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A GENERALIZED WHITE NOISE SPACE APPROACH TO STOCHASTIC INTEGRATION FOR A CLASS OF GAUSSIAN STATIONARY INCREMENT PROCESSES

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Abstract. Given a Gaussian stationary increment processes, we show that a Skorokhod-Hitsuda stochastic integral with respect to this process, which obeys the Wick-Itô calculus rules, can be naturally defined using ideas taken from Hida's white noise space theory. We use the Bochner-Minlos theorem to associate a probability space to the process, and define the counterpart of the S-transform in this space. We then use this transform to define the stochastic integral and prove an associated Itô formula.

Keywords: stochastic integral, white noise space, fractional Brownian motion.

Mathematics Subject Classification: 60H40, 60H05.

1. INTRODUCTION

In this paper we develop a stochastic calculus for the family of centered Gaussian processes with covariance function of the form

$$K_m(t,s) = \int_{\mathbb{R}} \frac{e^{i\xi t}-1}{\xi}\frac{e^{-i\xi s}-1}{\xi} m(\xi)d\xi,$$

where m is a positive measurable even function subject to $\int_{\mathbb{R}} \frac{m(\xi)}{\xi^2+1}d\xi < \infty$.

Note that $K_m(t,s)$ can also be written as

$$K_m(t,s) = r(t) + r(s) - r(t-s),$$

where

$$r(t) = \int_{\mathbb{R}} \frac{1-\cos(t\xi)}{\xi^2} m(\xi)d\xi.$$

This family includes in particular the fractional Brownian motion, which corresponds (up to a multiplicative constant) to $m(\xi) = |\xi|^{1-2H}$, where $H \in (0,1)$. We note that complex-valued functions of the form

$$K(t,s) = r(t) + \overline{r(s)} - r(t-s) - r(0),$$

where r is a continuous function, have been studied in particular by von Neumann, Schoenberg and Krein. Such a function is positive definite if and only if r can be written in the form

$$r(t) = r_0 + i\gamma t + \int_{\mathbb{R}} \left\{ e^{i\xi t} - 1 - \frac{i\xi t}{\xi^2+1} \right\} \frac{d\sigma(\xi)}{\xi^2},$$

where σ is an increasing right continuous function subject to $\int_{\mathbb{R}} \frac{d\sigma(\xi)}{\xi^2+1} < \infty$. See [14,16], and see [1] for more information on these kernels.

As in [1], our starting point is the (in general unbounded) operator T_m on the Lebesgue space of complex-valued functions $\mathbf{L}_2(\mathbb{R})$ defined by

$$\widehat{T_m f}(\xi) = \sqrt{m}(\xi)\widehat{f}(\xi), \tag{1.1}$$

with domain

$$\mathcal{D}(T_m) = \left\{ f \in \mathbf{L}_2(\mathbb{R})\,;\, \int_{\mathbb{R}} m(\xi)|\widehat{f}(\xi)|^2 d\xi < \infty \right\},$$

where $\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi t} f(t)dt$ denotes the Fourier transform. Clearly, the Schwartz space $\mathscr{S}$ of smooth rapidly decreasing functions belong to the domain of T_m. The indicator functions

$$\mathbf{1}_t = \begin{cases} \mathbf{1}_{[0,t]}, & t \geq 0, \\ \mathbf{1}_{[t,0]}, & t \leq 0 \end{cases}$$

also belong to $\mathcal{D}(T_m)$. In [1], and with some restrictions on m, a centered Gaussian process B_m with covariance function $K_m(t,s) = (T_m\mathbf{1}_t, T_m\mathbf{1}_s)_{\mathbf{L}_2(\mathbb{R})}$ was constructed in Hida's white noise space. In the present paper we chose a different path. We build from T_m the characteristic functional

$$C_m(s) = e^{-\frac{\|T_m s\|^2_{\mathbf{L}_2(\mathbb{R})}}{2}}. \tag{1.2}$$

It has been proved in [3] that C_m is continuous from $\mathscr{S}$ into $\mathbb{R}$. Restricting C_m to real-valued functions and using the Bochner-Minlos theorem, we obtain an analog of the white noise space in which the process B_m is built in a natural way. Stochastic calculus with respect to this process is then developed using an $\mathcal{S}$-transform approach.

The S-transform of an element $X(\omega)$ of the white noise space is defined by

$$\mathcal{S}X(s) = \mathbb{E}\left[X(\cdot)e^{\langle \cdot, s\rangle} \right] e^{-\frac{1}{2}\|s\|_{\mathbf{L}_2(\mathbb{R})}}.$$

An S-transform approach to stochastic integration in the white noise setting can be found in [10], [15, Section 13.3] and in [10]. The main idea is to define the Hitsuida-Skorohod integral of a stochastic process $X(t)$ with respect to the Brownian motion $B(t)$ over a Borel set E by

$$\int_E X(t)\delta B(t) \triangleq \mathcal{S}^{-1}\left(\int_E \mathcal{S}\left(X(t)\right)(s)s(t)dt\right).$$

Namely, the integral of $X(t)$ over the set E is the unique stochastic process $\Phi(t)$ such that for any $t \geq 0$ and $s \in \mathscr{S}$,

$$\left(\mathcal{S}\Phi(t)\right)(s) = \int_E \mathcal{S}\left(X(t)\right)(s)s(t)dt.$$

Since $s(t) = \frac{d}{dt}\left(s, \mathbf{1}_t\right)_{\mathbf{L}_2(\mathbb{R})}$, it suggests to extend the last definition of the integral by replacing the inner product in $\mathbf{L}_2(\mathbb{R})$ by a different one. In the present work, this inner product is determined by the spectrum of the process through the operator T_m. We note that when $m(\xi) = |\xi|^{1-2H}$, and $H \in (\frac{1}{2}, 1)$, the operator T_m reduces, up to a multiplicative constant, to the operator M_H defined in [9] and in [5]. The set L^2_ϕ presented in [8, equation (3)] is the domain of T_m and the functional C_m was used with the Bochner-Minlos theorem in [6, (3.5), p. 49]. In view of this, our work generalized the stochastic calculus for fractional Brownian motion presented in these works to the aforementioned family of Gaussian processes.

There are two main ideas in this paper. The first is the construction of a probability space in which a stationary increment process with spectral density m is naturally defined. This result, being a concrete example of Kolmogorov's extension theorem on the existence of a Gaussian process with a given spectral density, is interesting in its own right. The second main result deals with developing stochastic integration with respect to the fundamental process in this space. We take an approach based on the analog of the S-transform in our setting, and show that this stochastic integral coincides with the one already defined in [2] but in the framework of Hida's white noise space.

The paper consists of five sections besides the introduction. In Section 2 we construct an analog of Hida's white noise space using the characteristic function C_m. In Section 3 the associated fundamental process B_m is being defined and studied. The analog of the S-transform is defined and studied in Section 4. In Section 5 we define a Wick-Itô type stochastic integral with respect to B_m, and prove an associated Itô formula. In the last section we explain the relation of this integral to previous works on white noise based stochastic integrals.

2. THE M NOISE SPACE

We set $\mathscr{S}_\mathbb{R}$ to be the space of real-valued Schwartz functions, and $\Omega = \mathscr{S}'_\mathbb{R}$. We denote by $\mathcal{B}$ the associated Borel sigma algebra. Throughout this paper, we denote by $\langle \cdot, \cdot \rangle$

the duality between $\mathscr{S}'_{\mathbb{R}}$ and $\mathscr{S}_{\mathbb{R}}$, and by $(\cdot,\cdot)$ the inner product in $\mathbf{L}_2(\mathbb{R})$. In case there is no danger of confusion, the $\mathbf{L}_2(\mathbb{R})$ norm will be denoted as $\|\cdot\|$.

