

POSITIVITY AND STABILITY OF FRACTIONAL DESCRIPTOR TIME-VARYING DISCRETE-TIME LINEAR SYSTEMS

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The Weierstrass–Kronecker theorem on the decomposition of the regular pencil is extended to fractional descriptor time-varying discrete-time linear systems. A method for computing solutions of fractional systems is proposed. Necessary and sufficient conditions for the positivity of these systems are established.

Keywords: fractional system, descriptor system, time-varying system, positive system, discrete-time system.

1. Introduction

A dynamic system is called positive if its trajectory starting from any nonnegative initial condition remains forever in the positive orthant for all nonnegative inputs. An overview of the state of the art in positive system theory is given by Farina and Rinaldi (2000) as well as Kaczorek (2001; 2011; 1998a; 2015b; 1997). Models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

The Lyapunov, Bohl and Perron exponents as well as stability of time-varying discrete-time linear systems were investigated by Czornik (2014), Czornik *et al.* (2012; 2013; 2014a; 2014b), Czornik and Niezabitowski (2013a; 2013b; 2013c) as well as Niezabitowski (2014). Positive standard and descriptor systems and their stability were analyzed by Kaczorek (2001; 2011; 1998a; 2015b), along with positive linear systems with different fractional orders (Kaczorek, 2011; 2012) and singular discrete-time linear systems (Kaczorek, 1998a; 2015a). Switched discrete-time systems were considered by Zhang *et al.* (2014a; 2014b) and Zhong (2013), while extremal norms for positive linear inclusions by Rami *et al.* (2012). Positivity and stability of time-varying discrete-time linear systems were addressed by Kaczorek (2015c).

In this paper the Weierstrass–Kronecker decomposition theorem will be applied to fractional descriptor time-varying discrete-time linear systems with regular pencils to find their solutions, and necessary and sufficient conditions for positivity and sufficient

conditions for stability will be established.

The paper is organized as follows. In Section 2 the Weierstrass–Kronecker decomposition theorem is applied to find solutions to standard fractional descriptor time-varying discrete-time linear systems. Necessary and sufficient conditions for the positivity of descriptor systems are established in Section 3. The solution of fractional positive descriptor systems is analyzed in Section 4. Concluding remarks are given in Section 5.

The following notation will be used: \mathbb{R} , the set of real numbers; $\mathbb{R}^{n \times m}$, the set of $n \times m$ real matrices; $\mathbb{R}_+^{n \times m}$, the set of $n \times m$ matrices with nonnegative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n \times 1}$; I_n , the $n \times n$ identity matrix.

2. Standard fractional descriptor systems

Consider the fractional descriptor time-varying discrete-time linear system

$$E(i)\Delta^\alpha x_{i+1} = A(i)x_i + B(i)u_i, \quad i \in \mathbb{Z}_+ = \{0, 1, \dots\}, \quad (1a)$$

$$y_i = C(i)x_i, \quad (1b)$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$ are the state, input and output vectors, respectively, $A(i) \in \mathbb{R}^{n \times n}$, $B(i) \in \mathbb{R}^{n \times m}$, $C(i) \in \mathbb{R}^{p \times n}$ are matrices with entries depending on $i \in \mathbb{Z}_+$, and the fractional difference of the order α is

defined by

$$\Delta^\alpha x_i = \sum_{j=0}^i (-1)^j \binom{\alpha}{j} x_{i-j}, \quad (1c)$$

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j = 0, \\ \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!} & \text{for } j = 1, 2, \dots \end{cases} \quad (1d)$$

It is assumed that $\det E(i) = 0, i \in \mathbb{Z}_+$, and

$$\det[E(i)\lambda - A(i)] \neq 0 \quad (2)$$

for some $\lambda \in \mathbb{C}$ (the field of complex numbers) and $i \in \mathbb{Z}_+$.

Substituting (1c) into (1a), we obtain

$$E(i)x_{i+1} = [E(i)\alpha - A(i)]x_i + \sum_{j=2}^{i+1} c_j E(i)x_{i-j+1} + B(i)u_i, \quad (3a)$$

where

$$c_j = (-1)^{j+1} \binom{\alpha}{j}. \quad (3b)$$

It is well-known (Kaczorek, 2015b; 1998b) that if (2) holds, then there exists a pair of nonsingular matrices $P(i), Q(i) \in \mathbb{R}^{n \times n}$ such that

$$P(i)[E(i)\lambda - A(i)]Q(i) = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} \lambda - \begin{bmatrix} A_1(i) & 0 \\ 0 & I_{n_2} \end{bmatrix}, \quad (4)$$

$i \in \mathbb{Z}_+$, where $u_1 = \deg \det[E(i)\lambda - A(i)]$, $A_1(i) \in \mathbb{R}^{n_1 \times n_1}$, $N \in \mathbb{R}^{n_2 \times n_2}$ is a nilpotent matrix of index μ (i.e., $N^\mu = 0$ and $N^{\mu-1} \neq 0$).

