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On the existence of optimal consensus control for the fractional Cucker–Smale model

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This paper addresses the nonlinear Cucker–Smale optimal control problem under the interplay of memory effect. The aforementioned effect is included by employing the Caputo fractional derivative in the equation representing the velocity of agents. Sufficient conditions for the existence of solutions to the considered problem are proved and the analysis of some particular problems is illustrated by two numerical examples.

Key words: fractional calculus, fractional differential systems, flocking model, consensus, optimal control

1. Introduction

Flocking behavior is a well-established concept in biology, robotics and control theory, as well as economics and sociology. In the biological context, we can mention fish schools, insect colonies, bird flocks [7]. Distribution of wealth in a modern society [9] or the formation of choices and opinions [18,36] are examples of collective behavior in the socio-economic context. Likewise, increasing efforts are devoted to the investigation of the coordination and cooperation among mul-

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tiple mobile agents [24,30,31]. Thirteen years ago, Cucker and Smale proposed a dynamics model of collective behavior that was motivated by Vicek's model [38]. Namely, in their seminal papers [10,11], they postulated the model that describes the emergence of consensus in a group of moving agents (e.g., flocking in a swarm of birds). In the proposed model, the state of each agent is characterized by a pair (x_i, v_i), representing variables which we refer to as position and velocity, respectively. Then, the birds influence each other according to a decreasing function of their mutual space distance. More precisely, the system evolves following the second-order dynamic

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= \frac{1}{N}\sum_{j=1}^{N} \eta\left(\left\|x_j(t) - x_i(t)\right\|^2_{l_2^3}\right)\left(v_j(t) - v_i(t)\right), \qquad i = 1, \ldots, N, \end{aligned} \tag{1}$$

where $\eta(r)$ is a communication rate that decays as the distance between agents increases. The asymptotic behavior of the model, called flocking or consensus, consists in the fact that for $t \to \infty$, all agent velocities become equal, with fixed relative positions. The emergence of consensus either by a sufficiently cohesive initial condition (x_0, v_0) or a strong interaction $\eta(r)$ was studied in [10, 11, 21, 23]. Afterward, many modifications of the classical Cucker–Smale model were proposed. The original setting of the model was extended to a collision avoiding flocking control protocol [12] and to the scenario of guiding agents with a preferred velocity direction [13]. Shen [33] considered a non-symmetric structure of interactions. The Cucker–Smale model with the presence of noise was analyzed in [14]. In [16,17] was addressed the situation in which each interaction between agent is subject to random failure. Fractional Cucker–Smale models which were obtained by replacing the usual time derivative by a fractional time derivative were studied in [19, 20, 22]. Since the formation of consensus in the Cucker–Smale model strongly depends on the communication rate function and the initial configuration of the system, it is relevant to consider external control strategies that will facilitate the consensus. Taking account of existing works, in [4, 8] were designed optimal control protocols for the Cucker–Smale system under the prespecified cost functional. In the mentioned papers, a finite time optimal control was considered with the minimization criteria that is a sum of a norm of control and a distance from consensus. The obtained control either induces consensus on an initial configuration of the system that would otherwise diverge or accelerates the rate of convergence for initial data that would naturally converge to consensus.

In this paper, we follow the approaches previously described. Namely, we study the consensus control of the Cucker–Smale type system with optimization performance. However, differently from the previous papers, we study the Cucker–Smale flocking dynamics under the interplay of memory effect. As a mathematical model incorporating the memory effect, we use the fractional Cucker–Smale model proposed in [19]. To be specific, a modification of the Cucker–Smale model was obtained by replacing the usual time derivative by the Caputo fractional time derivative only in the second equation of system (1). In this way, the first equation of the system could be still treated as a position of an agent and simultaneously we incorporate the memory factor into the consensus process. We prove the existence of an optimal controller for a problem with the fractional Cucker–Smale model and the cost functional that minimizes the distance to consensus and control.

There have been much research that shows that fractional-order models own better description memory and hereditary properties of various processes than classical models with integer order derivatives [2, 6, 15, 25, 28, 35, 37]. Since fractional derivatives are non-local operators, the long-range interactions in time (memory) could be modeled [1, 34].

The reminder of the paper is organized as follows. Section 2 shows some preliminaries from fractional calculus. Section 3 presents the fractional Cucker–Smale model. Then, Section 4 is devoted to the study of the existence and uniqueness of solution to controlled fractional Cucker–Smale system. Our main result, that is the existence of optimal consensus control for controlled fractional Cucker–Smale model, is proved in Section 5. Numerical examples are given in Section 6. Finally, some conclusions are drawn in the last section.

2. Preliminaries

In this section, we present necessary definitions and properties concerning fractional derivatives and integrals (cf. [27, 32]). We shall assume that $[a, b] \subset \mathbb{R}$ is any bounded interval.

Let $\alpha > 0$ and $f \in L^1([a, b], \mathbb{R}^n)$. By the left- and the right-sided Riemann–Liouville integral of the function f of order α we mean functions

$$\left(I^{\alpha}_{a+} f\right)(t) := \frac{1}{\Gamma(\alpha)} \int\limits_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\, \mathrm{d}\tau, \qquad t \in [a, b]\ a.e.$$

and

$$\left(I^{\alpha}_{b-} f\right)(t) := \frac{1}{\Gamma(\alpha)} \int\limits_t^b \frac{f(\tau)}{(\tau-t)^{1-\alpha}}\, \mathrm{d}\tau, \qquad t \in [a, b]\ a.e.,$$

respectively. In view of convergence (cf. [32, Theorem 2.7])

$$\lim_{\alpha \to 0^+} \left(I^{\alpha}_{a+} f\right)(t) = f(t), \qquad t \in [a, b]\ a.e.$$

it is natural to put

$$\left(I^0_{a+}f\right)(t) = f(t), \qquad t \in [a,b] \ a.e.$$

Let $1 \leqslant p < \infty$. By $I^\alpha_{a+}(L^p([a,b],\mathbb{R}^n))$ (briefly $I^\alpha_{a+}(L^p)$) we denote the space

$$I^\alpha_{a+}(L^p) := \{f : [a,b] \to \mathbb{R}^n; \quad f = I^\alpha_{a+}g \ a.e. \ \text{on} \ [a,b], \quad g \in L^p([a,b],\mathbb{R}^n)\}.$$

Then, we identify functions belonging to the space $I^\alpha_{a+}(L^p)$ and equal almost everywhere on $[a,b]$. Let $\alpha \in (0,1]$ and $f \in L^1([a,b],\mathbb{R}^n)$. The left-sided Riemann–Liouville derivative $D^\alpha_{a+}f$ of order α of f is defined by

$$\left(D^\alpha_{a+}f\right)(t) := \frac{\mathrm{d}}{\mathrm{d}t}\left(I^{1-\alpha}_{a+}f\right)(t), \qquad t \in [a,b] \ a.e.,$$

provided that the function $I^{1-\alpha}_{a+}f$ is absolutely continuous on $[a,b]$.

Remark 1 *If $\alpha = 1$, then $D^\alpha_{a+}f = \frac{\mathrm{d}}{\mathrm{d}t}f$.*

The following composition properties hold.

Proposition 1 [27, Lemmas 2.4, 2.5 (a)] *Let $0 < \alpha \leqslant 1$.*

(a) If $f \in L^1([a,b],\mathbb{R}^n)$, then

$$\left(D^\alpha_{a+}I^\alpha_{a+}f\right)(t) = f(t), \qquad t \in [a,b] \ a.e.$$

(b) If $f \in I^\alpha_{a+}(L^1)$, then

$$\left(I^\alpha_{a+}D^\alpha_{a+}f\right)(t) = f(t), \qquad t \in [a,b] \ a.e.$$

Let $f \in C([a,b],\mathbb{R}^n)$. By the left-sided Caputo derivative of order α of the function f on the interval $[a,b]$ we mean a function ${}^CD^\alpha_{a+}f$ given by

$$\left({}^CD^\alpha_{a+}f\right)(t) := D^\alpha_{a+}(f(\cdot) - f(a))(t), \qquad t \in [a,b] \ a.e.,$$

provided that the derivative in the Riemann–Liouville sense on the right side exists.

