUCTION

Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$ $(n \geq 2)$. In this paper, we investigate the following semipositone nonlinear elliptic boundary value problem

$$\begin{cases} -\Delta u(x) = a(x) + \lambda f(x, u(x)), & x \in \Omega \text{ (in the distributional sense)}, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is sign-changing measurable function. The function $a : \Omega \to [0, \infty)$, is required to belong to the Kato class $K(\Omega)$ introduced in [15] for $n \geq 3$ and [14] for $n = 2$ as follows:

$$K(\Omega) = \left\{ q \in \mathcal{B}(\Omega),\ \lim_{r \to 0} \left(\sup_{x \in \Omega} \int\limits_{\Omega \cap B(x,r)} \frac{\delta(y)}{\delta(x)} G_\Omega(x, y)|q(y)|dy \right) = 0 \right\},$$

where $\mathcal{B}(\Omega)$ be the set of Borel measurable functions in Ω, $G_\Omega(x, y)$ is the Green's function of the Laplace operator in Ω and $\delta(x) = d(x, \partial\Omega)$ denotes the Euclidean distance from x to $\partial\Omega$.

The class $K(\Omega)$ properly contains $L^p(\Omega)$ with $1 < p \leq \infty$ and it has been proved that it is well adapted to study some nonlinear elliptic problems in Ω, see for instance [13–15, 21] and their references.

The nonlinearity $f(x,u(x))$ in (1.1) can be sign-changing (that is, we have the so-called semipositone boundary value problems in the literature). Semipositone BVPs occur in models for steady-state diffusion with reactions [3], and the study of such problems for elliptic and ordinary differential equations has been considered for many years, see for example [2, 5, 6, 8–12, 19, 23] and the references therein. Note that the mathematical analysis of nonnegative solutions to semipositone problems is more challenging since the nonlinearity is allowed to be sign-changing.

In [5], the authors studied the existence and multiplicity of solutions to the following perturbed bifurcation problems

$$\begin{cases} -\Delta u(x) = \lambda(u - u^3) - \varepsilon, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\lambda > 0$ and $\varepsilon > 0$.

By using sub-super solutions techniques, they obtained some local results of perturbed bifurcation theory. The existence of global of solutions was proved by using degree theory arguments.

In [1], by using the method of sub-super solutions, the authors have proved the existence of positive solution to the semipositone BVP

$$\begin{cases} -\Delta u(x) = \lambda m(x) f(u(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $\lambda > 0$, $m \in C(\Omega)$ and $m(x) \geq m_0 > 0$ for $x \in \Omega$, $f \in C^1([0,r))$ is a nondecreasing function for some $r > 0$ such that $f(0) < 0$ and there exists $\alpha \in (0,r)$ such that $(t-\alpha)f(t) \geq 0$ for $t \in [0,r]$.

In [4], the authors considered the following boundary value problem

$$\begin{cases} -\Delta u(x) = f(u(x)) + h(x), & x \in D, \\ u(x) = 0, & x \in \partial D, \end{cases} \tag{1.4}$$

where D is an open, bounded and connected subset of $\mathbb{R}^n$, $n \in \{1,2,3\}$, with a C^2-boundary, $h \in L^2(D)$ and $f : \mathbb{R} \to \mathbb{R}$ is of class C^1.

They have proved that problem (1.4) has exactly one solution provided that f' satisfy some convenient assumptions. Their approach is based on global invertibility result.

In [24], by using a variational method approach, the authors proved the existence and regularity of solutions for a class of degenerate elliptic equations of the form

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) + g(x,u) = b(x,u,\nabla u) + h(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded $C^{1,1}$-domain of $\mathbb{R}^n$ $(1 < p < n)$, $h \in L^m(\Omega)$, with $1 < m < \frac{n}{p}$, $a : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $b : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : b : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying some adequate conditions.