The matrices $P(i), Q(i), A_1(i)$ can be found for example, with the use of elementary row and column operations (Kaczorek, 1998b).

Premultiplying (1a) by the matrix $P(i)$, introducing the new state vector

$$\bar{x}_i = Q^{-1}(i)x_i = \begin{bmatrix} \bar{x}_{1,i} \\ \bar{x}_{2,i} \end{bmatrix}, \quad (5)$$

$$\bar{x}_{1i} = \begin{bmatrix} \bar{x}_{11,i} \\ \bar{x}_{12,i} \\ \vdots \\ \bar{x}_{1n_1,i} \end{bmatrix},$$

$$\bar{x}_{2i} = \begin{bmatrix} \bar{x}_{21,i} \\ \bar{x}_{22,i} \\ \vdots \\ \bar{x}_{2n_2,i} \end{bmatrix},$$

and using (4), we obtain

$$\bar{x}_{1,i+1} = A_{1\alpha}(i)\bar{x}_{1,i} + \sum_{j=2}^{i+1} c_j \bar{x}_{1,i-j+1} + B_1(i)u_i, \quad (6a)$$

$$N\bar{x}_{2,i+1} = (N\alpha + I_{n_2})\bar{x}_{2,i} + \sum_{j=2}^{i+1} c_j N\bar{x}_{2,i-j+1} + B_2(i)u_i, \quad (6b)$$

where

$$A_{1\alpha}(i) = A_1(i) + \alpha I_{n_1} \in \mathbb{R}^{n_1 \times n_1},$$

$$P(i)B(i) = \begin{bmatrix} B_1(i) \\ B_2(i) \end{bmatrix},$$

$$B_1(i) \in \mathbb{R}^{n_1 \times m}, \quad B_2(i) \in \mathbb{R}^{n_2 \times m}, \quad (6c)$$

Theorem 1. The solution $\bar{x}_{1,i}$ of Eqn. (6a) for a known admissible initial condition $\bar{x}_{10} \in \mathbb{R}^{n_1}$ and input $u_i \in \mathbb{R}^m, i \in \mathbb{Z}_+$, is given by

$$\bar{x}_{1,i} = \Phi_1(i, 0)\bar{x}_{1,0} + \sum_{k=0}^{i-1} \Phi_1(i, k+1)B_1(k)u_k, \quad (7a)$$

$i \in \mathbb{Z}_+$, where

$$\Phi_1(i, k+1) = A_{1\alpha}(k)\Phi_1(k, 0) + \sum_{j=2}^{k+1} c_j \Phi_1(k-j+1, 0),$$

$$\Phi(0, 0) = I_n, \quad (7b)$$

and c_j is defined by (3b).

Proof. Using (6c), (7a) and (7b), we obtain

$$A_{1\alpha}(i)\bar{x}_{1,i} + B_1(i)u_i + \sum_{j=2}^{i+1} c_j \bar{x}_{1,i-j+1}$$

$$= A_{1\alpha}(i) \left[\Phi_1(i, 0)\bar{x}_{1,0} + \sum_{k=0}^{i-1} \Phi_1(i, k+1)B_1(k)u_k \right]$$

$$+ B_1(i)u_i + \sum_{j=2}^{i+1} c_j \bar{x}_{1,i-j+1}$$

$$= \Phi_1(i+1, 0)\bar{x}_{1,0} + \sum_{k=0}^i \Phi_1(i+1, k+1)B_1(k)u_k$$

$$= \bar{x}_{1,i+1}.$$

■

To simplify the notation, it is assumed that the matrix

N in (6b) has the form

$$N = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{n_2 \times n_2}. \quad (8)$$