Remark 2 *If $\alpha = 1$, then ${}^CD^\alpha_{a+}f = \frac{\mathrm{d}}{\mathrm{d}t}f$. Moreover, if both derivatives $D^\alpha_{a+}f$ and ${}^CD^\alpha_{a+}f$ exist and $f(a) = 0$, then they coincide.*

Remark 3 *If $f(t) = const$, then ${}^CD^\alpha_{a+}f = 0$.*

Let us denote by ${}_C AC^{\alpha,p}_{a+}([a,b],\mathbb{R}^n)$ (briefly ${}_C AC^{\alpha,p}_{a+}$), where $p > \dfrac{1}{\alpha}$, the set of all functions $f : [a,b] \to \mathbb{R}^n$ that have the representation

$$f(t) = c_a + \left(I^{\alpha}_{a+}\varphi\right)(t), \qquad t \in [a,b] \ a.e.,$$

with some $c_a \in \mathbb{R}^n$ and $\varphi \in L^p([a,b],\mathbb{R}^n)$. From [5, Property 4] it follows that if $f \in {}_C AC^{\alpha,p}_{a+}$, then it is continuous on $[a,b]$ and $f(a) = c_a$. Consequently, there exists the Caputo derivative ${}^C D^{\alpha}_{a+} f$ and (cf. Proposition 1)

$$\left({}^C D^{\alpha}_{a+} f\right)(t) = D^{\alpha}_{a+}(f - f(a))(t) = (D^{\alpha}_{a+} I^{\alpha}_{a+}\varphi)(t) = \varphi(t), \quad t \in [a,b] \ a.e.$$

Remark 4 *If* $\alpha = 1$, *then* ${}_C AC^{\alpha,p}_{a+} = AC^p$, *where*

$$AC^p = AC^p([a,b],\mathbb{R}^n) = \left\{ f \in AC([a,b],\mathbb{R}^n) : \ \frac{\mathrm{d}}{\mathrm{d}t} f \in L^p([a,b],\mathbb{R}^n) \right\}.$$

From Proposition 1 and [5, Property 4] we immediately obtain the following result.

Proposition 2 *Let* $0 < \alpha \leqslant 1$ *and* $p > \dfrac{1}{\alpha}$.

(a) If $f \in L^p([a,b],\mathbb{R}^n)$, *then*

$$\left({}^C D^{\alpha}_{a+} I^{\alpha}_{a+} f\right)(t) = f(t), \qquad t \in [a,b] \ a.e.$$

(b) If $f \in {}_C AC^{\alpha,p}_{a+}$, *then*

$$\left(I^{\alpha}_{a+}\, {}^C D^{\alpha}_{a+} f\right)(t) = f(t) - f(a), \qquad t \in [a,b] \ a.e.$$

Let $\|\cdot\|_{l^n_2}$ denote an Euclidean norm in $\mathbb{R}^n$. In the space ${}_C AC^{\alpha,p}_{a+}$, we introduce the norm given by

$$\|f\|_{{}_C AC^{\alpha,p}_{a+}} := \left(\|f(a)\|^p_{l^n_2} + \left\| {}^C D^{\alpha}_{a+} f \right\|^p_{L^p} \right)^{\frac{1}{p}}. \tag{2}$$

It is easy to check that the space ${}_C AC^{\alpha,p}_{a+}$ with norm (2) is complete. In particular, ${}_C AC^{\alpha,2}_{a+}$ with the inner product

$$\langle f, g\rangle_{{}_C AC^{\alpha,2}_{a+}} := \langle f(a), g(a)\rangle_{\mathbb{R}^n} + \int_a^b \left\langle \left({}^C D^{\alpha}_{a+} f\right)(t),\ \left({}^C D^{\alpha}_{a+} g\right)(t) \right\rangle_{\mathbb{R}^n} \mathrm{d}t$$

is a Hilbert space.

The following preliminary result will be useful in the sequel.

Lemma 1 *Let* $\alpha \in (0,1]$ *and* $p > \frac{1}{\alpha}$. *If* $(f^l)_{l\in\mathbb{N}} \subset {}_C AC^{\alpha,p}_{a+}$ *is a sequence such that* $f^l \rightharpoonup f^0$ *weakly in* ${}_C AC^{\alpha,p}_{a+}$, *then* $f^l \to f^0$ *strongly in* $L^p([a,b],\mathbb{R}^n)$ *and* ${}^C D^\alpha_{a+} f^l \rightharpoonup {}^C D^\alpha_{a+} f^0$ *weakly in* $L^p([a,b],\mathbb{R}^n)$.

Proof. Let $f^l \rightharpoonup f^0$ weakly in ${}_C AC^{\alpha,p}_{a+}$. It is clear that linear mappings

$$ {}_C AC^{\alpha,p}_{a+} \ni f \longrightarrow {}^C D^\alpha_{a+} f \in L^p([a,b],\mathbb{R}^n), $$

$$ {}_C AC^{\alpha,p}_{a+} \ni f \longrightarrow f(a) \in \mathbb{R}^n $$

are continuous. Consequently, ${}^C D^\alpha_{a+} f^l \rightharpoonup {}^C D^\alpha_{a+} f^0$ weakly in $L^p([a,b],\mathbb{R}^n)$ and $f^l(a) \rightharpoonup f^0(a)$ weakly (so also strongly) in $\mathbb{R}^n$. Moreover, since the operator I^α_{a+} is completely continuous (cf. [29, Lemma 1.1]), we have

$$ I^\alpha_{a+}\, {}^C D^\alpha_{a+} f^l \longrightarrow I^\alpha_{a+}\, {}^C D^\alpha_{a+} f^0 \quad \text{strongly in } L^p([a,b],\mathbb{R}^n). $$

Thus

$$ f^l = f^l(a) + I^\alpha_{a+}\, {}^C D^\alpha_{a+} f^l \longrightarrow f^0(a) + I^\alpha_{a+}\, {}^C D^\alpha_{a+} f^0 = f^0 \quad \text{strongly in } L^p([a,b],\mathbb{R}^n). $$

The proof is completed. □

Let $\mathbb{E}^{\alpha,p}_{a+}([a,b],\mathbb{R}^n\times\mathbb{R}^n)$ (briefly $\mathbb{E}^{\alpha,p}_{a+}$) denote the space

$$ \mathbb{E}^{\alpha,p}_{a+} := AC^p\left([a,b],\mathbb{R}^n\right) \times {}_C AC^{\alpha,p}_{a+}\left([a,b],\mathbb{R}^n\right). $$

It is a Banach space with the norm

$$ \|z\|_{\mathbb{E}^{\alpha,p}_{a+}} := \left(\|z_1\|^p_{AC^p} + \|z_2\|^p_{{}_C AC^{\alpha,p}_{a+}} \right)^{\frac{1}{p}}, \qquad z = (z_1, z_2) \in \mathbb{E}^{\alpha,p}_{a+}, $$

as a Cartesian product of the Banach spaces AC^p and ${}_C AC^{\alpha,p}_{a+}$. In $\mathbb{E}^{\alpha,p}_{a+}$ we introduce a Bielecki type norm in the following way:

$$ \begin{aligned} \|z\|_{\mathbb{E}^{\alpha,p}_{a+},k} &:= \left(\|z_1\|^p_{AC^p,k} + \|z_2\|^p_{{}_C AC^{\alpha,p}_{a+},k} \right)^{\frac{1}{p}} \\ &= \left(\|z_1(a)\|^p_{l^n_2} + \int_a^b e^{-kpt} \|\dot{z}_1(t)\|^p_{l^n_2}\,\mathrm{d}t + \|z_2(a)\|^p_{l^n_2} \right. \\ &\qquad \left. + \int_a^b e^{-kpt} \|\left({}^C D^\alpha_{a+} z_2\right)(t)\|^p_{l^n_2}\,\mathrm{d}t \right)^{\frac{1}{p}}, \end{aligned} \tag{3} $$

where $k > 0$ is an arbitrary fixed constant. It is clear that

$$\min\left\{1, e^{-kb}\right\} \|z\|_{\mathbb{E}^{\alpha,p}_{a+}} \leqslant \|z\|_{\mathbb{E}^{\alpha,p}_{a+},k} \leqslant \max\left\{1, e^{-ka}\right\} \|z\|_{\mathbb{E}^{\alpha,p}_{a+}},$$

so, the space $\mathbb{E}^{\alpha,p}_{a+}$ with norm (3) is complete.

3. The fractional Cucker–Smale model

Let us consider the following system of N interacting agents

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ {}^C D^{\alpha}_{0^+} v_i(t) = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \eta\left(\|x_j(t) - x_i(t)\|^2_{l^d_2}\right)(v_j(t) - v_i(t)), \quad t \in [0,T] \; a.e., \\ (x_i(0), v_i(0)) = (x_{i0}, v_{i0}), \qquad i = 1,\ldots,N \end{cases} \tag{4}$$

where $\alpha \in (0,1]$. The state $(x,v) \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$ consists of the main state $x = (x_1,\ldots,x_N)$ and a consensus parameter $v = (v_1,\ldots,v_N)$, where $x_i \in \mathbb{R}^d$ represents the main state of the agent i, $i = 1,\ldots,N$, while $v_i \in \mathbb{R}^d$ denotes its consensus parameter. Note that we propose a modification of the Cucker–Smale model employing fractional operators but only to the second equation of system (4). In this way, the first equation of the system could be still treated as a position of an agent and simultaneously we include the memory factor to the consensus process. The coefficient

$$\eta\left(\|x_j - x_i\|^2_{l^d_2}\right),$$

where $\eta \in C^1([0,+\infty),(0,+\infty))$, is non-increasing function, called a rate of communication or an interaction potential, describes the influence of j-th agent on the dynamics of the i-th agent. It means that the interaction between agents is a function of the distance between them. Observe that, similarly to the integer-order Cucker–Smale model [8], the mean consensus parameter

$$\bar{v} = \frac{1}{N}\sum_{j=1}^{N} v_j(t)$$

is an invariant of dynamics (4) and therefore, by Remark 3, it is constant in time.