Theorem 2.1. *There exists a unique probability measure μ_m on $(\Omega,\mathcal{B})$ such that*

$$e^{-\frac{\|T_m s\|^2}{2}} = \int_\Omega e^{i\langle\omega,s\rangle} d\mu_m(\omega), \quad s\in\mathscr{S}_{\mathbb{R}}.$$

Proof. The function $C_m(s)$ is positive definite on $\mathscr{S}_{\mathbb{R}}$ since

$$C_m(s_1-s_2) = \exp\left\{-\frac{1}{2}\|T_m s_1\|^2\right\} \times \exp\left\{(T_m s_1, T_m s_2)\right\} \times \exp\left\{-\frac{1}{2}\|T_m s_2\|^2\right\},$$

and the middle term is positive definite since an exponent of a positive definite function is still positive definite. Moreover, the operator T_m is continuous from $\mathscr{S}$ (and hence from $\mathscr{S}_{\mathbb{R}}$) into $\mathbf{L}_2(\mathbb{R})$. This was proved in [3], and we repeat the argument for completeness. As in [3], we set $K=\int_{\mathbb{R}}\frac{m(u)}{1+u^2}du$ and $s^\sharp(u)=\overline{s(-u)}$. For $s\in\mathscr{S}$, we have

$$\begin{aligned}\|T_m s\|^2_{\mathbf{L}_2(\mathbb{R})} &= \int_{\mathbb{R}} |\widehat{s}(u)|^2 m(u)du = \int_{\mathbb{R}} |(1+u^2)\widehat{s}(u)|^2\frac{m(u)}{1+u^2}du \le \\ &\le K\left(\int_{\mathbb{R}} |s\star s^\sharp|(\xi)d\xi + \int_{\mathbb{R}} |s'\star(s^\sharp)'|(\xi)d\xi\right) \le \\ &\le K\left(\left(\int_{\mathbb{R}}|s(\xi)|d\xi\right)^2 + \left(\int_{\mathbb{R}}|s'(\xi)|d\xi\right)^2\right),\end{aligned}$$

where we have denoted convolution by $\star$. Therefore C_m is a continuous map from $\mathscr{S}_{\mathbb{R}}$ into $\mathbb{R}$, and the existence of μ_m follows from the Bochner-Minlos theorem. ☐

The triplet $(\Omega,\mathcal{B},\mu_m)$ will be used as our probability space.

Proposition 2.2. *For any $s\in\mathscr{S}_{\mathbb{R}}$ it holds that*

$$\mathbb{E}[\langle\omega,s\rangle^2] = \|T_m s\|^2. \tag{2.1}$$

Proof. We have

$$e^{-\frac{1}{2}\|T_m s\|^2} = \int_\Omega e^{i\langle\omega,s\rangle} d\mu_m(\omega). \tag{2.2}$$

Expanding both sides of (2.2) in power series for ϵs we obtain

$$\mathbb{E}\left[\langle\omega,s\rangle\right] = \int_\Omega \langle\omega,s\rangle d\mu_m(\omega) = 0. \tag{2.3}$$

and

$$\mathbb{E}\left[\langle \omega, s\rangle^2\right] = \int_\Omega \langle \omega, s\rangle^2 d\mu_m(\omega) = \|T_m s\|^2. \tag{2.4}$$

□

We now want to extend the isometry (2.1) to any function in the domain of T_m. This extension involves two separate steps: first, an approximation procedure, and next complexification. For the approximation step we introduce an inner product defined by the operator T_m. For f and g in $\mathcal{D}(T_m)$ we define the inner product

$$(f, g)_m \triangleq \int_\mathbb{R} \widehat{f}\widehat{g}^* m d\xi.$$

Note that $\mathcal{D}(T_m)$ is consist of those functions f in $\mathbf{L}_2(\mathbb{R})$ that satisfy

$$\|f\|_m^2 \triangleq (f, f)_m < \infty.$$

We define the space $\mathbf{L}\mathscr{S}_m$ and $\mathbf{L}_m$ to be the closure of $\mathscr{S}$ and $\mathcal{D}(T_m)$ in the norm $\|\cdot\|_m$, respectively.

Proposition 2.3. *We have*

$$\mathbf{L}_m = \mathbf{L}\mathscr{S}_m.$$

Proof. Let $f \in \mathcal{D}_m$ be orthogonal to any $s \in \mathscr{S}$ in the norm $\|\cdot\|_m$, i.e.

$$0 = (s, f)_m = \int_\mathbb{R} \widehat{s}\widehat{f}^* m d\xi, \quad \forall s \in \mathscr{S}.$$

It follows that $\widehat{f}^* m = 0$ almost everywhere since it defines the zero distribution on $\mathscr{S}$. But that also means

$$\int_\mathbb{R} \widehat{f}\widehat{f}^* m d\xi = 0,$$

so f is zero in $\mathbf{L}_m$. □

Theorem 2.4. *The isometry* (2.1) *extends to any* $f \in \mathbf{L}_2(\mathbb{R})$*, where* f *is real-valued and in the domain of* T_m*.*

Proof. We first note that, for f in the domain of T_m, we have

$$\overline{T_m f} = T_m \overline{f}. \tag{2.5}$$

Indeed, since m is even and real, we have

$$\widehat{T_m \overline{f}} = \sqrt{m}(\widehat{f})^\sharp = (\sqrt{m}\widehat{f})^\sharp = \left(\widehat{T_m f}\right)^\sharp = \widehat{\overline{T_m f}}.$$

Let now f be real-valued and in $\mathcal{D}(T_m) \subset \mathbf{L}_m$. It follows from Proposition 2.3 that there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of elements in $\mathscr{S}$ such that

$$\lim_{n\to\infty} \|s_n - f\|_m = 0. \tag{2.6}$$

In view of (2.5), and since f is real-valued, we have

$$\lim_{n\to\infty} \|\overline{s_n} - f\|_m = \lim_{n\to\infty} \|\overline{T_m s_n} - T_m f\|_{\mathbf{L_2}(\mathbb{R})} = \lim_{n\to\infty} \|T_m \overline{s_n} - T_m f\|_{\mathbf{L_2}(\mathbb{R})} = 0. \tag{2.7}$$

Together with (2.6) this last equation leads to

$$\lim_{n\to\infty} \|T_m(\text{Re } s_n) - T_m f\|_{\mathbf{L_2}(\mathbb{R})} = 0. \tag{2.8}$$

In particular $(T_m(\text{Re } s_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbf{L_2}(\mathbb{R})$. By (2.1), $(\langle\omega, \text{Re } s_n\rangle)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{W}_m$. We denote by $\langle\omega, f\rangle$ its limit. It is easily checked that the limit does not depend on the given sequence for which (2.6) holds. □

We denote by $\mathcal{D}_\mathbb{R}(T_m)$ the elements in the domain of T_m which are real-valued.

Let $f, g \in \mathcal{D}_\mathbb{R}(T_m)$. The polarization identity applied to

$$\mathbb{E}[\langle\omega, f\rangle^2] = \|T_m f\|^2, \quad f \in \mathcal{D}_\mathbb{R}(T_m), \tag{2.9}$$

leads to

$$\mathbb{E}\left[\langle\omega, f\rangle\langle\omega, g\rangle\right] = \text{Re } \left(T_m f, T_m g\right).$$

In view of (2.5), $T_m f$ and $T_m g$ are real and so we obtain the following result.

Proposition 2.5. *Let $f, g \in \mathcal{D}_\mathbb{R}(T_m)$. It holds that*

$$\mathbb{E}\left[\langle\omega, f\rangle\langle\omega, g\rangle\right] = \left(T_m f, T_m g\right). \tag{2.10}$$

Proposition 2.6. *$\{\langle\omega, f\rangle,\ f \in \mathcal{D}_\mathbb{R}(T_m)\}$ is a Gaussian process in the sense that for any $f_1, \ldots, f_n \in \mathcal{D}_\mathbb{R}(T_m)$ and $a_1, \ldots, a_n \in \mathbb{R}$, the random variable $\sum_{i=1}^n a_i\langle\omega, f_i\rangle$ has a normal distribution.*

Proof. By (2.2), for $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} \mathbb{E}[e^{i\lambda\sum_{i=1}^n a_i\langle\omega,f_i\rangle}] &= \int_\Omega e^{i\lambda\sum_{i=1}^n a_i\langle\omega,f_i\rangle} d\mu_m(\omega) = \\ &= \int_\Omega e^{i\langle\omega,\lambda\sum_{i=1}^n a_i f_i\rangle} d\mu_m(\omega) = \\ &= e^{-\frac{1}{2}\lambda^2\|\sum_{i=1}^n a_i T_m f_i\|^2}. \end{aligned} \tag{2.11}$$