From (6b) and (8), we have

$$\begin{aligned} & \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{21,i+1} \\ \bar{x}_{22,i+1} \\ \vdots \\ \bar{x}_{2n_2,i+1} \end{bmatrix} \\ &= \begin{bmatrix} 1 & \alpha & 0 & \dots & 0 \\ 0 & 1 & \alpha & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \alpha \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} \bar{x}_{21,i} \\ \bar{x}_{22,i} \\ \vdots \\ \bar{x}_{2n_2,i} \end{bmatrix} \\ &+ \sum_{j=2}^{i+1} \begin{bmatrix} 0 & c_j & 0 & \dots & 0 \\ 0 & 0 & c_j & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c_j \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{21,i-j+1} \\ \bar{x}_{22,i-j+1} \\ \vdots \\ \bar{x}_{2n_2,i-j+1} \end{bmatrix} \\ &+ \begin{bmatrix} B_{21}(i) \\ \vdots \\ B_{2n_2}(i) \end{bmatrix} u_i, \quad i \in \mathbb{Z}_+, \end{aligned} \quad (9)$$

and

$$\begin{aligned} 0 &= \bar{x}_{2n_2,i} + B_{2n_2}(i)u_i, \\ \bar{x}_{2n_2,i+1} &= \bar{x}_{2n_2-1,i} + \alpha \bar{x}_{2n_2,i} \\ &+ \sum_{j=2}^{i+1} c_j \bar{x}_{2n_2,i-j+1} + B_{2n_2-1}(i)u_i, \\ &\vdots \\ \bar{x}_{22,i+1} &= \bar{x}_{21,i} + \alpha \bar{x}_{22,i} \\ &+ \sum_{j=2}^{i+1} c_j \bar{x}_{22,i-j+1} + B_{21}(i)u_i, \quad i \in \mathbb{Z}_+. \end{aligned} \quad (10)$$

Solving Eqns. (10) with respect to the components of

the vector $\bar{x}_{2,i}$, we obtain

$$\begin{aligned} \bar{x}_{2n_2,i} &= -B_{2n_2}(i)u_i, \\ \bar{x}_{2n_2-1,i} &= -B_{2n_2}(i+1)u_{i+1} + \alpha B_{2n_2}(i)u_i \\ &+ \sum_{j=2}^{i+1} c_j B_{2n_2}(i-j+1)u_{i-j+1} \\ &- B_{2n_2-1}(i)u_i, \\ &\vdots \\ \bar{x}_{21,i} &= -B_{2n_2}(i+n_2-1)u_{i+n_2-1} \\ &+ \alpha B_{21}(i+n_2-2)u_i \\ &+ \sum_{j=2}^{i+1} c_j B_{2n_2}(i+n_2-j-1)u_{i-j+1} \\ &+ \dots - B_{21}(i)u_i. \end{aligned} \quad (11)$$

The admissible initial conditions for the system (6b) are given by (11) for $i = 0$.

The solution of Eqn. (6b) for known $u_i \in \mathbb{R}^m$ and admissible initial conditions $\bar{x}_{20} \in \mathbb{R}^{n_2}$ is given by (11). The discussion can be easily extended to the case when the matrix N in (6b) has the form

$$N = \text{blockdiag}[N_1, \dots, N_q], \quad q > 1, \quad (12)$$

and N_k for $k = 1, 2, \dots, q$ has the form (8).

Example 1. Consider the fractional descriptor time-varying system described by Eqn. (1a) with the matrices

$$\begin{aligned} E(i) &= \begin{bmatrix} 0 & 0 & 0 & \frac{e^{2i}}{\cos(i)+2} \\ -\frac{(i+2)(\sin(i)+1)}{i+1} & e^i & 0 & -\frac{e^{2i}(e^{-i}+1)}{\cos(i)+2} \\ \frac{i+2}{i+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ B(i) &= \begin{bmatrix} \frac{1}{\cos(i)+2} \\ e^{-i} - \frac{e^{-i}+1}{\cos(i)+2} \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \frac{2i(i+2)(\cos(i)+1)(\sin(i)+1)}{i+1} - \sin(i)(\sin(i)+1) \\ \sin(i) - \frac{2i(i+2)(\cos(i)+1)}{i+1} \\ \frac{2i(i+2)}{i+1} \end{bmatrix}, \\ A(i) &= \begin{bmatrix} 0 & 0 & a_{13}(i) & 0 \\ a_{21}(i) & a_{22}(i) & a_{23}(i) & a_{24}(i) \\ a_{31}(i) & 0 & 0 & a_{34}(i) \\ 0 & 0 & 0 & a_{44}(i) \end{bmatrix}, \end{aligned} \quad (13)$$