Definition 1 *Consensus is a state in which all agents have the same consensus parameter equals $\bar{v}$.*

Remark 5 *If every agent moves with the same consensus parameter equal* $\bar{v}$*, then the dynamics originating from* (x_0, v_0) *is given by rigid translation* $x(t) = x_0 + t\bar{v}$ *that is called a rigid flock.*

Let us introduce matrices $A, D : \mathbb{R}^{Nd} \to \mathbb{R}^{N\times N}$ given by:

$$A(x) = \begin{bmatrix} \frac{1}{N}\eta(0) & \frac{1}{N}\eta\left(\|x_2 - x_1\|^2_{l_2^d}\right) & \dots & \frac{1}{N}\eta\left(\|x_N - x_1\|^2_{l_2^d}\right) \\ \frac{1}{N}\eta\left(\|x_1 - x_2\|^2_{l_2^d}\right) & \frac{1}{N}\eta(0) & \dots & \frac{1}{N}\eta\left(\|x_N - x_2\|^2_{l_2^d}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N}\eta\left(\|x_1 - x_N\|^2_{l_2^d}\right) & \frac{1}{N}\eta\left(\|x_2 - x_N\|^2_{l_2^d}\right) & \dots & \frac{1}{N}\eta(0) \end{bmatrix}_{N\times N},$$

$$D(x) = \begin{bmatrix} \frac{1}{N}\sum_{j=1}^{N}\eta\left(\|x_j - x_1\|^2_{l_2^d}\right) & 0 & \dots & 0 \\ 0 & \frac{1}{N}\sum_{j=1}^{N}\eta\left(\|x_j - x_2\|^2_{l_2^d}\right) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{N}\sum_{j=1}^{N}\eta\left(\|x_j - x_N\|^2_{l_2^d}\right) \end{bmatrix}_{N\times N},$$

and define a matrix $L : \mathbb{R}^{Nd} \to \mathbb{R}^{Nd\times Nd}$ in the following way:

$$L(x) := (D(x) - A(x)) \otimes I_d, \quad x \in \mathbb{R}^{Nd},$$

where $\otimes$ denotes the Kronecker product of matrices and I_d is a d-dimensional identity matrix. Then, we can write system (4) in the following matrix form:

$$\begin{cases} \dot{x}(t) = v(t), \\ {}^C D^{\alpha}_{0^+} v(t) = -L(x(t))v(t), \quad t \in [0, T] \ a.e., \\ (x(0), v(0)) = (x_0, v_0). \end{cases} \tag{5}$$

Lemma 2 *Assume that*

(*A*) *the function* $(\cdot)\dot{\eta}(\cdot)$ *is bounded on* $[0, \infty)$ *provided that the limit* $\lim_{r\to\infty}(r\dot{\eta}(r))$ *does not exist.*

Then the function L satisfies a globally Lipschitz condition, it means there exists $C > 0$ such that[1]

$$\|L(x) - L(y)\|_{Nd\times Nd} \leqslant C\|x - y\|_{l_2^{Nd}}, \quad x, y \in \mathbb{R}^{Nd}. \tag{6}$$

Proof. First, we shall prove that the function $\widetilde{L} : \mathbb{R}^{Nd} \to \mathbb{R}^{N\times N}$ given by $\widetilde{L} = D - A$ satisfies a globally Lipschitz condition. In order to prove this fact it is sufficient to show that each component of matrix functions A and D satisfies a globally Lipschitz condition. To that end, we shall show that derivatives $\dfrac{\partial A_{il}(x)}{\partial x}, \dfrac{\partial D_{il}(x)}{\partial x} : \mathbb{R}^{Nd} \to \mathbb{R}^{Nd}$ are continuous and bounded on $\mathbb{R}^{Nd}$. Indeed, we have

$$\frac{\partial A_{il}(x)}{\partial x} = \left(\frac{\partial A_{il}(x)}{\partial x_1}, \ldots, \frac{\partial A_{il}(x)}{\partial x_N}\right),$$

whereby

$$\frac{\partial A_{il}(x)}{\partial x_k} = \begin{cases} -\dfrac{2}{N}\dot{\eta}\left(\|x_l - x_i\|^2_{l_2^d}\right)(x_l - x_i), & k = i, \\ \dfrac{2}{N}\dot{\eta}\left(\|x_l - x_i\|^2_{l_2^d}\right)(x_l - x_i), & k = l, \\ \mathbf{0}_d & k \in \{1, \ldots, N\} \setminus \{i, l\} \end{cases} \qquad i, l = 1, \ldots, N,$$

(here $\mathbf{0}_d$ denotes the d-dimensional zero vector) and

$$\frac{\partial D_{il}(x)}{\partial x} = \left(\frac{\partial D_{il}(x)}{\partial x_1}, \ldots, \frac{\partial D_{il}(x)}{\partial x_N}\right),$$

whereby

$$\frac{\partial D_{il}(x)}{\partial x_k} = \mathbf{0}_{Nd}, \qquad i \neq l, \ k = 1, \ldots, N,$$

$$\frac{\partial D_{ii}(x)}{\partial x_k} = \begin{cases} -\dfrac{2}{N}\displaystyle\sum_{j=1}^{N}\dot{\eta}\left(\|x_j - x_i\|^2_{l_2^d}\right)(x_j - x_i), & k = i, \\ \dfrac{2}{N}\dot{\eta}\left(\|x_k - x_i\|^2_{l_2^d}\right)(x_k - x_i), & k \neq i, \end{cases} \qquad i = 1, \ldots, N.$$

[1] The symbol $\|\cdot\|_{n\times n}$ denotes the Frobenius norm of a square matrix $A = [a_{ij}]_{i,j=1,\ldots,n}$.

Of course, $\dfrac{\partial A_{il}(x)}{\partial x}$, $\dfrac{\partial D_{il}(x)}{\partial x}$, $i, l = 1, \ldots, N$, are continuous on $\mathbb{R}^{Nd}$. Moreover, let us note that

$$\begin{aligned}\|\dot{\eta}\left(\|x_l - x_i\|^2_{l_2^d}\right)(x_l - x_i)\|_{l_2^d} &\leqslant \left|\dot{\eta}\left(\|x_l - x_i\|^2_{l_2^d}\right)\right| \|(x_l - x_i)\|_{l_2^d} \\ &\leqslant \begin{cases} \left|\dot{\eta}\left(\|x_l - x_i\|^2_{l_2^d}\right)\right| \|(x_l - x_i)\|^2_{l_2^d} & \text{if } \|(x_l - x_i)\|_{l_2^d} > 1 \\ \left|\dot{\eta}\left(\|x_l - x_i\|^2_{l_2^d}\right)\right| & \text{if } \|(x_l - x_i)\|_{l_2^d} \leqslant 1 \end{cases}\end{aligned}$$

for $i, l = 1, \ldots, N$. Consequently, from assumption (A) and [3, Corollary 10] it follows that the derivatives $\dfrac{\partial A_{il}(x)}{\partial x}$, $\dfrac{\partial D_{il}(x)}{\partial x}$, $i, l = 1, \ldots, N$, are bounded on $\mathbb{R}^{Nd}$.

Now, we show that condition (6) holds. Indeed, let $\widetilde{C} > 0$ be a Lipschitz constant for the function $\widetilde{L}$. Then we have

$$\begin{aligned}\|L(x) - L(y)\|_{Nd\times Nd} &= \left\|\left(\widetilde{L}(x) - \widetilde{L}(y)\right) \otimes I_d\right\|_{Nd\times Nd} \\ &= \left\|\widetilde{L}(x) - \widetilde{L}(y)\right\|_{N\times N} \|I_d\|_{d\times d} \\ &\leqslant \widetilde{C}\sqrt{d}\|x - y\|_{l_2^{Nd}}, \quad x, y \in \mathbb{R}^{Nd}.\end{aligned}$$

The proof is completed. □

4. A fractional controlled Cucker–Smale model

In this section, we introduce a control to the fractional Cucker–Smale model (5). Namely, we study the existence and uniqueness of a solution to the following system:

$$\begin{cases} \dot{x}(t) = v(t), & \\ {}^C D^{\alpha}_{0^+} v(t) = -L(x(t))v(t) + u(t), & t \in [0, T] \ a.e., \\ u(t) \in M \subset \mathbb{R}^{Nd}, & t \in [0, T], \\ (x(0), v(0)) = (x_0, v_0), & \end{cases} \tag{7}$$

where $u = (u_1, \ldots, u_N) : [0, T] \to \mathbb{R}^{Nd}$ is a control function and

$$M := \left\{z \in \mathbb{R}^{Nd} : \quad \|z\|_{l_2^{Nd}} \leqslant K\right\}$$

for a given $K > 0$. Define

$$\mathcal{U}^p_M := \{u \in L^p([0, T], \mathbb{R}^m); \quad u(t) \in M, \ \ t \in [0, T]\}$$

for $1 \leqslant p < \infty$, and

$$\mathbb{E}^{\alpha,p}_{0+,z_0} := \{z = (x, v) \in \mathbb{E}^{\alpha,p}_{0+}([0,T], \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}) : \quad z(0) = z_0\}$$

for $p > \dfrac{1}{\alpha}$ and $z_0 = (x_0, v_0) \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$.