□

In particular, we have that for any $\xi_1, \ldots, \xi_n \in \mathcal{D}_\mathbb{R}(T_m)$ such that $T_m\xi_1, \ldots, T_m\xi_n$ are orthonormal in $\mathbf{L}_2(\mathbb{R})$ and for any $\phi \in \mathbf{L_2}(\mathbb{R}^\mathbf{n})$

$$\mathbb{E}\left[\phi\left(\langle\omega, \xi_1\rangle, \ldots, \langle\omega, \xi_1\rangle\right)\right] = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \phi(x_1, \ldots, x_n) \prod_{i=1}^n e^{-\frac{1}{2}{x_i}^2} dx_1 \cdot \ldots \cdot dx_n. \tag{2.12}$$

Definition 2.7. We set $\mathcal{G}$ to be the σ-field generated by the Gaussian elements

$$\{\langle\omega, f\rangle, f \in \mathcal{D}_{\mathbb{R}}(T_m)\},$$

and denote

$$\mathcal{W}_m \triangleq \mathbf{L_2}(\Omega, \mathcal{G}, \mu_m).$$

Note that $\mathcal{G}$ may be significantly smaller than $\mathcal{B}$, the Borel σ-field of Ω. For example, if $m \equiv 0$, then T_m is the zero operator and $\mathcal{G} = \{\emptyset, \Omega, 0, \Omega\backslash\{0\}\}$. We will see in the following section that the time derivative, in the sense of distributions, of the fundamental stochastic process B_m in the space $\mathcal{W}_m$ has spectral density $m(\xi)$. It is therefore seems appropriate to refer $\mathcal{W}_m$ as the *m-noise space*. In the case $m(\xi) \equiv 1$, T_m is the identity over $\mathbf{L_2}(\mathbb{R})$ and μ_m is the white noise measure used for example in [10, (1.4), p. 3]. Moreover, by Theorem 1.9 on p. 7 therein, $\mathcal{G}$ equals the Borel sigma algebra and so the 1-noise space coincides with Hida's white noise space.

Remark 2.8. For two spectral functions m_1 and m_2, a somewhat obvious question is whether μ_{m_1} and μ_{m_2} are equivalent or singular with respect to each other (recall that any two Gaussian measures on the same locally convex space are either equivalent or mutually singular [18, Theorem 2.7.2]). Although this question is irrelevant to our approach, we point out that a sufficient simple condition for equivalence is if T_{m_1} and T_{m_2} are unitary equivalent.

3. THE PROCESS B_M

We now define our fundamental stationary increment process $B_m : \mathbb{R} \longrightarrow \mathcal{W}_m$ via

$$B_m(t) \triangleq B_m(t, \omega) \triangleq \langle\omega, \mathbf{1}_t\rangle.$$

This process plays the role of the Brownian motion for the Itô formula in the space $\mathcal{W}_m$. Note that this is the same definition as the Brownian motion in [12], the difference being the probability measure assigned to $(\Omega, \mathcal{B})$.

Theorem 3.1. *B_m has the following properties:*

(1) *B_m is a centered Gaussian random process.*
(2) *For $t, s \in \mathbb{R}$, the covariance of $B_m(t)$ and $B_m(s)$ is*

$$K_m(t,s) = \int_{\mathbb{R}} \frac{e^{i\xi t}-1}{\xi}\frac{e^{-i\xi s}-1}{\xi} m(\xi)d\xi = (T_m\mathbf{1}_t, T_m\mathbf{1}_s). \tag{3.1}$$

(3) *The process B_m has a continuous version under the condition*

$$\int_{\mathbb{R}} \frac{m(\xi)}{1+|\xi|}d\xi < \infty. \tag{3.2}$$

Proof. (1) follows from (2.11) and (2.3). To prove (2), we see that by (2.10) we have

$$\begin{aligned}\mathbb{E}\left[B_m(t)B_m(s)\right] &= \mathbb{E}\left[\langle\omega, \mathbf{1}_t\rangle\langle\omega, \mathbf{1}_s\rangle\right] = \\ &= \mathrm{Re}\ \left(T_m\mathbf{1}_t, T_m\mathbf{1}_s\right) = \left(T_m\mathbf{1}_t, T_m\mathbf{1}_s\right),\end{aligned}$$

since this last expression is real.

To prove (3) we use similar arguments to [3, Theorem 10.2]. For $t, s \in \mathbb{R}$,

$$\mathbb{E}\left[\left(B_m(t) - B_m(s)\right)^2\right] = \mathbb{E}\left[\langle\cdot, \mathbf{1}_{[s,t]}\rangle^2\right] = 2\int_{\mathbb{R}} \frac{1-\cos\left((t-s)\xi\right)}{\xi^2} m(\xi)d\xi,$$

where the last equality follows by vanishing imaginary part of (3.1). We now compute

$$\begin{aligned}2\int_0^1 \frac{1-\cos(t\xi)}{\xi^2} m(\xi)d\xi = 2\int_0^1 t^2 \frac{2\sin\left(\frac{t\xi}{2}\right)^2}{\xi^2 t^2} m(\xi)d\xi \leq \\ \leq C_1 t^2 \quad (C_1 > 0 \text{ independent of } t).\end{aligned}$$

Using the mean-value theorem for the function $\xi \to \cos(t\xi)$ we have

$$1 - \cos(t\xi) = t\xi\sin(t\theta_t), \quad \theta_t \in [0, \xi].$$

Thus,

$$\int_1^\infty \frac{1-\cos(t\xi)}{\xi^2} m(\xi)d\xi = t\int_1^\infty \sin(t\theta_t)\frac{m(\xi)}{\xi}d\xi \leq t\int_1^\infty \frac{m(\xi)}{\xi}d\xi \leq C_2 t,$$

where we have used (3.2) in the last move. Since $B_m(t) - B_m(s)$ is zero mean Gaussian, we obtain

$$\mathbb{E}\left[\left(B_m(t) - B_m(s)\right)^4\right] = C_3\mathbb{E}\left[\left(B_m(t) - B_m(s)\right)^2\right] \leq C_4\left(t-s\right)^2.$$

Thus B_m satisfies Kolmogorov-Čentsov test for the existence of a continuous version. □

Our next goal is to define stochastic integration with respect to the process B_m in the space $\mathcal{W}_m$. The definition of the Wiener integral with respect to B_m for $f \in \mathcal{D}(T_m)$ is straightforward in view of the Hilbert spaces isomorphism (2.4) and given by

$$\int_0^\tau f(t)dB_m(t) \triangleq \langle\omega, \mathbf{1}_\tau f\rangle. \tag{3.3}$$

Note that since

$$\int_{\mathbb{R}} m(\xi)|\widehat{f}(\xi)|^2 d\xi \leq \sup_{\xi\in\mathbb{R}}(1+\xi^2)|\widehat{f}(\xi)|^2 \int_{\mathbb{R}} \frac{m(\xi)}{1+\xi^2}d\xi,$$

a sufficient condition for a function $f \in \mathbf{L_2}(\mathbb{R})$ to be in the domain of T_m is

$$\sup_{\xi \in \mathbb{R}} (1+\xi^2)|\widehat{f}(\xi)|^2 \leq \infty.$$

This is satisfied in particular if f is differentiable with derivative in $\mathbf{L_2}(\mathbb{R})$.

Recall that in the white noise space one may define the Skorokhod-Hitsuda stochastic integral of X_t on the interval $[a, b]$ as

$$\int_a^b X_t dB(t) = \int_a^b X_t \diamond \dot{B}_m dt,$$

where $\dot{B}_m$ denotes the time derivative of the Brownian motion and $\diamond$ denotes the Wick product. The chaos decomposition of the white noise space is used in order to define the Wick product and appropriate spaces of stochastic distributions, where $\dot{B}_m$ lives. Chaos decomposition for $\mathcal{W}_m$ can be obtained by a similar procedure to the one explained in [9,13, section 3] for the fractional Brownian motion. A space of stochastic distributions that contains $\dot{B}_m$ and is closed under the Wick product can similarly be defined.

A somewhat alternative approach, which uses only the expectation and the Lebesgue integral on the real line, is achieved by using the S-transform [10]. As we shall see below, an analogue of the S-transform can be naturally defined in the space $\mathcal{W}_m$, thus allows us to introduce Skorokhod-Hitsuda integral for $\mathcal{W}_m$ valued processes.