In [17], under some natural hypotheses, using tools from Morse theory (in particular, critical groups), the authors proved an existence theorem (for the superlinear case) and a multiplicity theorem (for the linear resonant case) to the following two phase Robin problem

$$\begin{cases} -\operatorname{div}(a_0(x)\left|Du\right|^{p-2} Du) - \Delta_q u + \xi(x)\left|u\right|^{p-2} u = f(x,u), & x \in \Omega, \\ \frac{\partial u}{\partial n_\theta} + \beta(x)\left|u\right|^{p-2} u = 0, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

where $1 < q < p \leq n$.

For more recently related results, we refer the reader to [16] and [22].

In this paper, we study the semipositone BVP (1.1) and provide sufficient conditions to guarantee that this problem has a unique nonnegative continuous solution. The global behavior of this solution is also given. Our approach relies on the properties of the Kato class $K(\Omega)$ and fixed point theorem in some convenient space.

The rest of this paper is organized as follows. In Section 2, we collect some basic properties of the Kato class $K(\Omega)$ and give sharp estimates on some potential functions. In Section 3, we present our main result and give its proof with some examples.

Next we adopt the following notations:

(i) $\mathcal{B}^+(\Omega)$ (resp. $K^+(\Omega)$) denotes the collection of all nonnegative functions in $\mathcal{B}(\Omega)$ (resp. in $K(\Omega)$).
(ii) For $f, g \in \mathcal{B}^+(\Omega)$, we say that $f \approx g$ in Ω, if there exists $c > 0$ such that $\frac{1}{c} f(x) \leq g(x) \leq c f(x)$, for all $x \in \Omega$.
(iii) We refer to the set $C(\overline{\Omega})$ of all continuous functions in $\overline{\Omega}$ and let $C_0(\Omega)$ be the subclass of $C(\overline{\Omega})$ consisting of functions which vanish continuously on $\partial\Omega$. Note that $C_0(\Omega)$ is a Banach space equipped with norm

$$\|g\|_\infty := \sup_{x\in\Omega} |g(x)|.$$

(iv) For $g \in \mathcal{B}^+(\Omega)$, we let

$$Vg(x) := \int_\Omega G_\Omega(x,y) g(y) dy, \quad \text{for } x \in \Omega.$$

We recall that if $g \in L^1_{\text{loc}}(\Omega)$ and $Vg \in L^1_{\text{loc}}(\Omega)$, then we have (see [7, p. 52])

$$-\Delta(Vg) = g, \text{ in } \Omega \quad \text{(in the distributional sense)}. \tag{1.7}$$

Note that if $g \in \mathcal{B}^+(\Omega)$ such that $Vg(x_0) < \infty$ for some $x_0 \in \Omega$, then we have $Vg \in L^1_{\text{loc}}(\Omega)$ (see [7, Lemma 2.9]).

The letter c will denote a generic positive constant which may vary from line to line.

2. BASIC PROPERTIES OF THE KATO CLASS $K(\Omega)$

We recall that for $y \in \Omega$, the Green's function $G_\Omega(\cdot, y)$ of the Laplacian in Ω is defined as the solution, in the distributional sense, of the following problem

$$\begin{cases} -\Delta G_\Omega(\cdot, y) = \delta_y, & \text{in } \Omega, \\ G_\Omega(x, y) = 0, & x \in \partial\Omega, \end{cases}$$

where δ_y denotes the Dirac measure at y.

By [15] for $n \geq 3$ and [20] for $n = 2$, we have on $\Omega \times \Omega$,

$$G_\Omega(x, y) \approx \begin{cases} \dfrac{\delta(x)\delta(y)}{|x-y|^{n-2}\left(|x-y|^2 + \delta(x)\delta(y)\right)}, & \text{if } n \geq 3 \\ \log\left(1 + \dfrac{\delta(x)\delta(y)}{|x-y|^2}\right), & \text{if } n = 2. \end{cases} \tag{2.1}$$

Lemma 2.1. *Let $q \in K(\Omega)$ and $x_0 \in \overline{\Omega}$. Then the following assertions hold.*

(i)

$$\lim_{r \to 0}\left(\sup_{x \in \Omega} \int_{\Omega \cap B(x_0, r)} G_\Omega(x, y)|q(y)|dy\right) = 0.$$

(ii) *The function $x \mapsto \delta(x)q(x)$ is in $L^1(\Omega)$. In particular $x \to q(x) \in L^1_{\text{loc}}(\Omega)$.*

(iii) $\|q\|_\Omega := \sup_{x \in \Omega} \int_\Omega \frac{\delta(y)}{\delta(x)} G_\Omega(x, y)|q(y)|dy < \infty$.