Definition 2 *By a solution to control system (7), corresponding to any fixed control $u \in \mathcal{U}^p_M$, we mean a function $z = (x, v) \in \mathbb{E}^{\alpha,p}_{0+,z_0}$ satisfying system* (7) *a.e. on* $[0,T]$.

From Proposition 2 we immediately obtain the following result.

Proposition 3 *The function $z = (x, v) \in \mathbb{E}^{\alpha,p}_{0+,z_0}$ is a solution to control system (7), corresponding to any control $u \in \mathcal{U}^p_M$ if and only if it satisfies the integral equation*

$$\begin{cases} x(t) = x_0 + (I^1_{0+}v)(t) \\ v(t) = v_0 - I^\alpha_{0^+}[L(x(t))v(t)] + (I^\alpha_{0^+}u)(t) \end{cases} \qquad t \in [0,T].$$

We are now in a position to prove the existence of a unique solution to system (7).

Theorem 1 *Let $\alpha \in (0,1]$ and $p > \dfrac{1}{\alpha}$. If condition (A) from Lemma 2 is satisfied, then control system (7) has a unique solution $z = (x, v) \in \mathbb{E}^{\alpha,p}_{0+,z_0}$, corresponding to any control $u \in \mathcal{U}^p_M$.*

Proof. Let us fix $u \in \mathcal{U}^p_M$. In view of Proposition 3 it is sufficient to prove that the mapping $T_u : \mathbb{E}^{\alpha,p}_{0+,z_0} \to \mathbb{E}^{\alpha,p}_{0+,z_0}$ given by

$$T_u(z) = T_u(x, v) = \left(x_0 + I^1_{0+}v, v_0 - I^\alpha_{0^+}[L(x)v] + I^\alpha_{0^+}u\right)$$

has a unique fixed point (the fact that T_u is well defined is obvious).

Indeed, let us consider a metric space $\left(\mathbb{E}^{\alpha,p}_{0+}, \rho_{p,k}(\cdot,\cdot)\right)$, where $\rho_{p,k}$ is a Bielecki type metric induced by the norm $\|\cdot\|_{\mathbb{E}^{\alpha,p}_{0+,k}}$. Since $\mathbb{E}^{\alpha,p}_{0+,z_0}$ is a closed subset of $\mathbb{E}^{\alpha,p}_{0+}$, it follows that $\left(\mathbb{E}^{\alpha,p}_{0+,z_0}, \rho_{p,k,z_0}(\cdot,\cdot)\right)$, where

$$\rho_{p,k,z_0}(z,\widetilde{z}) := \rho_{p,k}(z,\widetilde{z})_{|\,\mathbb{E}^{\alpha,p}_{0+,z_0}}$$
$$= \left(\int_0^T e^{-kpt}\left(\left\|\frac{\mathrm{d}}{\mathrm{d}t}(z_1(t) - \widetilde{z}_1(t))\right\|^p_{l^n_2} + \left\|{}^C D^\alpha_{0+}(z_2(t) - \widetilde{z}_2(t))\right\|^p_{l^n_2}\right)\mathrm{d}t\right)^{\frac{1}{p}}$$

for $(z,\widetilde{z}) = ((z_1,z_2),(\widetilde{z}_1,\widetilde{z}_2)) \in \mathbb{E}^{\alpha,p}_{0+,z_0}$, is a complete metric space. We shall show that T_u is a contraction. Let $(z,\widetilde{z}) = ((x,v),(\widetilde{x},\widetilde{v})) \in \mathbb{E}^{\alpha,p}_{0+,z_0}$. Then, using Lemma 2, Proposition 2 and [26, Lemma 1], we obtain

$$
\begin{aligned}
&\left(\rho_{p,k,z_0}\left(T_u(z) - T_u\left(\widetilde{z}\right)\right)\right)^p \\
&= \int\limits_0^T e^{-kpt}\left(\|v(t)-\widetilde{v}(t)\|^p_{l_2^{Nd}} + \|L(x(t))v(t) - L(\widetilde{x}(t))\widetilde{v}(t)\|^p_{l_2^{Nd}}\right)\mathrm{d}t \\
&\leqslant \int\limits_0^T e^{-kpt}\left\|I^\alpha_{0+}\,{}^CD^\alpha_{0+}(v(t)-\widetilde{v}(t))\right\|^p_{l_2^{Nd}}\mathrm{d}t \\
&\quad + 2^{p-1}\int\limits_0^T e^{-kpt}\left(\|(L(x(t)) - L(\widetilde{x}(t)))v(t)\|^p_{l_2^{Nd}} + \|L(\widetilde{x}(t))(v(t)-\widetilde{v}(t))\|^p_{l_2^{Nd}}\right)\mathrm{d}t \\
&\leqslant \left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^{p-1}\int\limits_0^T e^{-kpt} I^\alpha_{0+}\left\|{}^CD^\alpha_{0+}(v(t)-\widetilde{v}(t))\right\|^p_{l_2^{Nd}}\mathrm{d}t \\
&\quad + 2^{p-1}\left(C\max_{t\in[0,T]}\|v(t)\|_{l_2^{Nd}}\right)^p\int\limits_0^T e^{-kpt}\left\|I^1_{0+}\frac{\mathrm{d}}{\mathrm{d}t}(x(t)-\widetilde{x}(t))\right\|^p_{l_2^{Nd}}\mathrm{d}t \\
&\quad + 2^{p-1}\left(\max_{r\in[0,+\infty)}\|L(r)\|_{Nd\times Nd}\right)^p\int\limits_0^T e^{-kpt}\|I^\alpha_{0+}\,{}^CD^\alpha_{0+}(v(t)-\widetilde{v}(t))\|^p_{l_2^{Nd}}\mathrm{d}t \\
&\leqslant \left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^{p-1}\int\limits_0^T e^{-kpt} I^\alpha_{0+}\left\|{}^CD^\alpha_{0+}(v(t)-\widetilde{v}(t))\right\|^p_{l_2^{Nd}}\mathrm{d}t \\
&\quad + (2T)^{p-1}\left(C\max_{t\in[0,T]}\|v(t)\|_{l_2^{Nd}}\right)^p\int\limits_0^T e^{-kpt} I^1_{0+}\left\|\frac{\mathrm{d}}{\mathrm{d}t}(x(t)-\widetilde{x}(t))\right\|^p_{l_2^{Nd}}\mathrm{d}t \\
&\quad + \left(\frac{2T^\alpha}{\Gamma(\alpha+1)}\right)^{p-1}\left(\max_{r\in[0,+\infty)}\|L(r)\|_{Nd\times Nd}\right)^p\int\limits_0^T e^{-kpt} I^\alpha_{0+}\left\|{}^CD^\alpha_{0+}(v(t)-\widetilde{v}(t))\right\|^p_{l_2^{Nd}}\mathrm{d}t \\
&= J_1 + J_2,
\end{aligned}
$$

where

$$J_1 = \left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^{p-1} \Bigg[1 + 2^{p-1}\left(\max_{r\in[0,+\infty)} \|L(r)\|_{Nd\times Nd}\right)^p\Bigg] \int\limits_0^T e^{-kpt} I^\alpha_{0+} \left\| {}^C D^\alpha_{0+}(v(t)-\widetilde{v}(t))\right\|^p_{l_2^{Nd}} \,\mathrm{d}t$$

and

$$J_2 = (2T)^{p-1}\left(C \max_{t\in[0,T]} \|v(t)\|_{l_2^{Nd}}\right)^p \int\limits_0^T e^{-kpt} I^1_{0+} \left\|\frac{\mathrm{d}}{\mathrm{d}t}(x(t)-\widetilde{x}(t))\right\|^p_{l_2^{Nd}} \,\mathrm{d}t.$$

Let us note that since (cf. [26, proof of Theorem 4])

$$(I^\alpha_{T-} e^{-kp\cdot})(t) \leqslant \frac{1}{(kp)^\alpha} e^{-kpt}, \qquad t\in[0,T] \; a.e.,$$

it follows, by [27, Lemma 2.7] that

$$\begin{aligned}
&\int\limits_0^T e^{-kpt} I^\alpha_{0+} \left\| {}^C D^\alpha_{0+}(v(t)-\widetilde{v}(t))\right\|^p_{l_2^{Nd}} \,\mathrm{d}t \\
&\qquad = \int\limits_0^T (I^\alpha_{T-} e^{-kp\cdot})(t) \left\| {}^C D^\alpha_{0+}(v(t)-\widetilde{v}(t))\right\|^p_{l_2^{Nd}} \,\mathrm{d}t \\
&\qquad \leqslant \frac{1}{(kp)^\alpha} \int\limits_0^T e^{-kpt} \left\| {}^C D^\alpha_{0+}(v(t)-\widetilde{v}(t))\right\|^p_{l_2^{Nd}} \,\mathrm{d}t
\end{aligned}$$

and

$$\begin{aligned}
\int\limits_0^T e^{-kpt} I^1_{0+} \left\|\frac{\mathrm{d}}{\mathrm{d}t}(x(t)-\widetilde{x}(t))\right\|^p_{l_2^{Nd}} \,\mathrm{d}t &\leqslant \int\limits_0^T (I^1_{T-} e^{-kp\cdot})(t) \left\|\frac{\mathrm{d}}{\mathrm{d}t}(x(t)-\widetilde{x}(t))\right\|^p_{l_2^{Nd}} \,\mathrm{d}t \\
&\leqslant \frac{1}{kp} \int\limits_0^T e^{-kpt} \left\|\frac{\mathrm{d}}{\mathrm{d}t}(x(t)-\widetilde{x}(t))\right\|^p_{l_2^{Nd}} \,\mathrm{d}t.
\end{aligned}$$