4. THE S_M TRANSFORM

We now define the analog of the S transform in the space $\mathcal{W}_m$ and study its properties.

For $s \in \mathscr{S}_{\mathbb{R}}$ we define the analog of the Wick exponential in the space $\mathcal{W}_m$:

$$e^{\diamond\langle\omega,s\rangle} \triangleq e^{\langle\omega,s\rangle - \frac{1}{2}\|T_m s\|^2}$$

Definition 4.1. The S_m transform of $\Phi \in \mathcal{W}_m$ is defined by

$$(\mathcal{S}_m \Phi)(s) \triangleq \int_\Omega e^{\diamond\langle\omega,s\rangle} \Phi(\omega) d\mu_m(\omega) = \mathbb{E}\left[e^{\diamond\langle\omega,s\rangle}\Phi(\omega)\right], \quad s \in \mathscr{S}_{\mathbb{R}}.$$

Theorem 4.2. *Let $\Phi, \Psi \in \mathcal{W}_m$. If $(\mathcal{S}_m\Phi)(s) = (\mathcal{S}_m\Psi)(s)$ for all $s \in \mathscr{S}$, then $\Phi = \Psi$.*

Proof. We follow the same arguments as in [4, Theorem 2.2] with some small changes. By linearity of the S_m transform, it is enough to prove

$$(\forall s \in \mathscr{S}\colon\ (\mathcal{S}_m\Phi)(s) = 0) \Rightarrow \Phi = 0.$$

Let $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathscr{S}_{\mathbb{R}}$ be a countable dense set in $\mathbf{L_2}(\mathbb{R})$ and denote by $\mathcal{G}_n$ the σ-field generated by $\{\langle\omega,\xi_1\rangle, \ldots, \langle\omega,\xi_n\rangle\}$. We may choose $\{\xi_n\}_{n\in\mathbb{N}}$ such that $\{T_m\xi_n\}_{n\in\mathbb{N}}$ are orthonormal. For every $n \in \mathbb{N}$, $E\left[\Phi|\mathcal{G}_n\right] = \phi_n\left(\langle \omega, \xi_1\rangle, \dots, \langle \omega, \xi_n\rangle\right)$ for some measurable function $\phi_n : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\mathbb{E}\Phi = \underset{\mathbb{R}^n}{\int \cdots \int} \phi_n(x) e^{-\frac{1}{2}x'x} dx < \infty,$$

where x' denotes the transpose of x; see for instance [7, Proposition 2.7, p. 7]. Thus, for $t = (t_1, \dots, t_n) \in \mathbb{R}^n$, using (2.12) we obtain

$$\begin{aligned} 0 &= \int_\Omega e^{\diamond \langle \omega, \sum_{k=1}^n t_k \xi_k\rangle} \Phi(\omega) d\mu_m = \int_\Omega e^{\diamond \langle \omega, \sum_{k=1}^n t_k \xi_k\rangle} \mathbb{E}\left[\Phi|\mathcal{G}_n\right] d\mu_m(\omega) = \\ &= e^{-\frac{1}{2}\sum_{k=1}^n t_k^2 \|T_m \xi_k\|^2} \int_\Omega e^{\sum_{k=1}^n t_k \langle \omega, \xi_k\rangle} \phi_n\left(\langle \omega, \xi_1\rangle, \dots, \langle \omega, \xi_n\rangle\right) d\mu_m(\omega) = \\ &= e^{-\frac{1}{2}\sum_{k=1}^n t_k^2 \|T_m \xi_k\|^2} \frac{1}{(2\pi)^{\frac{n}{2}}} \underset{\mathbb{R}^n}{\int \cdots \int} e^{\sum_{k=1}^n t_k x_k} \phi_n\left(x_1, \dots, x_n\right) e^{-\frac{1}{2}\sum_{k=1}^n x_k^2} dx_1 \dots dx_n = \\ &= \underset{\mathbb{R}^n}{\int \cdots \int} \phi_n(x)\, e^{-\frac{1}{2}(x-t)'(x-t)} dx. \end{aligned}$$

Since the last expression is a convolution integral of ϕ_n with a positive eigenvector of the Fourier transform, by properties of the Fourier transform we get that $\phi_n = 0$ for all $n \in \mathbb{N}$. Since $\bigcup_{n \in \mathbb{N}} \mathcal{G}_n = \mathcal{G}$, we have $\Phi = 0$. □

Note that Theorem 4.2 also proves that the set of linear combinations of random variables of the form

$$e^{\langle \omega, f\rangle}, \quad f \in \mathcal{D}_\mathbb{R}\left(T_m\right),$$

is a dense subset in $\mathcal{W}_m$.

Definition 4.3. A stochastic polynomial is a random variable of the form

$$p\left(\langle \omega, f_1\rangle, \dots, \langle \omega, f_2\rangle\right), \quad f_1, \dots, f_n \in \mathcal{D}_\mathbb{R}\left(T_m\right),$$

for some polynomial p in n variables. We denote the set of stochastic polynomials by $\mathscr{P}$.

Corollary 4.4. *The set of stochastic polynomials is dense in $\mathcal{W}_m$.*

Proof. We first note that the stochastic polynomials indeed belong to $\mathcal{W}_m$ because the random variables $\langle \omega, f\rangle$ are Gaussian and hence have moments of any order.

Let $\Phi \in \mathcal{W}_m$ be such that $\mathbb{E}\left[\Phi p\right] = 0$ for each $p \in \mathscr{P}$. Then for any $f \in \mathcal{D}_\mathbb{R}(T_m)$,

$$E\left[e^{\langle \omega, f\rangle}\Phi(\omega)\right] = E\left[\sum_{n=0}^\infty \frac{\langle \omega, f\rangle^n}{n!}\Phi(\omega)\right] = \sum_{n=0}^\infty \frac{\mathbb{E}\left[\langle \omega, f\rangle^n \Phi(\omega)\right]}{n!} = 0, \tag{4.1}$$

where interchanging of summation is justified by Fubini's theorem since

$$\sum_{n=0}^{\infty} \mathbb{E}\left[| \frac{\langle \omega, f\rangle^n}{n!}\Phi(\omega) | \right] \leq \sum_{n=0}^{\infty} \frac{1}{(n!)} \sqrt{\mathbb{E}\left[\langle \omega, f\rangle^{2n}\right]\mathbb{E}\left[\Phi(\omega)^2\right]} \leq$$
$$\leq \sum_{n=0}^{\infty} \sqrt{\frac{(2n-1)!!}{(n!)^2}} \| T_m f \|^n \sqrt{\mathbb{E}\left[\Phi(\omega)^2\right]} \leq$$
$$\leq \sum_{n=0}^{\infty} \frac{2^n}{n!} \| T_m f \|^n \sqrt{\mathbb{E}\left[\Phi(\omega)^2\right]} =$$
$$= e^{2\|T_m f\|^2} \cdot \sqrt{\mathbb{E}\left[\Phi(\omega)^2\right]} < \infty.$$

(We have used the Cauchy-Schwarz inequality and the moments of a Gaussian distribution).