(iv) *Assume that Ω is the unit ball $B(0, 1)$ and q is a radial function in $B(0, 1)$, then*

$$q \in K(B(0, 1)) \Leftrightarrow \int_0^1 r(1-r)\,|q(r)|\,dr < \infty.$$

Proof. See [15] for $n \geq 3$ and [14] for $n = 2$. □

Remark 2.2 ([21])**.** Let $q \in K(\Omega)$, then

$$\alpha_q := \sup_{x, y \in \Omega} \int_\Omega \frac{G_\Omega(x, z)G_\Omega(z, y)}{G_\Omega(x, y)}|q(z)|dz < \infty. \tag{2.2}$$

In fact, we have

$$\alpha_q < \infty \text{ if and only if } \|q\|_\Omega < \infty.$$

The next Proposition is due to [15] for $n \geq 3$ and [14] for $n = 2$.

Proposition 2.3. *Let $q \in K(\Omega)$. Then the function*

$$x \to Vq(x) := \int_\Omega G_\Omega(x, y)q(y)dy \text{ belongs to } C_0(\Omega). \tag{2.3}$$

Proof. Let $\varepsilon > 0$, $x_0 \in \overline{\Omega}$ and $q \in K(\Omega)$. From Lemma 2.1(i) there exists $r > 0$ such that

$$\sup_{z\in\Omega} \int\limits_{\Omega\cap B(x_0,r)} G_\Omega(z,y)|q(y)|dy \le \frac{\varepsilon}{4}.$$

Assume that $x_0 \in \Omega$ and let $x \in B(x_0, \frac{r}{2}) \cap \Omega$, then

$$\begin{aligned} |Vq(x) - Vq(x_0)| &\le 2\sup_{z\in\Omega} \int\limits_{\Omega\cap B(x_0,r)} G_\Omega(z,y)|q(y)|dy \\ &\quad + \int\limits_{\Omega_0} |G_\Omega(x,y) - G_\Omega(x_0,y)|\,|q(y)|dy, \\ &\le \frac{\varepsilon}{2} + \int\limits_{\Omega_0} |G_\Omega(x,y) - G_\Omega(x_0,y)|\,|q(y)|dy, \end{aligned}$$

where $\Omega_0 = \Omega \cap B^c(x_0, r)$.

Or for all $x \in B(x_0, \frac{r}{2}) \cap \Omega$ and $y \in \Omega_0$, we have $|x - y| \ge \frac{r}{2}$. Therefore by (2.1) we deduce that

$$|G_\Omega(x,y) - G_\Omega(x_0,y)| \le c\frac{\delta(x)\delta(y)}{|x-y|^n} \le \frac{c}{r^n}\delta(y),$$

where c is some positive constant.

Now, since $(x,y) \mapsto G_\Omega(x,y)$ is continuous on $\big(B(x_0, \frac{r}{2}) \cap \Omega\big) \times \Omega_0$, we get by Lemma 2.1(ii) and Lebesgue's dominated convergence theorem,

$$\int\limits_{\Omega_0} |G_\Omega(x,y) - G_\Omega(x_0,y)|\,|q(y)|dy \to 0 \quad \text{as } x \to x_0.$$

It follows that there exists $\delta > 0$ with $\delta < \frac{r}{2}$ such that if $x \in B(x_0, \delta) \cap \Omega$,

$$\int\limits_{\Omega_0} |G_\Omega(x,y) - G_\Omega(x_0,y)|\,|q(y)|dy \le \frac{\varepsilon}{2}.$$