Consequently,

$$\left(\rho_{p,k,z_0}\left(T_u(z)-T_u(\widetilde{z})\right)\right)^p \leqslant c_{k,p}\left(\|\dot{x}-\dot{\widetilde{x}}\|^p_{L^p,k}+\|{}^C D^\alpha_{a+}(v-\widetilde{v})\|^p_{L^p,k}\right),$$

where

$$c_{k,p}=\frac{1}{(kp)^\alpha}\max\left\{\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^{p-1}\left[1+2^{p-1}\left(\max_{r\in[0,+\infty)}\|L(r)\|_{Nd\times Nd}\right)^p\right],\right.$$
$$\left.(2T)^{p-1}\left(C\max_{t\in[0,T]}\|v(t)\|_{l_2^{Nd}}\right)^p\right\}.$$

Hence

$$\rho_{p,k,z_0}\left(T_u(z)-T_u(\widetilde{z})\right)\leqslant (c_{k,p})^{\frac{1}{p}}\left(\|\dot{x}-\dot{\widetilde{x}}\|^p_{L^p,k}+\|{}^C D^\alpha_{a+}(v-\widetilde{v})\|^p_{L^p,k}\right)^{\frac{1}{p}}$$
$$=(c_{k,p})^{\frac{1}{p}}\rho_{p,k,z_0}\left(z-\widetilde{z}\right).$$

Since $(c_{k,p})^{\frac{1}{p}}\in(0,1)$ for sufficiently large k, we conclude, by the Banach contraction principle, that the operator T_u has a unique fixed point. The proof is completed. $\square$

The following lemma will be used in the next section.

Lemma 3 *If all assumptions of Theorem 1 are satisfied, then there exist constants $C_1, C_2 > 0$ such that*

$$\|x^u(t)\|_{l_2^{Nd}}\leqslant C_1,\quad t\in[0,T],\qquad u\in\mathcal{U}^p_M,$$

and

$$\|v^u(t)\|_{l_2^{Nd}}\leqslant C_2,\quad t\in[0,T],\qquad u\in\mathcal{U}^p_M,$$

where $(x^u,v^u)\in\mathbb{E}^{\alpha,p}_{0+,z_0}$ is a solution of (7) corresponding to a control $u\in\mathcal{U}^p_M$.

Proof. From Proposition 3 we conclude that

$$\|v^u(t)\|_{l_2^{Nd}}\leqslant\|v_0\|_{l_2^{Nd}}+I^\alpha_{0+}\|L(x^u(t))v^u(t)\|_{l_2^{Nd}}+I^\alpha_{0+}\|u(t)\|_{l_2^{Nd}}$$
$$\leqslant C_3+C_4 I^\alpha_{0+}\|v^u(t)\|_{l_2^{Nd}}$$

for all $t\in[0,T]$ and $u\in\mathcal{U}^p_M$, where $C_3=\|v_0\|_{l_2^{Nd}}+\dfrac{KT^\alpha}{\Gamma(\alpha+1)}$, $C_4=\max\limits_{r\in[0,+\infty)}\|L(r)\|_{Nd\times Nd}$. Using a fractional version of the Gronwall inequality (cf. [39, Corollary 2]) we obtain

$$\|v^u(t)\|_{l_2^{Nd}}\leqslant C_2,\qquad t\in[0,T]\ a.e.,\quad u\in\mathcal{U}^p_M,$$

where $C_2 := C_3 E_\alpha(C_4 T^\alpha)$ (here E_α is the Mittag-Leffler function defined by $E_\alpha(w) = \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(k\alpha+1)}$). Hence, by Proposition 3, we get

$$\|x^u(t)\|_{l_2^{Nd}} \leqslant \|x_0\|_{l_2^{Nd}} + I_{0+}^1 \|v^u(t)\|_{l_2^{Nd}} \leqslant C_1, \qquad t \in [0,T] \ a.e., \quad u \in \mathcal{U}_M^p,$$

where $C_1 := \|x_0\|_{l_2^{Nd}} + TC_2$. The proof is completed. □

5. Existence of optimal solutions

In this section, our main goal is to enforce consensus in system (7) using the optimal control strategy. To do so we minimize the following cost functional

$$J(z,u) = \int_0^T \left(\sum_{i=1}^N \left\| v_i(t) - \frac{1}{N}\sum_{j=1}^N v_j(t) \right\|_{l_2^d}^2 + \gamma \sum_{i=1}^N \|u_i(t)\|_{l_2^d} \right) \mathrm{d}t, \tag{8}$$

where $z = (x,v)$ and $\gamma > 0$, subject to system (7). The cost functional consists of two parts: flocking and sparsity. Flocking part measures the distance to consensus, while sparsity part measures the norm of control function. In other words, we design an external control that enforces consensus in the system with a minimal amount of intervention.

In order to get an existence result for the optimization problem raised above, we assume that $\alpha \in \left(\frac{1}{2}, 1\right]$.

Let us note that, due to uniqueness of a solution to (7) we can equivalently consider the reduced cost functional $\widetilde{J} : \mathcal{U}_M^2 \to \mathbb{R}^+$ given by

$$\widetilde{J}(u) := J(z^u, u).$$

Our aim is to find $\hat{u} \in \mathcal{U}_M^2$ satisfying condition

$$\widetilde{J}(\hat{u}) \leqslant \widetilde{J}(u), \qquad u \in \mathcal{U}_M^2.$$

Then $\hat{u}$ is called an optimal control for problem described by equations (7) and (8), $z^{\hat{u}} \in \mathbb{E}_{0+,z_0}^{\alpha,2}$ is called the optimal state associated with $\hat{u}$, and the pair $(z^{\hat{u}}, \hat{u})$ is called an optimal solution to (7) and (8).

We start with two preparatory lemmas.

Lemma 4 *If all assumptions of Theorem 1 are satisfied, $u^0 \in \mathcal{U}_M^2$ and $(u^l)_{l\in\mathbb{N}} \subset \mathcal{U}_M^2$ is a sequence of controls such that*

$$u^l \underset{l\to\infty}{\rightharpoonup} u^0 \quad \textit{weakly in} \ \ L^2([0,T],\mathbb{R}^{Nd}),$$

then the sequence $(z^l)_{l\in\mathbb{N}} := (x^l, v^l)_{l\in\mathbb{N}} \subset \mathbb{E}^{\alpha,2}_{0+,z_0}$ of corresponding solutions of (7) is convergent strongly in $L^2([0,T],\mathbb{R}^{Nd}) \times L^2([0,T],\mathbb{R}^{Nd})$ to a solution $z^0 := (x^0, v^0)$, corresponding to u^0.

Proof. Let $u^0 \in \mathcal{U}_M^2$ and $(u^l)_{l\in\mathbb{N}} \subset \mathcal{U}_M^2$ be a sequence of controls such that

$$u^l \underset{l\to\infty}{\rightharpoonup} u^0 \quad \text{weakly in} \ \ L^2([0,T],\mathbb{R}^{Nd}).$$

Consider the sequence $(z^l)_{l\in\mathbb{N}} := (x^l, v^l)_{l\in\mathbb{N}} \subset \mathbb{E}^{\alpha,2}_{0+,z_0}$ of corresponding solutions of (7). Then, using Lemma 3, we assert that

$$\begin{aligned}\|x^l\|_{AC^2} &= \left(\|x_0\|^2_{l_2^{Nd}} + \int_0^T \|\dot{x}^l(t)\|^2_{l_2^{Nd}}\,\mathrm{d}t\right)^{\frac12} = \left(\|x_0\|^2_{l_2^{Nd}} + \int_0^T \|v^l(t)\|^2_{l_2^{Nd}}\,\mathrm{d}t\right)^{\frac12} \\ &\leqslant \sqrt{\|x_0\|^2_{l_2^{Nd}} + C_2^2 T}, \quad l \in \mathbb{N}\end{aligned}$$

and

$$\begin{aligned}\|v^l\|_{{}_CAC^{\alpha,2}_{0+}} &= \left(\|v_0\|^2_{l_2^{Nd}} + \int_0^T \|({}^C D^\alpha_{0+} v^l)(t)\|^2_{l_2^{Nd}}\,\mathrm{d}t\right)^{\frac12} \\ &= \left(\|v_0\|^2_{l_2^{Nd}} + \int_0^T \|u^l(t) - L(x^l(t))v^l(t)\|^2_{l_2^{Nd}}\,\mathrm{d}t\right)^{\frac12} \\ &\leqslant \sqrt{\|v_0\|^2_{l_2^{Nd}} + 2T\left(K^2 + \left(C_2 \max_{r\in[0,+\infty)} \|L(r)\|_{Nd\times Nd}\right)^2\right)}, \quad l \in \mathbb{N}.\end{aligned}$$