We have shown that $\mathbb{E}\left[e^{\langle \omega, f\rangle}\Phi(\omega)\right] = 0$ for any $f \in \mathcal{D}_{\mathbb{R}}(T_m)$, so by Theorem 4.2 we obtain $\Phi = 0$ in $\mathcal{W}_m$. □

Lemma 4.5. *Let $f, g \in \mathcal{D}_{\mathbb{R}}(T_m)$. Then*

$$E[e^{\diamond\langle \omega, f\rangle} \ e^{\diamond\langle \omega, g\rangle}] = e^{(T_m f, T_m g)}.$$

Proof.

$$E[: e^{\langle \omega, f\rangle} :] = e^{-\frac{1}{2}\|T_m f\|^2} E[e^{\langle \omega, f\rangle}] = 1, \tag{4.2}$$

since $E[e^{\langle \omega, f\rangle}]$ is the moment generating function of the Gaussian random variable $\langle \omega, f\rangle$ with variance $\|T_m f\|^2$ evaluated at 1. Thus we get

$$\mathbb{E}[e^{\diamond\langle \omega, f\rangle} \ e^{\diamond\langle \omega, g\rangle}] = e^{(T_m f, T_m g)}\mathbb{E}[e^{\diamond\langle \omega, f+g\rangle}] = e^{(T_m f, T_m g)}.$$ □

The following formula is useful in calculating the S_m transform of the multiplication of two random variables, and can be easily proved using Lemma 4.5.

$$\mathcal{S}_m \left(e^{\diamond\langle \omega, f\rangle} \ e^{\diamond\langle \omega, g\rangle} \right) = e^{(T_m s, T_m f)} e^{(T_m s, T_m f)} e^{(T_m s, T_m g)}, \quad f, g \in \mathcal{D}_{\mathbb{R}}(T_m). \tag{4.3}$$

Proposition 4.6. *Let $\{\Phi_n\}$ be a sequence in $\mathcal{W}_m$ that converges in $\mathcal{W}_m$ to Φ. Then for any $s \in \mathscr{S}_{\mathbb{R}}$ the sequence of real numbers $\{\mathcal{S}_m(\Phi_n)(s)\}$ converges to $\mathcal{S}_m(\Phi)(s)$.*

Proof. For any $s \in \mathscr{S}_{\mathbb{R}}$,

$$|\mathcal{S}_m \Phi_n(s) - \mathcal{S}_m \Phi(s)| = \left|\mathbb{E}\left[e^{\diamond\langle \omega, s\rangle}(\Phi_n - \Phi)\right]\right| \leq \sqrt{\mathbb{E}\left[\left(e^{\diamond\langle \omega, s\rangle}\right)^2\right]} \cdot \sqrt{\mathbb{E}\left[(\Phi_n - \Phi)^2\right]}.$$

By direct calculation $\mathbb{E}\left[\left(e^{\diamond\langle \omega, s\rangle}\right)^2\right] = e^{\|T_m s\|^2}$ and since $\mathbb{E}\left[(\Phi_n - \Phi)^2\right] \longrightarrow 0$, the claim follows. □

We can find the S_m transform of powers of $\langle \omega, f\rangle$ for $f \in \mathcal{D}_{\mathbb{R}}(T_m)$ by the formula for Hermite polynomials.

Corollary 4.7. *For $f \in \mathcal{D}_{\mathbb{R}}(T_m)$ and $s \in \mathcal{S}_{\mathbb{R}}$, we have that*

$$(T_m s, T_m f)^n = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\left(-\frac{1}{2}\right)^k \left(\mathcal{S}_m \langle \omega, f\rangle^{n-2k}\right)(s)\ \|T_m f\|^{2k}}{k!(n-2k)!}, \tag{4.4}$$

in particular

$$(\mathcal{S}_m \langle \omega, f\rangle)(s) = (T_m f, T_m s) \tag{4.5}$$

and

$$(\mathcal{S}_m \langle \omega, f\rangle^2)(s) = (T_m f, T_m s)^2 + \|T_m s\|^2. \tag{4.6}$$

Proof. From Lemma 4.5 we get

$$(\mathcal{S}_m e^{\diamond \langle \omega, f\rangle})(s) = e^{(T_m s, T_m f)}.$$

Then

$$e^{-\frac{1}{2}\|T_m f\|^2} \mathcal{S}_m \left(\sum_{k=0}^{\infty} \frac{\langle \omega, f\rangle^k}{k!} \right)(s) = \sum_{k=0}^{\infty} \frac{(T_m s, T_m f)^k}{k!} \tag{4.7}$$

By the linearity of the S_m transform and Fubini's theorem, and replacing f by tf with $t \in \mathbb{R}$ we compare powers of t at both sides to get (4.4). ☐

This last corollary can be also formulated in terms of the Hermite polynomials. Recall that the n_{th} Hermite polynomial with parameter $t \in \mathbb{R}$ is defined by

$$h_n^{[t]}(x) \triangleq n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\left(-\frac{1}{2}\right)^k x^{n-2k} \cdot t^{2k}}{k!(n-2k)!} \tag{4.8}$$

(see for instance [15, p. 33]). For $f \in \mathcal{D}(T_m)$ we define

$$\tilde{h_n}(\langle \omega, f\rangle) \triangleq h_n^{[\|T_m f\|]}(\langle \omega, f\rangle) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\left(-\frac{1}{2}\right)^k \langle \omega, f\rangle^{n-2k} \cdot \|T_m f\|^{2k}}{k!(n-2k)!}, \tag{4.9}$$

and we also set $\tilde{h}_0 = 1$.

So by (4.4) we have that

$$\left(\mathcal{S}_m \tilde{h}_n(\langle \omega, f\rangle)\right)(s) = (T_m s, T_m f)^n. \tag{4.10}$$

Using equation 4.4 and Lemma 4.5, one can easily verify the following result.

Proposition 4.8. *Let $f \in \mathcal{D}_{\mathbb{R}}(T_m)$. It holds that*

$$e^{\diamond \langle \omega, f\rangle} = \sum_{k=0}^{\infty} \frac{\tilde{h}_k(\langle \omega, f\rangle)}{k!} \tag{4.11}$$

It is possible to define a Wick product in $\mathcal{W}_m$ using the S_m transform.

Definition 4.9. Let $\Phi, \Psi \in \mathcal{W}_m$. The Wick product of Φ and Ψ is the element $\Phi \diamond \Psi \in \mathcal{W}_m$ that satisfies

$$\mathcal{S}_m(\Phi \diamond \Psi)(s) = (\mathcal{S}_m\Phi)(s)(\mathcal{S}_m\Psi)(s), \quad \forall s \in \mathscr{S}_{\mathbb{R}},$$

if it exists.

As this definition suggests, in general the Wick product is not stable in $\mathcal{W}_m$. From (4.10), the Wick product of Hermite polynomials satisfies

$$\tilde{h}_n\left(\langle\omega, f\rangle\right) \diamond \tilde{h}_k\left(\langle\omega, f\rangle\right) = \tilde{h}_{n+k}\left(\langle\omega, f\rangle\right), \quad n, k \in \mathbb{N}, \quad f \in \mathcal{D}_{\mathbb{R}}(T_m).$$

5. STOCHASTIC INTEGRAL

We now use the S_m-transform to define a Wick-Itô type stochastic integral which can be seen as a version of Hitsuda-Skorokhod integral in $\mathcal{W}_m$, and prove an Itô formula for this integral. In the next section we also show that for particular choices of m, our definition of the stochastic integral coincides with previously defined Wick-Itô stochastic integrals for fractional Brownian motion; see [6, 8]. We set

$$\mathcal{B}_s(t) = \mathcal{S}_m\left(B_m(t)\right)(s).$$

By (4.5), we see that

$$\mathcal{B}_s(t) = (T_m s, T_m \mathbf{1}_t) = (T_m^* T_m s, \mathbf{1}_t)_{\mathbf{L}_2} = \int_{\mathbb{R}} m(\xi)\widehat{s}(\xi)\frac{e^{i\xi t} - 1}{\xi} d\xi. \tag{5.1}$$

This function is absolutely continuous with respect to Lebesgue measure and its derivative is

$$(\mathcal{B}_s(t))' = \int_{\mathbb{R}} m(\xi)\widehat{s}(\xi)e^{i\xi t} d\xi. \tag{5.2}$$

We note that when T_m is a bounded operator from $\mathbf{L}_2(\mathbb{R})$ into itself we have by a result of Lebesgue (see [17, p. 410]), $(\mathcal{B}_s(t))' = (T_m^* T_m s)(t)$ (a.e.).

Definition 5.1. Let $M \in \mathbb{R}$ be a Borel set and let $X : M \longrightarrow \mathcal{W}_m$ be a stochastic process. The process X will be called integrable over M if for any $s \in \mathscr{S}_{\mathbb{R}}$, $\left(\mathcal{S}_m X_t\right)(s)\mathcal{B}_s(t)'$ is integrable on M, and if there is a $\Phi \in \mathcal{W}_m$ such that

$$\mathcal{S}_m\Phi(s) = \int_M \left(\mathcal{S}_m X_t\right)(s)\mathcal{B}_s(t)' dt$$

for any $s \in \mathscr{S}_{\mathbb{R}}$. If X is integrable, Φ is uniquely determined by Theorem 4.2 and we denote it by $\int_M X_t dB_m(t)$.