Hence for $x \in B(x_0, \delta) \cap \Omega$, we have

$$|Vq(x) - Vq(x_0)| \le \varepsilon.$$

This implies that

$$\lim_{x\to x_0} Vq(x) = Vq(x_0).$$

If $x_0 \in \partial\Omega$ and $x \in B(x_0, \frac{r}{2}) \cap \Omega$, then we have

$$|Vq(x)| \le \sup_{z\in\Omega} \int\limits_{\Omega\cap B(x_0,r)} G_\Omega(z,y)|q(y)|dy + \int\limits_{\Omega_0} G_\Omega(x,y)|q(y)|dy.$$

Now, since $\lim_{x\to x_0} G_\Omega(x,y) = 0$, for all $y \in \Omega_0$, we deduce by similar arguments as above that

$$\lim_{x\to x_0} Vq(x) = 0.$$

Hence $Vq \in C_0(\Omega)$. □

Remark 2.4 (See [15] for $n \geq 3$ and [14] for $n = 2$)**.** The function $x \mapsto (\delta(x))^{-\alpha}$ belongs to $K(\Omega)$ if and only if $\alpha < 2$.

Proposition 2.5. *Let* $D = \operatorname{diam}(\Omega)$, $\alpha < 2$ *and* $g \in \mathcal{B}^+(\Omega)$ *satisfying*

$$g(x) \approx (\delta(x))^{-\alpha}, \text{ on } \Omega.$$

Then $x \to g(x) \in K(\Omega)$ *and we have*

$$Vg(x) \approx \begin{cases} (\delta(x))^{2-\alpha}, & \text{if } 1 < \alpha < 2,\\ (\delta(x))\log(\frac{2D}{\delta(x)}), & \text{if } \alpha = 1,\\ \delta(x), & \text{if } \alpha < 1.\end{cases}$$

Proof. See [13, Proposition 1]. □

3. MAIN RESULT

In order to study the semipositone BVP (1.1), we make the following assumptions.

$(H1)$ a is a nontrivial function in $K^+(\Omega)$.
$(H2)$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function with $f(x,0) = 0$, for all $x \in \Omega$.
$(H3)$ There exists a function $q \in K^+(\Omega)$ such that

$$|f(x,u) - f(x,v)| \leq q(x)\,|u - v|, \quad \text{for all } x \in \Omega \text{ and } u, v \in \mathbb{R}.$$

Define

$$\omega(x) := \int_\Omega G_\Omega(x,y)a(y)dy, \quad x \in \Omega. \tag{3.1}$$

Then by Proposition 2.3, ω belongs to $C_0(\Omega)$ and by Lemma 2.1 (ii) and (1.7), ω satisfies in the distributional sense

$$\begin{cases} -\Delta\omega = a, & \text{in } \Omega,\\ \omega(x) = 0, & x \in \partial\Omega.\end{cases} \tag{3.2}$$

We recall that

$$\alpha_q := \sup_{x,y\in\Omega} \int_\Omega \frac{G_\Omega(x,z)G_\Omega(z,y)}{G_\Omega(x,y)} q(z)dz$$

and by Remark 2.2, $\alpha_q < \infty$.

Lemma 3.1. *Let $p \in K^+(\Omega)$, then*

$$V(p\omega)(x) = \int_\Omega G_\Omega(x,y)p(y)\omega(y)dy \le \alpha_p \omega(x), \text{ for all } x \in \Omega, \tag{3.3}$$

where ω is given by (3.1).

Proof. By the Fubini–Tonelli theorem, for all $x \in \Omega$, we have

$$\begin{aligned} V(p\omega)(x) &= \int_\Omega a(z)\Big(\int_\Omega G_\Omega(x,y)G_\Omega(y,z)p(y)dy\Big)dz \\ &\le \alpha_p \int_\Omega G_\Omega(x,z)a(z)dz \\ &= \alpha_p\omega(x). \end{aligned}$$

□

Lemma 3.2. *Let $h, g \in \mathcal{B}(\Omega)$ such that $0 \le h \le g$. Assume that $Vg \in C_0(\Omega)$, then $Vh \in C_0(\Omega)$.*

Proof. From [18, pp. 19–20], Vh is a lower semicontinuous function in Ω. Since the function $Vg \in C_0(\Omega)$, then by writing,

$$Vg = Vh + V(g-h),$$

Vh becomes upper semicontinuous function in Ω. Hence $Vh \in C_0(\Omega)$. □

Now we state our main result.