This means that the sequence of norm $\left(\|z^l\|_{\mathbb{E}^{\alpha,2}_{0+}}\right)_{l\in\mathbb{N}}$ is bounded on $\mathbb{E}^{\alpha,2}_{0+}$. Consequently, since $\mathbb{E}^{\alpha,2}_{0+}$ is reflexive (as a Hilbert space), we conclude that there exist a subsequence $(z^{l_s})_{s\in\mathbb{N}} := (x^{l_s}, v^{l_s})_{s\in\mathbb{N}} \subset \mathbb{E}^{\alpha,2}_{0+,z_0}$ and a function $z^0 = (x^0, v^0) \in \mathbb{E}^{\alpha,2}_{0+}$ such that

$$z^{l_s} \underset{s\to\infty}{\rightharpoonup} z^0 \quad \text{weakly in } \mathbb{E}^{\alpha,2}_{0+}.$$

Then,

$$x^{l_s} \underset{s\to\infty}{\rightharpoonup} x^0 \quad \text{weakly in } AC^2 \qquad \text{and} \qquad v^{l_s} \underset{s\to\infty}{\rightharpoonup} v^0 \quad \text{weakly in } {}_CAC^{\alpha,2}_{0+}.$$

Lemma 1 implies

$$\begin{aligned} x^{l_s} &\xrightarrow{s\to\infty} x^0 \quad \text{strongly in } L^2([0,T],\mathbb{R}^{Nd}) \qquad \text{and} \\ v^{l_s} &\xrightarrow{s\to\infty} v^0 \quad \text{strongly in } L^2([0,T],\mathbb{R}^{Nd}). \end{aligned}$$

Supposing contrary and repeating the above argumentation we assert that

$$\begin{aligned} x^{l} &\xrightarrow{l\to\infty} x^0 \quad \text{strongly in } L^2([0,T],\mathbb{R}^{Nd}) \qquad \text{and} \\ v^{l} &\xrightarrow{l\to\infty} v^0 \quad \text{strongly in } L^2([0,T],\mathbb{R}^{Nd}). \end{aligned} \tag{9}$$

Now, we shall prove that $z^0 = (x^0, v^0)$ is a solution to (7) corresponding to u^0. Indeed, first let us note that from Lemma 2 we obtain

$$\begin{aligned} &\|L(x^0(t))v^0(t) - L(x^l(t))v^l(t)\|^2_{l_2^{Nd}} \\ &\quad \leqslant 2\Big(\Big\|L(x^l(t))v^0(t) - L(x^l(t))v^l(t)\Big\|^2_{l_2^{Nd}} + \Big\|L(x^0(t))v^0(t) - L(x^l(t))v^0(t)\Big\|^2_{l_2^{Nd}}\Big) \\ &\quad \leqslant 2\Bigg(\Big(\max_{r\in[0,+\infty)} \|L(r)\|_{Nd\times Nd}\Big)^2 \|v^0(t) - v^l(t)\|^2_{l_2^{Nd}} \\ &\qquad + \Big(\max_{t\in[0,T]} \|v^0(t)\|_{l_2^{Nd}}\Big)^2 \|L(x^0(t)) - L(x^l(t))\|^2_{Nd\times Nd}\Bigg) \\ &\quad \leqslant 2\Bigg(\Big(\max_{r\in[0,+\infty)} \|L(r)\|_{Nd\times Nd}\Big)^2 \|v^0(t) - v^l(t)\|^2_{l_2^{Nd}} \\ &\qquad + \Big(C\max_{t\in[0,T]} \|v^0(t)\|_{l_2^{Nd}}\Big)^2 \|x^0(t) - x^l(t)\|^2_{l_2^{Nd}}\Bigg), \qquad t \in [0,T]. \end{aligned}$$

Consequently,

$$L(x^l)v^l \xrightarrow[l\to\infty]{} L(x^0)v^0 \quad \text{strongly in } L^2([0,T],\mathbb{R}^{Nd}).$$

Moreover, Lemma 1 implies

$$\begin{gathered}\dot{x}^l \underset{l\to\infty}{\rightharpoonup} \dot{x}^0 \quad \text{weakly in } L^2([0,T],\mathbb{R}^{Nd}) \quad \text{and}\\ {}^C D^\alpha_{0+} v^l \underset{l\to\infty}{\rightharpoonup} {}^C D^\alpha_{0+} v^0 \quad \text{weakly in } L^2([0,T],\mathbb{R}^{Nd}).\end{gathered}$$

Hence

$$\dot{x}^l - v^l \underset{l\to\infty}{\rightharpoonup} \dot{x}^0 - v^0 \quad \text{weakly in } L^2([0,T],\mathbb{R}^{Nd})$$

and

$${}^C D^\alpha_{0+} v^l + L(x^l)v^l - u^l \underset{l\to\infty}{\rightharpoonup} {}^C D^\alpha_{0+} v^0 + L(x^0)v^0 - u^0 \quad \text{weakly in } L^2([0,T],\mathbb{R}^{Nd}).$$

On the other hand, since (x^l, v^l) is a solution to (7), we have

$$\dot{x}^l(t) - v^l(t) = 0 \quad \text{and} \quad {}^C D^\alpha_{0+} v^l(t) + L(x^l(t))v^l(t) - u^l(t) = 0, \quad t \in [0,T] \ a.e.,$$

so uniqueness of a weak limit implies

$$\dot{x}^0(t) - v^0(t) = 0 \quad \text{and} \quad {}^C D^\alpha_{0+} v^0(t) + L(x^0(t))v^0(t) - u^0(t) = 0, \quad t \in [0,T] \ a.e.$$

Furthermore, from (9) it follows that there exist subsequences $(x^{l_s})_{s\in\mathbb{N}}$ and $(v^{l_s})_{s\in\mathbb{N}}$ such that

$$x^{l_s}(t) \xrightarrow{s\to\infty} x^0(t) \quad \text{and} \quad v^{l_s}(t) \xrightarrow{s\to\infty} v^0(t), \quad t \in [0,T] \ a.e.$$

Since $(x^{l_s})_{s\in\mathbb{N}}$ and $(v^{l_s})_{s\in\mathbb{N}}$ are continuous on $[0,T]$, if follows that the above convergences hold for all $t \in [0,T]$. In particular,

$$x^{l_s}(0) \xrightarrow{s\to\infty} x^0(0) \qquad \text{and} \qquad v^{l_s}(0) \xrightarrow{s\to\infty} v^0(0).$$

On the other hand

$$x^{l_s}(0) = x_0 \qquad \text{and} \qquad v^{l_s}(0) = v_0,$$

so $x^0(0) = x_0$ and $v^0(0) = v_0$. This means that $z^0 = (x^0, v^0)$ is a solution to (7) corresponding to u^0. The proof is completed. □

Lemma 5 *If all assumptions of Theorem 1 are satisfied, $u^0 \in \mathcal{U}^2_M$ and $(u^l)_{l\in\mathbb{N}} \subset \mathcal{U}^2_M$ is a sequence of controls such that*

$$u^l \underset{l\to\infty}{\rightharpoonup} u^0 \quad \textit{weakly in } L^2([0,T],\mathbb{R}^{Nd}),$$

then

$$\widetilde{J}(u^0) \leqslant \liminf_{l\to\infty} \widetilde{J}(u^l).$$

In other words, the functional $\widetilde{J}$ is weakly sequentially lower semicontinuous on $\mathcal{U}^2_M$.