where

$$\begin{aligned}
 a_{13}(i) &= \frac{1}{\cos(i) + 2}, \\
 a_{21}(i) &= \frac{(i+2)(i+2\cos(i)+2\sin(i))}{(i+1)(\sin(i)+2)} \\
 &\quad + \frac{i\sin(i) + \cos(i)\sin(i)+3}{(i+1)(\sin(i)+2)}, \\
 a_{22}(i) &= 1 - 2e^i, \\
 a_{23}(i) &= -\frac{e^{-i} + 1}{\cos(i) + 2}, \\
 a_{24}(i) &= \frac{e^{2i}(i+2)(\cos(i)+1)(\sin(i)+1)}{i+1}, \\
 a_{31}(i) &= -\frac{i+2}{\sin(i)+2}, \\
 a_{34}(i) &= -\frac{e^{2i}(i+2)(\cos(i)+1)}{i+1}, \\
 a_{44}(i) &= \frac{e^{2i}(i+2)}{i+1}.
 \end{aligned}$$

The condition (2) is satisfied since

$$\begin{aligned}
 &\det[E(i)\lambda - A(i)] \\
 &= -\frac{(i+2)^2(2e^i + \lambda e^i - 1)(2\lambda + i + \lambda \sin(i) + 1)e^{2i}}{(i+1)^2(\cos(i)+2)(\sin(i)+2)} \\
 &\neq 0.
 \end{aligned} \tag{14}$$

In this case,

$$\begin{aligned}
 P(i) &= \begin{bmatrix} 1 + e^{-i} & 1 & 1 + \sin(i) & 0 \\ 0 & 0 & 1 & 1 + \cos(i) \\ 2 + \cos(i) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i+1}{i+2} \end{bmatrix}, \\
 Q(i) &= \begin{bmatrix} 0 & \frac{i+1}{i+2} & 0 & 0 \\ e^{-i} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-2i} \end{bmatrix}
 \end{aligned} \tag{15}$$

and

$$\begin{aligned}
 \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} &= P(i)E(i)Q(i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
 \begin{bmatrix} A_1(i) & 0 \\ 0 & I_{n_2} \end{bmatrix} &= P(i)A(i)Q(i)
 \end{aligned}$$

$$\begin{aligned}
 &= \begin{bmatrix} e^{-i} - 2 & 1 + \cos(i) & 0 & 0 \\ 0 & -\frac{i+1}{2+\sin(i)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
 \begin{bmatrix} B_1(i) \\ B_2(i) \end{bmatrix} &= P(i)B(i) = \begin{bmatrix} e^{-i} & 0 \\ 0 & \sin(i) \\ 1 & 0 \\ 0 & 2i \end{bmatrix},
 \end{aligned} \tag{16}$$

$$(n_1 = n_2 = 2).$$

Equations (6) have the form

$$\begin{aligned}
 \begin{bmatrix} \bar{x}_{11,i+1} \\ \bar{x}_{12,i+1} \end{bmatrix} &= \begin{bmatrix} e^{-i} - 2 & 1 + \cos(i) \\ 0 & -\frac{i+1}{2+\sin(i)} \end{bmatrix} \begin{bmatrix} \bar{x}_{11,i} \\ \bar{x}_{12,i} \end{bmatrix} \\
 &\quad + \sum_{j=2}^{i+1} c_j \begin{bmatrix} \bar{x}_{11,i-j+1} \\ \bar{x}_{12,i-j+1} \end{bmatrix} \\
 &\quad + \begin{bmatrix} e^{-i} & 0 \\ 0 & \sin(i) \end{bmatrix} \begin{bmatrix} u_{1,i} \\ u_{2,i} \end{bmatrix}
 \end{aligned} \tag{17a}$$

and

$$\begin{aligned}
 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{21,i+1} \\ \bar{x}_{22,i+1} \end{bmatrix} &= \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{x}_{21,i} \\ \bar{x}_{22,i} \end{bmatrix} \\
 &\quad + \sum_{j=2}^{i+1} c_j \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{21,i-j+1} \\ \bar{x}_{22,i-j+1} \end{bmatrix} \\
 &\quad + \begin{bmatrix} 1 & 0 \\ 0 & 2i \end{bmatrix} \begin{bmatrix} u_{1,i} \\ u_{2,i} \end{bmatrix}.
 \end{aligned} \tag{17b}$$

The solution of (17a) is given by (7a) and (7b) with the matrices $A_1(i)$ and $B_1(i)$ defined by (16).