Theorem 3.3. *Assume that $(H1)$–$(H3)$ hold. Then there exists $\lambda^* > 0$, such that for $\lambda \in (0, \lambda^*)$, BVP (1.1) has a unique nonnegative solution $u \in C_0(\Omega)$ satisfying*

$$\Big(\frac{2\lambda^* - 2\lambda}{2\lambda^* - \lambda}\Big)\omega(x) \le u(x) \le \Big(\frac{2\lambda^*}{2\lambda^* - \lambda}\Big)\omega(x), \quad \text{for all } x \in \overline{\Omega}. \tag{3.4}$$

Proof. Assume that $(H1)$–$(H3)$ hold. Consider the Banach space

$$E_\omega = \Big\{v \in C_0(\Omega) : \sup_{x\in\Omega}\frac{|v(x)|}{\omega(x)} < \infty\Big\},$$

equipped with the norm

$$\|v\|_\omega := \sup_{x\in\Omega}\frac{|v(x)|}{\omega(x)}.$$

In particular, (E_ω, d) becomes a complete metric space, with

$$d(u,v) := \|u - v\|_\omega.$$

Set $\lambda^* := \frac{1}{2\alpha_q}$. Clearly $\lambda^* > 0$ and for $\lambda \in (0, \lambda^*)$, let

$$F_\omega = \Big\{v \in E_\omega : \Big(\frac{1-2\lambda\alpha_q}{1-\lambda\alpha_q}\Big)\omega(x) \le v(x) \le \Big(\frac{1}{1-\lambda\alpha_q}\Big)\omega(x) \text{ for } x \in \overline{\Omega}\Big\}.$$

Since $\omega \in F_\omega$, we conclude that F_ω is a non-empty closed subset of (E_ω, d) and therefore (F_ω, d) is a complete metric space.

Define an operator T on F_ω by

$$Tv(x) = \omega(x) + \lambda \int_\Omega G_\Omega(x,y) f(y, v(y))dy, \text{ for } x \in \overline{\Omega}. \tag{3.5}$$

We claim that $T(F_\omega) \subset F_\omega$. Indeed, from $(H1)$–$(H3)$, for all $v \in F_\omega$, we have

$$|f(y, v(y))| \leq \Big(\frac{1}{1-\lambda\alpha_q}\Big) q(y)\omega(y) \tag{3.6}$$

$$\leq \Big(\frac{\|\omega\|_\infty}{1-\lambda\alpha_q}\Big) q(y). \tag{3.7}$$

Hence, by using (3.7), Proposition 2.3 and Lemma 3.2, we obtain that the function

$$x \to h(x) := \int_\Omega G_\Omega(x,y) f(y, v(y))dy \in C_0(\Omega).$$

So $T(F_\omega) \subset C_0(\Omega)$.

For $v \in F_\omega$, we obtain by (3.6) and Lemma 3.1,

$$\frac{-\lambda\alpha_q}{1-\lambda\alpha_q}\omega(x) \leq \lambda \int_\Omega G_\Omega(x,y) f(y, v(y))dy \leq \frac{\lambda\alpha_q}{1-\lambda\alpha_q}\omega(x).$$

Hence from (3.5), we conclude that $T(F_\omega) \subset F_\omega$.

Next, by using (3.5), $(H3)$ and (3.3), we obtain for $v_1, v_2 \in F_\omega$

$$\begin{aligned}
|Tv_1(x) - Tv_2(x)| &\leq \lambda \int_\Omega G_\Omega(x,y) \left| f(y, v_1(y)) - f(y, v_2(y)) \right| dy \\
&\leq \lambda \int_\Omega G_\Omega(x,y) q(y) \left| v_1(y) - v_2(y) \right| dy \\
&\leq \lambda d(v_1, v_2) \int_\Omega G_\Omega(x,y) q(y)\omega(y) dy \\
&\leq \lambda\alpha_q d(v_1, v_2)\omega(x).
\end{aligned}$$

Hence,

$$d(Tv_1, Tv_2) \leq \lambda\alpha_q d(v_1, v_2).$$

Since $\lambda\alpha_q < \frac{1}{2}$, then T is a contraction operator in F_ω. So, there exists a unique $u \in F_\omega$ such that

$$u(x) = \omega(x) + \lambda \int_\Omega G_\Omega(x,y) f(y, u(y))dy, \quad \text{for } x \in \overline{\Omega}. \tag{3.8}$$

It remains to prove that u is a solution of BVP (1.1).