Proof. We have

$$\widetilde{J}(u) = \int_0^T \sum_{i=1}^N \left\| v_i^u(t) - \frac{1}{N}\sum_{j=1}^N v_j^u(t) \right\|_{l_2^d}^2 \mathrm{d}t + \gamma \int_0^T \sum_{i=1}^N \|u_i(t)\|_{l_2^d} \mathrm{d}t = \widetilde{J}_1(u) + \widetilde{J}_2(u).$$

It is clear that $\widetilde{J}_2(u)$ is weakly lower semicontinuous on $\mathcal{U}_M^2$ as a convex and continuous functional. Now, let $u^0 \in \mathcal{U}_M^2$, $(u^l)_{l\in\mathbb{N}} \subset \mathcal{U}_M^2$ be a sequence of controls such that

$$u^l \underset{l\to\infty}{\rightharpoonup} u^0 \quad \text{weakly in } L^2([0,T],\mathbb{R}^{Nd})$$

and $(z^l)_{l\in\mathbb{N}} = (x^l, v^l)_{l\in\mathbb{N}} \subset \mathbb{E}^{\alpha,2}_{0+,z_0}$ – the sequence of corresponding solutions to (7). In particular, Lemma 4 implies that there exists a solution $z^0 = (x^0, v^0)$ to (7) corresponding to u^0 such that

$$v^l \underset{l\to\infty}{\longrightarrow} v^0 \quad \text{strongly in } L^2([0,T],\mathbb{R}^{Nd}).$$

Consequently, there exists a subsequence $(v^{l_s})_{s\in\mathbb{N}}$ such that

$$v^{l_s}(t) \underset{s\to\infty}{\longrightarrow} v^0(t), \quad t \in [0,T] \ a.e.$$

Since $(v^{l_s})_{s\in\mathbb{N}}$ is continuous on $[0,T]$, we conclude that the above convergence holds for all $t \in [0,T]$. Using Fatou's Lemma we assert that

$$\widetilde{J}_1(u^0) \leqslant \liminf_{l\to\infty} \widetilde{J}_1(u^{l_s}).$$

Supposing contrary and repeating the above argumentation we assert that

$$\widetilde{J}_1(u^0) \leqslant \liminf_{l\to\infty} \widetilde{J}_1(u^l).$$

Finally, we obtain

$$\liminf_{l\to\infty} \widetilde{J}(u^l) = \liminf_{l\to\infty} \left(\widetilde{J}_1(u^l) + \widetilde{J}_2(u^l)\right) \geqslant \liminf_{l\to\infty} \widetilde{J}_1(u^l) + \liminf_{l\to\infty} \widetilde{J}_2(u^l) \geqslant \widetilde{J}(u^0).$$

The proof is completed. □

Now, we shall formulate and prove a theorem on the existence of optimal solutions to problem described by equations (7) and (8).

Theorem 2 *Let $M \subset \mathbb{R}^m$ be a bounded, closed and convex set. If all assumptions of Theorem 1 are satisfied, then there exists an optimal solution $(z^{\hat{u}}, \hat{u}) \in \mathbb{E}^{\alpha,2}_{0+,z_0} \times \mathcal{U}_M^2$ to problem (7) and (8).*

Proof. Let $(u^l)_{l\in\mathbb{N}} \subset \mathcal{U}_M^2$ be a minimizing sequence of $\widetilde{J}$. Since $\mathcal{U}_M^2$ is a weakly sequentially compact set (as a convex, bounded and closed subset of a Hilbert space $L^2([0,T],\mathbb{R}^{Nd})$), we have that there exist $u^0 \in \mathcal{U}_M^2$ and a subsequence $(u^{l_s})_{s\in\mathbb{N}}$ such that

$$u^{l_s} \underset{s\to\infty}{\rightharpoonup} u^0 \quad \text{weakly in } \ L^2([0,T],\mathbb{R}^{Nd}).$$

From Lemma 5 it follows that

$$\widetilde{J}(u^0) \leqslant \liminf_{s\to\infty} \widetilde{J}(u^{l_s}) = \lim_{s\to\infty} \widetilde{J}(u^{l_s}) = \inf_{u\in\mathcal{U}_M} \widetilde{J}(u),$$

so u^0 is an optimal control for problem (7) and (8). Then, the pair (z^0, u^0), where z^0 is a solution to (7) corresponding to u^0, is an optimal solution to (7) and (8). The proof is completed. □

6. Illustrative examples

In this section, we present some numerical simulations that illustrate our theoretical results in one dimensional case, i.e., $d = 1$. Let us consider a fractional controlled Cucker–Smale system (7) with $N = 4$ agents and $\alpha = 0.6$, i.e.,

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ {}^C D_{0^+}^{0.6} v_i(t) = \dfrac{1}{4}\displaystyle\sum_{j=1}^{4} \eta\left((x_j(t)-x_i(t))^2\right)(v_j(t)-v_i(t)) \\ \qquad\qquad + u_i(t), \quad t\in[0,T] \ a.e., \quad i = 1,\ldots,4 \\ (x_1(0),v_1(0)) = (1,1), \quad (x_2(0),v_2(0)) = (2,2), \\ (x_3(0),v_3(0)) = (6,3), \quad (x_4(0),v_4(0)) = (10,4), \\ u(t)\in M\subset\mathbb{R}^4, \quad t\in[0,T], \end{cases} \tag{10}$$

where $M = \left\{z\in\mathbb{R}^4 : \ \|z\|_{l_2^4} \leqslant K\right\}$. Our goal is to enforce consensus in system (10) using the optimal control approach with the following cost functional:

$$J(z,u) = \int_0^T \left(\sum_{i=1}^{4}\left(v_i(t) - \frac{1}{4}\sum_{j=1}^{4} v_j(t)\right)^2 + \sum_{i=1}^{4}|u_i(t)|\right) \mathrm{d}t, \tag{11}$$

where $z = (x,v)$. In what follows we consider problem (10)–(11) with two different types of interaction potential η.

Example Suppose that rate of communication η in system (10) is given by

$$\eta\left((x_j(t) - x_i(t))^2\right) = \frac{1}{1 + (x_j(t) - x_i(t))^2}, \quad i, j = 1, \ldots, 4. \tag{12}$$

Note that, in this case, assumption (A) from Lemma 2 is satisfied and, by Theorem 1, controlled system (10) has a unique solution $z = (x, v) \in \mathbb{E}^{0.6,p}_{0+,z_0}$, corresponding to any control $u \in \mathcal{U}^p_M$, where $p > \frac{5}{3}$ and $z_0 = ((1, 2, 6, 10), (1, 2, 3, 4))$. Moreover, by Theorem 2, there exists an optimal solution $(z^{\hat{u}}, \hat{u}) \in \mathbb{E}^{0.6,2}_{0+,z_0} \times \mathcal{U}^2_M$ to problem (10)–(11). Figures 1–4 show the trajectories of the main state, the consensus parameter and the control function with different values of the bound K. Clearly, convergence of all agents to the consensus configuration depends on the amount of additional energy u_i.

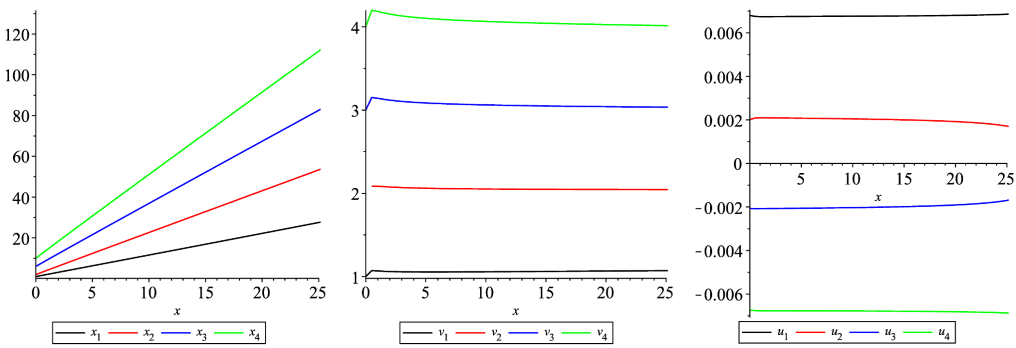

Figure 1: Trajectories of the main state x (left), the consensus prameter v (center) and the control u (right) for η given by (12) and $K = 0.01$

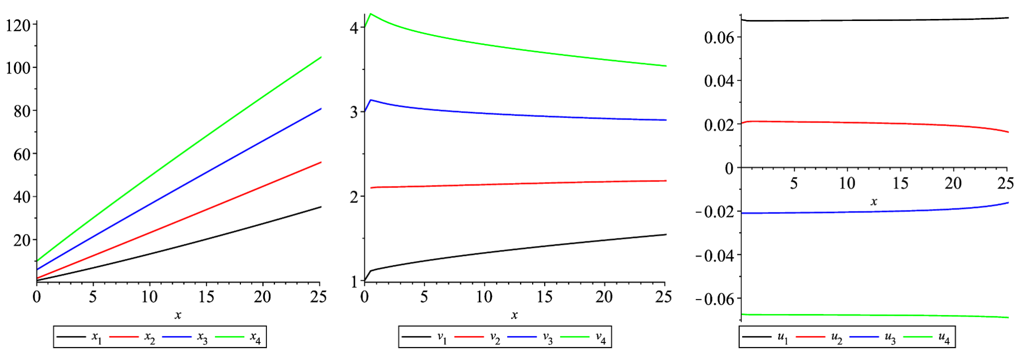

Figure 2: Trajectories of the main state x (left), the consensus prameter v (center) and the control u (right) for η given by (12) and $K = 0.1$