If $T = Id_{L_2(\mathbb{R})}$, this definition coincides with the *Hitsuda-Skorokhod* integral [10, Chapter 8]. See also Section 5.

Note that since

$$|\mathcal{B}_s(t)'| \leq \int_{\mathbb{R}} m(\xi)|\widehat{s}(\xi)|d\xi \leq \sup_{\xi} |(1+\xi^2)\widehat{s}(\xi)| \int_{\mathbb{R}} \frac{m(\xi)}{1+\xi^2} d\xi < \infty,$$

for any $s \in \mathscr{S}$ there exists a constant K_s such that

$$\Big| \int_M \mathcal{S}_m X_t(s) \mathcal{B}_s(t)' dt \Big| \leq K_s \int_M \left| \mathbb{E}\left[X_t e^{\diamond \langle \omega, s \rangle} \right] \right| dt \leq K_s \mathbb{E}\left[\left(e^{\diamond \langle \omega, s \rangle} \right)^2 \right] \int_M \mathbb{E}[X_t^2] dt.$$

Thus a sufficient condition for the integrability of $\mathcal{S}_m X_t(s)\mathcal{B}_s(t)'$ is $\int_M \mathbb{E}[X_t^2]dt < \infty$.

Proposition 5.2. *Any non-random $f \in \mathcal{D}_{\mathbb{R}}(T_m)$ is integrable and we have*

$$\int_0^\tau f(t) dB_m(t) = \langle \omega, \mathbf{1}_\tau f \rangle. \tag{5.3}$$

Proof. In virtue of (4.5) and the definition of the stochastic integral, we need to show that

$$\int_0^\tau f(t) \mathcal{B}_s(t)' dt = (T_m s, T_m \mathbf{1}_\tau f).$$

Using formula (5.2) and Fubini's theorem, we have

$$\begin{aligned} \int_0^\tau f(t)\mathcal{B}_s(t)'dt &= \int_0^\tau f(t) \left(\int_{\mathbb{R}} m(\xi)\widehat{s}(\xi) e^{i\xi t} d\xi \right) dt = \\ &= \int_{\mathbb{R}} m(\xi)\widehat{s}(\xi) \left(\int_0^\tau f(t) e^{it\xi} dt \right) d\xi = \\ &= \int_{\mathbb{R}} m(\xi)\widehat{s}(\xi) \left(\widehat{f\mathbf{1}_\tau}(\xi) \right) d\xi = (T_m s, T_m \mathbf{1}_\tau f). \end{aligned}$$

□

Proposition 5.3. *The stochastic integral has the following properties:*

(1) *For $0 \leq a < b \in \mathbb{R}$,*

$$B_m(b) - B_m(a) = \int_a^b dB_m(t).$$

(2) *Let $X : M \longrightarrow \mathcal{W}_m$ an integrable process. Then*

$$\int_M X_t dB_m(t) = \int_{\mathbb{R}} \mathbf{1}_M X_t dB_m(t).$$

(3) *Let $X : M \longrightarrow \mathcal{W}_m$ an integrable process. Then*

$$\mathbb{E}\left[\int\limits_M X_t dB_m(t)\right] = \mathcal{S}_m\left(\int\limits_M X_t dB_m(t)\right)(0) = 0.$$

(4) *The Wick product and the stochastic integral can be interchanged: Let $X : \mathbb{R} \longrightarrow \mathcal{W}_m$ an integrable process and assume that for $Y \in \mathcal{W}_m$, $Y \diamond X_t$ is integrable. Then*

$$Y \diamond \int\limits_{\mathbb{R}} X_t dB_m(t) = \int\limits_{\mathbb{R}} Y \diamond X_t dB_m(t).$$

Proof. The proof of the first three items is easy and we omit it. The last item is proved in the following way:

$$\begin{aligned}\mathcal{S}_m\Big(Y \diamond \int\limits_{\mathbb{R}} X_t dB_m(t)\Big)(s) &= (S_m Y)(s) \int\limits_M (\mathcal{S}_m X_t)(s) dB_m = \\ &= \int\limits_M (S_m Y)(s)(\mathcal{S}_m X_t)(s) dB_m = \\ &= \mathcal{S}_m\Big(\int\limits_{\mathbb{R}} Y \diamond X_t dB_m(t)\Big)(s).\end{aligned}$$

□

Example 5.4. For $\tau \geq 0$, by equation (4.6), we have

$$\int\limits_0^\tau (T_m s, T_m \mathbf{1}_t) \frac{d}{dt} (T_m s, T_m \mathbf{1}_t)\, dt = \frac{1}{2} (T_m s, T_m \mathbf{1}_\tau)^2 = \frac{1}{2} \mathcal{S}_m \left(\langle \omega, \mathbf{1}_t \rangle - \|T_m \mathbf{1}_t\|^2\right)(s).$$

Then B_m is integrable on the interval $[0, \tau]$, and we get

$$\int\limits_0^\tau B_m(t) dB_m(t) = \frac{1}{2} B_m(\tau)^2 - \frac{1}{2} \|T_m \mathbf{1}_\tau\|^2.$$

This reduces to the well known Itô integral identity in the case where $m \equiv 1$, i.e. where T_m is the identity operator.

Example 5.5. Let $\widetilde{h_n}$ be defined by (4.9). A similar argument to the one in (5.2) will show that for any f such that $f\mathbf{1}_t \in \mathcal{D}(T_m)$, the function $t \mapsto (T_m s, T_m f \mathbf{1}_t)$ is differentiable with derivative

$$\frac{d}{dt} (T_m s, T_m f \mathbf{1}_t) = f(t) \int\limits_{\mathbb{R}} m(\xi) \widehat{s}(\xi) e^{-it\xi} d\xi = f(t) \mathcal{B}_s(t)'.$$

By a similar argument to Proposition 5.2, we get

$$\begin{aligned}\frac{1}{n+1}\mathcal{S}_m\left(\tilde{h}_{n+1}\left(\langle\omega,\mathbf{1}_\tau f\rangle\right)(t)\right)(s) &= \frac{1}{n+1}\left(T_m s, T_m\mathbf{1}_\tau f\right) = \\ &= \int_0^\tau \left(T_m s, T_m f\mathbf{1}_t\right)^n f(t)\mathcal{B}_s(t)' dt = \\ &= \mathcal{S}_m\left(\int_0^\tau f(t)\tilde{h}_n\left(\langle\omega,\mathbf{1}_t f\rangle\right) dB_m(t)\right)(s),\end{aligned}$$

thus

$$\int_0^\tau f(t)\tilde{h}_n\left(\langle\omega,\mathbf{1}_t f\rangle\right) dB_m(t) = \frac{1}{n+1}\tilde{h}_{n+1}\left(\langle\omega,\mathbf{1}_\tau f\rangle\right). \tag{5.4}$$

It follows from (5.4) that for any polynomial p and f with $\mathbf{1}_t f \in \mathcal{D}(T_m)$ the process $p(\langle\omega,\mathbf{1}_t f\rangle)$ is integrable. This result can be easily extended to the process

$$t \mapsto e^{\langle\omega,\mathbf{1}_t f\rangle}, \quad \mathbf{1}_t f \in \mathcal{D}(T_m),$$

and we also obtain the following corollary.