To this end, by using (3.7) and Lemma 2.1 (ii), we derive that the function $y \to f(y,u(y)) \in L^1_{\text{loc}}(\Omega)$ and from (3.8) the function $x \to \int_\Omega G_\Omega(x,y)f(y,u(y))dy \in L^1_{\text{loc}}(\Omega)$.

Hence, from (1.7) and (3.2), we conclude that u is a solution of BVP (1.1) satisfying (3.4). □

Remark 3.4. Let us consider the function $f(\cdot,u) = 2u + \sin u$. Note that f satisfies conditions $(H2)$–$(H3)$ and we have

$$u \leq \; f(\cdot,u) \leq 3u, \text{ for } u \geq 0. \tag{3.9}$$

In this case problem (1.1) has no positive solution if λ is large.

Indeed, let φ_1 denote the normalized positive eigenfunction corresponding to the first eigenvalue λ_1 of $(-\Delta)$. Thus, by multiplication with φ_1 in (1.1) and integrating, we find

$$\lambda_1 \int_\Omega u\varphi_1 dx = \int_\Omega a\varphi_1 dx + \lambda \int_\Omega f(x,u)\varphi_1 dx > \lambda \int_\Omega u\varphi_1 dx.$$

This relation shows that problem (1.1) has no positive solution if $\lambda \geq \lambda_1$.

Example 3.5. Let $\alpha, \beta < 2$. By Theorem 3.3, there exists $\lambda^* > 0$, such that for $\lambda \in (0,\lambda^*)$, the following BVP

$$\begin{cases} -\Delta u(x) = \frac{1}{(\delta(x))^\alpha} + \lambda \frac{1}{(\delta(x))^\beta} \sin(u(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

has a unique nonnegative solution $u \in C_0(\Omega)$ satisfying

$$\Big(\frac{2\lambda^* - 2\lambda}{2\lambda^* - \lambda}\Big)\omega(x) \leq u(x) \leq \Big(\frac{2\lambda^*}{2\lambda^* - \lambda}\Big)\omega(x), \quad \text{for all } x \in \overline{\Omega}, \tag{3.10}$$

where $\omega(x) = V(\frac{1}{(\delta(\cdot))^\alpha})(x)$.

In particular, from (3.10) and Proposition 2.5, we deduce that

$$u(x) \approx \begin{cases} (\delta(x))^{2-\alpha}, & \text{if } 1 < \alpha < 2, \\ (\delta(x))\log(\frac{2D}{\delta(x)}), & \text{if } \alpha = 1, \\ \delta(x), & \text{if } \alpha < 1, \end{cases}$$

where $D = \operatorname{diam}(\Omega)$.

Example 3.6. Let $\gamma \in \mathbb{R}$, q and a be two functions in $K^+(\Omega)$. Then by Theorem 3.3, there exists $\lambda^* > 0$, such that for $\lambda \in (0,\lambda^*)$, the following BVP

$$\begin{cases} -\Delta u(x) = a(x) + \lambda q(x)\,(\delta(x))^\gamma\,(\cos(\frac{u(x)}{(\delta(x))^\gamma}) - 1), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

has a unique nonnegative solution $u \in C_0(\Omega)$ satisfying

$$\Big(\frac{2\lambda^* - 2\lambda}{2\lambda^* - \lambda}\Big)V(a)(x) \leq u(x) \leq \Big(\frac{2\lambda^*}{2\lambda^* - \lambda}\Big)V(a)(x), \text{ for all } x \in \overline{\Omega}.$$

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