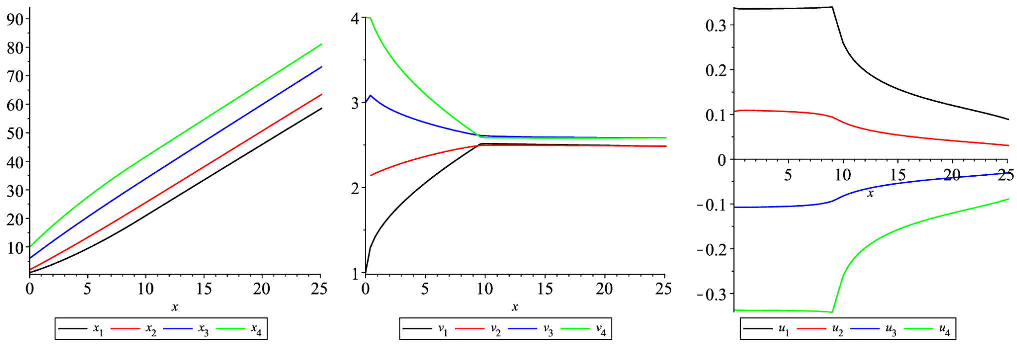

Figure 3: Trajectories of the main state x (left), the consensus prameter v (center) and the control u (right) for η given by (12) and $K = 0.5$

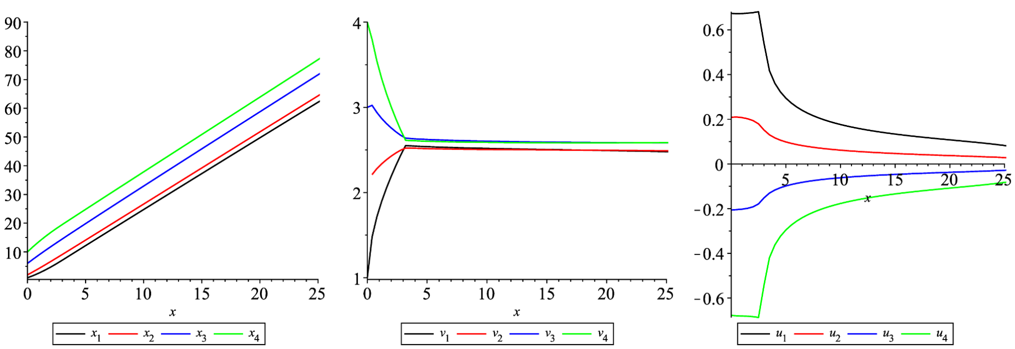

Figure 4: Trajectories of the main state x (left), the consensus prameter v (center) and the control u (right) for η given by (12) and $K = 1$

Example 2. Now, let us consider the following rate of communication:

$$\eta\left((x_j(t) - x_i(t))^2\right) = \exp\left(-(x_j(t) - x_i(t))^2\right), \qquad i, j = 1, \ldots, 4. \tag{13}$$

Similarly as in Example 1, by Theorem 1, for $p > \dfrac{5}{3}$ and $z_0 = ((1, 2, 6, 10), (1, 2, 3, 4))$, the control system (10) has a unique solution $z = (x, v) \in \mathbb{E}^{0.6,p}_{0+,z_0}$, corresponding to any control $u \in \mathcal{U}^p_M$ and, by Theorem 2, there exists an optimal solution $\left(z^{\hat{u}}, \hat{u}\right) \in \mathbb{E}^{0.6,2}_{0+,z_0} \times \mathcal{U}^2_M$ to problem (10)–(11). In Figures 5–8, we can observe that trajectories of the main state, the consensus parameter and the control depend on the bound of the control K. It is visible that dependently on K the resulting optimal control is able to steer the system to achieve consensus.

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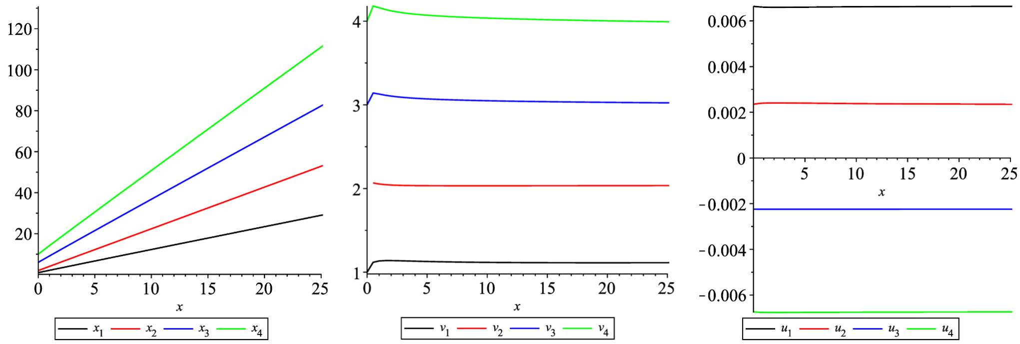

Figure 5: Trajectories of the main state x (left), the consensus prameter v (center) and the control u (right) for η given by (13) and $K = 0.01$

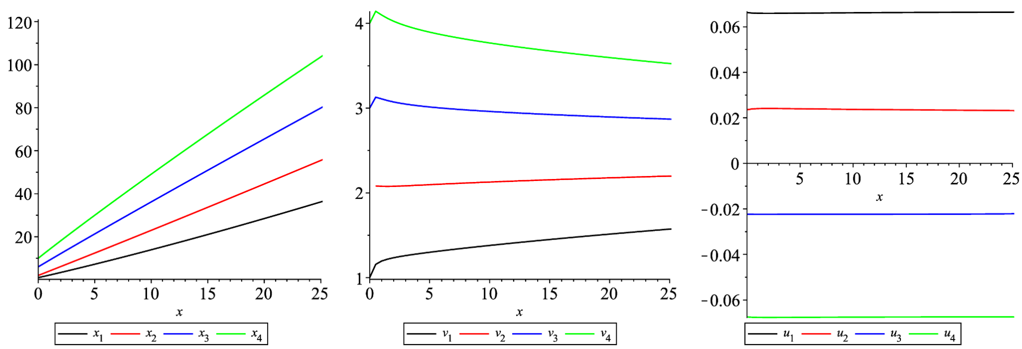

Figure 6: Trajectories of the main state x (left), the consensus prameter v (center) and the control u (right) for η given by (13) and $K = 0.1$

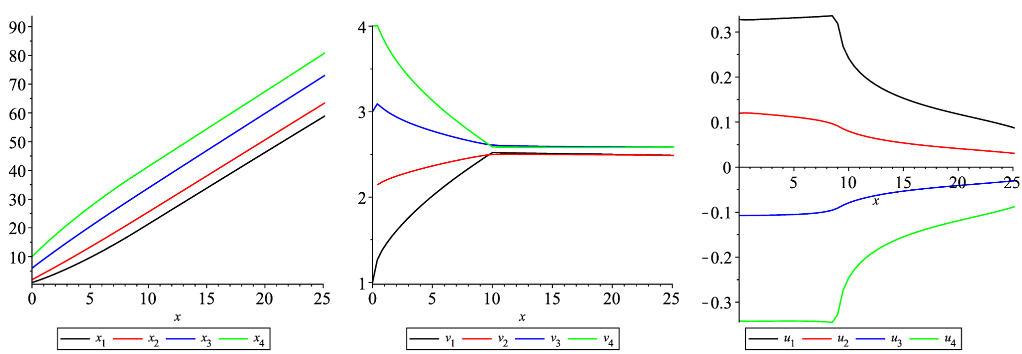

Figure 7: Trajectories of the main state x (left), the consensus prameter v (center) and the control u (right) for η given by (13) and $K = 0.5$

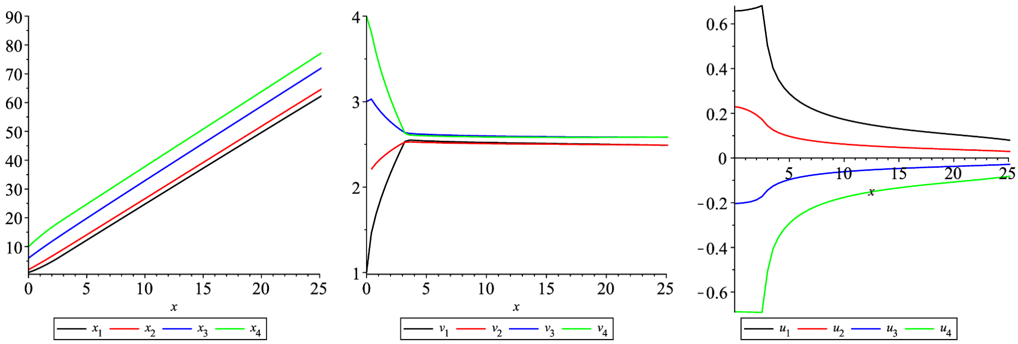

Figure 8: Trajectories of the main state x (left), the consensus prameter v (center) and the control u (right) for η given by (13) and $K = 1$

7. Conclusions

In this paper, we studied the nonlinear Cucker–Smale optimal control problem, where in the velocity equation the usual integer order derivative was replaced by the Caputo fractional derivative. As a consequence, our model described a group behavior phenomena taking into account the velocities of agents in all previous times. We established sufficient conditions that guarantee the existence of optimal solutions to the considered problem. In numerical examples, the aforementioned criteria allowed us to choose appropriate communication rate functions for the Cucker–Smale optimal control problem.

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