Corollary 5.6.

$$\int_0^\tau f(t)e^{\diamond\langle\omega,\mathbf{1}_t f\rangle} dB_m(t) = e^{\diamond\langle\omega,\mathbf{1}_t f\rangle} - 1.$$

Example 5.7. Let $f \in \mathcal{D}_{\mathbb{R}}(T_m)$. Using (4.3) we can obtain

$$\begin{aligned}\mathcal{S}_m\left(e^{\diamond\langle\omega,f\rangle}\int_0^\tau e^{\diamond\langle\omega,\mathbf{1}_t\rangle} dB_m(t)\right)(s) &= \mathcal{S}_m\left(e^{\diamond\langle\omega,f\rangle}\ e^{\diamond\langle\omega,\mathbf{1}_\tau\rangle} - e^{\diamond\langle\omega,f\rangle}\right)(s) = \\ &= e^{(T_m s, T_m f)}\left(e^{(T_m s, T_m\mathbf{1}_\tau)}e^{(T_m f, T_m\mathbf{1}_\tau)} - 1\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\mathcal{S}_m\left(\int_0^\tau e^{\diamond\langle\omega,f\rangle}\ e^{\diamond\langle\omega,\mathbf{1}_t\rangle} dB_m(t)\right)(s) = \\ &= e^{(T_m s, T_m f)}\int_0^\tau e^{(T_m s, T_m\mathbf{1}_t)}e^{(T_m f, T_m\mathbf{1}_t)}\frac{d}{dt}\left(T_m s, T_m\mathbf{1}_t\right) dt = \\ &= e^{(T_m s, T_m f)}\left(e^{(T_m s, T_m\mathbf{1}_\tau)}e^{(T_m f, T_m\mathbf{1}_\tau)} - 1\right) - \\ &\quad - \int_0^\tau e^{(T_m s, T_m\mathbf{1}_t)}e^{(T_m f, T_m\mathbf{1}_t)}\frac{d}{dt}\left(T_m f, T_m\mathbf{1}_t\right) dt.\end{aligned}$$

So in general for an integrable stochastic process X and a random variable Y we have the somewhat undesirable result

$$Y\int_0^\tau X_t dB_m(t) \neq \int_0^\tau YX_t dB_m(t).$$

Compare it with property (4) in Proposition 5.3.

6. ITÔ'S FORMULA

In this section we prove an Itô's formula. We begin by proving a version of the classical Girsanov theorem in our setting.

Theorem 6.1. *Let $f \in \mathcal{D}(T_m)$, and let μ be the measure defined by $\mu(A) = \mathbb{E}[e^{\diamond\langle\omega,f\rangle}\mathbf{1}_A]$. The process*

$$\tilde{B}_m(t) \triangleq B_m(t) - (T_m f, T_m \mathbf{1}_t)$$

is Gaussian under the probability law μ and satisfies

$$\mathbb{E}_\mu[\tilde{B}_m(t)\tilde{B}_m(s)] = (T_m\mathbf{1}_t, T_m\mathbf{1}_s).$$

Proof. We will first prove that for all $t \geq 0$, $\tilde{B}_m(t)$ is a Gaussian random variable relative to the measure μ by considering its moment generating function $\mathbb{E}_\mu\left[e^{\lambda\tilde{B}_m(t)}\right]$, $\lambda \in \mathbb{R}$,

$$\begin{aligned}\mathbb{E}_\mu\left[e^{\lambda\tilde{B}_m(t)}\right] &= \mathbb{E}\left[e^{\langle\omega,f\rangle-\frac{1}{2}\|T_m f\|^2}e^{\lambda\langle\omega,\mathbf{1}_t\rangle-\lambda(T_m f,T_m\mathbf{1}_t)}\right] = \\ &= e^{-\lambda(T_m f,Tm\mathbf{1}_t)}\ e^{-\frac{1}{2}\|T_m f\|^2}\mathbb{E}\left[e^{\langle\omega,f+\lambda\mathbf{1}_t\rangle}\right].\end{aligned} \tag{6.1}$$

Since $\langle\omega, f+\lambda\mathbf{1}_t\rangle$ is a zero mean Gaussian random variable with variance

$$\|T_m\left(f+\lambda\mathbf{1}_t\right)\|^2 = \|T_m f\|^2 + \lambda^2\|T_m\mathbf{1}_t\|^2 + 2\lambda\left(T_m f, T_m\mathbf{1}_t\right),$$

its moment generating function evaluated at 1 is given by

$$\mathbb{E}\left[e^{\langle\omega,T_m f+\lambda\mathbf{1}_t\rangle}\right] = e^{\frac{1}{2}\|T_m f\|^2}e^{\frac{1}{2}\lambda^2\|T_m\mathbf{1}_t\|^2}e^{\lambda(T_m f,T_m\mathbf{1}_t)}, \tag{6.2}$$

and we conclude from (6.1) that

$$\mathbb{E}_\mu\left[e^{\lambda\tilde{B}_m(t)}\right] = e^{\frac{1}{2}\lambda^2\|T_m\mathbf{1}_t\|^2}. \tag{6.3}$$

Thus for all $t \geq 0$, $\tilde{B}_m(t)$ is a zero mean Gaussian random variable on $(\Omega, \mathcal{G}, \mu)$. Similar arguments will show that any linear combination of time samples is a Gaussian variable, and so $\widetilde{B}_m(t), t \geq 0$ is a Gaussian process. Finally, by the polarization formula,

$$\mathbb{E}_\mu[\tilde{B}_m(t)\tilde{B}_m(s)] = (T_m\mathbf{1}_t, T_m\mathbf{1}_s).$$

□

We now interpret integrals of the type $\int_0^\tau \Phi(t)dt$, where for every $t \in [0,\tau]$, $\Phi(t) \in \mathcal{W}_m$, as Pettis integrals, that is as

$$\mathbb{E}\left[\left(\int_0^\tau \Phi(t)dt\right)\Psi\right] = \int_0^\tau \mathbb{E}[\Phi(t)\Psi]dt, \quad \forall \Psi \in \mathcal{W}_m,$$

under the hypothesis that the function $t \mapsto \mathbb{E}[\Phi(t)\Psi]$ belongs to $\mathbf{L}_1([0,\tau],dt)$ for every $\Psi \in \mathcal{W}_m$. See [11, pp. 77–78]. We note that if X is moreover pathwise integrable and such that the pathwise intregral belongs to $\mathcal{W}_m$, then

$$\int_0^\tau \mathbb{E}[|X_t|]dt < \infty,$$

and we can apply Fubini's theorem to show that both integrals coincide. It is also clear from the definition of the Pettis integral that it commutes with the S_m transform.

We introduce the conditions

$$\mathbb{E}\left[|F(t,X_t)|e^{\diamond\langle\omega,s\rangle}\right] < \infty, \tag{6.4}$$

$$\mathbb{E}\left[|\frac{\partial F}{\partial t}(t,X_t)|e^{\diamond\langle\omega,s\rangle}\right] < \infty, \tag{6.5}$$

$$\mathbb{E}\left[|\frac{\partial F}{\partial x}(t,X_t)|e^{\diamond\langle\omega,s\rangle}\right] < \infty, \tag{6.6}$$

for $F \in C^{1,2}([0,\infty),\mathbb{R})$.

We now develop an Itô formula for a class of stochastic processes of the form

$$X_t(\omega) = \int_0^\tau f(t)dB_m(t) = \langle\omega, \mathbf{1}_\tau f\rangle, \quad \tau \geq 0, \quad \mathbf{1}_\tau f \in \mathcal{D}(T_m). \tag{6.7}$$

Theorem 6.2. *Let $F \in C^{1,2}([0,\infty),\mathbb{R})$ satisfying (6.4)–(6.6), and assume that $\|T_m \mathbf{1}_t f\|^2$ is absolutely continuous with respect to the Lebesgue measure as a function of t. Then we have*

$$F(\tau,X_\tau) - F(0,0) = \int_0^\tau \frac{\partial}{\partial t}F(t,X_t)dt + \int_0^\tau f(t)\frac{\partial}{\partial x}F(t,X_t)\,dB_m(t) + \tag{6.8}$$
$$+\frac{1}{2}\int_0^\tau \frac{d}{dt}\|T_m\mathbf{1}_t f\|^2 \frac{\partial^2}{\partial x^2}F(t,X_t)dt$$

in $\mathcal{W}_m$.

The proof is based on the proof for the Itô formula in the S-transform approach to Hitsuda-Skorokhod integration in the standard white noise space found in [15, Section 13.5].

Proof. Let $s \in \mathscr{S}_\mathbb{R}$ and $f \in \mathcal{D}\ (T_m)$. It follows from Theorem 6.1 that for every $t \in [0,\tau]$, $X_t(\omega) = \langle \omega, \mathbf{1}_t f\rangle$ is normally distributed under the measure

$$\mu_s(A) \triangleq \mathbb{E}\left[\mathbf{1}_A \exp\left\{\langle \omega, s\rangle - \frac{1}{2}\|T_m s\|^2\right\}\right] = \mathbb{E}\left[\mathbf{1}_A e^{\diamond\langle \omega, s\rangle}\right]$$

with mean $(T_m s, T_m \mathbf{1}_t f)$ and variance $\|T_m \mathbf{1}_t f\|^2$. Thus,

$$\begin{aligned}(\mathcal{S}_m F(t, X_t))(s) &= \mathbb{E}\left[e^{\diamond\langle \omega, s\rangle} F(t, X_t)\right] = \\ &= \int_\mathbb{R} F\left(t, u + (T_m \mathbf{1}_t f, T_m s)\right) \rho\left(\|T_m \mathbf{1}_t f\|^2, u\right) du,\end{aligned} \tag{6.9}$$

where $\rho(w,u) = \frac{1}{\sqrt{2\pi w}} e^{-\frac{u^2}{2w}}$ and satisfies

$$\frac{\partial}{\partial w}\rho = \frac{1}{2}\frac{\partial^2}{\partial u^2}\rho. \tag{6.10}$$

Integrating by parts we obtain

$$\int_\mathbb{R} F(t,u)\frac{\partial^2}{\partial u^2}\rho(w,u)du = \int_\mathbb{R} \frac{\partial^2}{\partial u^2} F(t,u)\rho(w,u)du. \tag{6.11}$$

In view of (6.4)–(6.6) we may differentiate under the integral sign by (6.9), (6.10) and (6.11) and obtain for $0 \le t \le \tau$,

$$\begin{aligned}&\frac{d}{dt}\mathcal{S}_m\left(F(t, X_t)\right)(s) = \\ &= \int_\mathbb{R} \frac{\partial}{\partial t} F\left(t, u + (T_m 1_t f, T_m s)\right) \rho\left(\|T_m 1_t f\|^2, u\right) du + \\ &\quad + \int_\mathbb{R} \frac{\partial}{\partial x} F\left(t, u + (T_m \mathbf{1}_t f, T_m s)\right) \frac{d}{dt}\left(T_m \mathbf{1}_t f, T_m s\right) \rho\left(\|T_m \mathbf{1}_t f\|^2, u\right) du + \\ &\quad + \int_\mathbb{R} F\left(t, u + (T_m \mathbf{1}_t f, T_m s)\right) \frac{d}{dt}\|T_m \mathbf{1}_t f\|^2 \frac{\partial}{\partial t}\rho\left(\|T_m \mathbf{1}_t f\|^2\right) du = \\ &= \mathcal{S}_m\left(\frac{\partial}{\partial t} F\left(t, X_t\right)\right)(s) + \frac{d}{dt}\left(T_m s, T_m 1_t f\right) \mathcal{S}_m\left(\frac{\partial}{\partial x} F\left(t, X_t\right)\right)(s) + \\ &\quad + \frac{1}{2}\frac{d}{dt}\|T_m \mathbf{1}_t f\|^2 \cdot \mathcal{S}_m\left(\frac{\partial^2}{\partial x^2} F\left(t, X_t\right)\right)(s).\end{aligned}$$

Hence,

$$\mathcal{S}_m\left(F\left(\tau,X_\tau\right)-F\left(0,0\right)\right)(s)=\int_0^\tau \mathcal{S}_m\left(\frac{\partial}{\partial t}F\left(t,X_t\right)\right)(s)dt+ \tag{6.12}$$

$$+\int_0^\tau \frac{d}{dt}\left(T_m s, T_m \mathbf{1}_t f\right)\mathcal{S}_m\left(\frac{\partial}{\partial x}F\left(t,X_t\right)\right)(s)dt+$$

$$+\frac{1}{2}\int_0^\tau \frac{d}{dt}\|T_m 1_t f\|^2\cdot \mathcal{S}_m\left(\frac{\partial^2}{\partial x^2}F\left(t,X_t\right)\right)(s)dt.$$

By the definition of the stochastic integral,

$$\mathcal{S}_m\left(\int_0^\tau f(t)\frac{\partial}{\partial x}F\left(t,X_t\right)dB_m(t)\right)(s)=\int_0^\tau \mathcal{S}_m\left(\frac{\partial}{\partial x}F\left(t,X_t\right)\right)(s)f(t)\mathcal{B}_s(t)dt,$$

which in view of Example 5.5 equals

$$\int_0^\tau \frac{d}{dt}\left(T_m s, T_m \mathbf{1}_t f\right)\mathcal{S}_m\left(\frac{\partial}{\partial x}F\left(t,X_t\right)\right)(s)dt.$$

We may now use Fubini's theorem to interchange the S_m-transform and the pathwise integral, and obtain that the S_m-transform of the right hand side of (6.8) is exactly the right hand side of (6.12), which proves the theorem. □

7. RELATION TO OTHER WHITE-NOISE EXTENSIONS OF WICK-ITÔ INTEGRAL

Recall that the white noise space corresponds to $m(\xi)\equiv 1$, so denoting it $\mathcal{W}_1$ is consistent with our notation, and $\mathcal{S}_1$ is the classical S-transform of the white noise space. We define a map $\widetilde{T_m}:\mathcal{W}_m\longrightarrow\mathcal{W}_1$ by describing its action on the dense set of stochastic polynomials in $\mathcal{W}_m$:

$$\widetilde{T_m}\langle\omega,f\rangle^n=\langle\omega,T_m f\rangle^n,\quad f\in\mathcal{D}\left(T_m\right).$$

Note that since the range of T_m is contained in $\mathcal{D}(T_1)=\mathcal{D}(Id_{L_2(\mathbb{R})})=\mathbf{L}_2(\mathbb{R})$, this map is well defined. It is easy to see that $\widetilde{T_m}$ is an isometry between Hilbert spaces. By continuity, we obtain that

$$\widetilde{T_m}e^{\langle\omega,f\rangle}=e^{\langle\omega,T_m f\rangle},\quad f\in\mathcal{D}\left(T_m\right),$$

hence

$$\left(\mathcal{S}_1\widetilde{T_m}e^{\langle\omega,f\rangle}\right)(T_m s)=e^{(T_m s,T_m f)}=\left(\mathcal{S}_m e^{\langle\omega,f\rangle}\right)(s).$$

So this relation between S_1 and S_m can be extended such that for any $\Phi \in \mathcal{W}_m$,

$$\left(\mathcal{S}_1 \widetilde{T_m} \Phi\right)(T_m s) = (\mathcal{S}_m \Phi)(s).$$

Let $X : [0, \tau] \longrightarrow \mathcal{W}_m$ be a stochastic process. We have defined its Itô integral as the unique element $\Phi \in \mathcal{W}_m$ (if exists) having S_m-transform

$$(\mathcal{S}_m \Phi)(s) = \int_0^\tau (X_t)(s) \frac{d}{dt}(T_m s, T_m \mathbf{1}_t)(s) dt.$$

This suggests that if we define in the white noise the process $\tilde{B}_m$ as $\langle \omega, T_m \mathbf{1}_t \rangle$ and stochastic integral with respect to $\tilde{B}_m$ as the unique element $\Phi \in \mathcal{W}_1$(if exists) having S_1-transform

$$(\mathcal{S}_1 \Phi)(s) = \int_0^\tau (X_t)(s) \frac{d}{dt}(s, T_m \mathbf{1}_t))(s) dt, \tag{7.1}$$

both definitions coincide in the sense that

$$\widetilde{T_m} \int_0^\tau X_t dB_m(t) = \int_0^\tau \widetilde{T_m} X_t d\tilde{B}_m(t). \tag{7.2}$$

Recall that the fractional Brownian motion can be obtained in our setting by taking $m(\xi) = C_H |\xi|^{1-2H}$, $H \in (0, 1)$, where $C_H = \frac{\Gamma(1+2H)\sin(\pi H)}{2\pi}$. The resulting T_m was defined in [9] and denoted M_H there. In the white noise space, the fractional Brownian motion can be defined by the continuous version of the process $\{\langle \omega, M_H \mathbf{1}_t \rangle\}_{t \geq 0}$.
An approach that is based on the definition described in (7.1) for the fractional Brownian motion was given in [4]. Due to Theorem 3.4 there, under appropriate conditions our definition of the Hitsuda-Skorokhod integral in the case of $T_m = M_H$ coincides in the sense of (7.2) with the Hitsuda-Skorokhod integral defined there. Stochastic integration in the white noise setting for the family of stochastic processes considered in this paper can be found in [2], and its equivalence to the integral described here can be obtained by a similar argument to that of Theorem 3.4 in [4].

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