From (17b), we have

$$\begin{aligned}
 \bar{x}_{22,i} &= -2iu_{2,i}, \\
 \bar{x}_{21,i} &= -2(i+1)u_{2,i+1} + \alpha 2iu_{2,i} \\
 &\quad + \sum_{j=2}^{i+1} c_j 2(i-j+1)u_{2,i-j+1} - u_{1,i}, \quad i \in \mathbb{Z}_+.
 \end{aligned} \tag{18}$$

The solution of Eqn. (1a) with (13) is given by

$$x(i) = \begin{bmatrix} x_1(i) \\ x_2(i) \\ x_3(i) \\ x_4(i) \end{bmatrix} = Q(i) \begin{bmatrix} \bar{x}_{11,i} \\ \bar{x}_{12,i} \\ \bar{x}_{21,i} \\ \bar{x}_{22,i} \end{bmatrix}, \quad i \in \mathbb{Z}_+, \tag{19}$$

where $Q(i)$ is defined by (14) and the components of the state vector $\bar{x}(i)$ by (7a) and (7b) with $A_1(i)$ and $B_1(i)$ defined by (16). \blacklozenge

3. Positive systems

Definition 1. The fractional descriptor time-varying discrete-time linear system (1) is called (internally) *positive* if and only if $x_i \in \mathbb{R}_+^n$ and $y_i \in \mathbb{R}_+^p$, $i \in \mathbb{Z}_+$, for

any admissible initial conditions $x_0 \in \mathbb{R}_+^n$ and all inputs $u_i \in \mathbb{R}_+^m, i \in \mathbb{Z}_+$.

The matrix $Q(i) \in \mathbb{R}^{n \times n}, i \in \mathbb{Z}_+$, is called *monomial* if in each row and column only one entry is positive and the remaining entries are zero for $i \in \mathbb{Z}_+$.

It is well-known (Kaczorek, 2001) that $Q^{-1}(i) \in \mathbb{R}_+^{n \times n}, i \in \mathbb{Z}_+$, if and only if the matrix is monomial.

It is assumed that for the positive system (1) the decomposition (4) is possible for a monomial matrix $Q(i)$. In this case,

$$x_i = Q(i)\bar{x}_i \in \mathbb{R}_+^n \Leftrightarrow \bar{x}_i \in \mathbb{R}_+^n, \quad i \in \mathbb{Z}_+. \quad (20)$$

It is also well-known that premultiplication of Eqn. (1a) by the matrix $P(i)$ does not change its solution $x_i, i \in \mathbb{Z}_+$.

From (11) it follows that $\bar{x}_{2,i} \in \mathbb{R}_+^{n_2}, i \in \mathbb{Z}_+$, for $u_i \in \mathbb{R}_+^m, i \in \mathbb{Z}_+$, if and only if

$$-B_2(i) \in \mathbb{R}_+^{n_2 \times m} \quad \text{for } i \in \mathbb{Z}_+. \quad (21)$$

Therefore, the following theorem has been proved.

Theorem 2. *Let the decomposition (4) of the system be possible for a monomial matrix $Q(i) \in \mathbb{R}_+^{n \times n}, i \in \mathbb{Z}_+$. The substitution (6b) is positive if and only if the condition (21) is satisfied.*

Theorem 3. *Let the decomposition (4) of the system be possible for a monomial matrix $Q(i) \in \mathbb{R}_+^{n \times n}, i \in \mathbb{Z}_+$. The substitution (6a) for $0 < \alpha < 1$ is positive if and only if*

$$A_{1\alpha}(i) \in \mathbb{R}_+^{n_1 \times n_1}, \quad B_1(i) \in \mathbb{R}_+^{n_1 \times m}, \quad i \in \mathbb{Z}_+. \quad (22)$$

Proof. As for sufficiency, if $0 < \alpha < 1$, then from (3b) and (1d) we have

$$c_2 = (-1)^3 \frac{\alpha(\alpha-1)}{2} > 0 \quad (23a)$$

and

$$c_{j+1} = (-1)^{j+1} \binom{\alpha}{j+1} = \frac{(j-\alpha)}{j+1} c_j > 0, \quad j = 2, 3, \dots \quad (23b)$$

From (7) and (23), it follows that $\bar{x}_{1,i} \in \mathbb{R}_+^{n_1}, i \in \mathbb{Z}_+$, for $x_0 \in \mathbb{R}_+^n$ and $u_i \in \mathbb{R}_+^m, i \in \mathbb{Z}_+$, if the condition (22) is satisfied. The necessity can be shown in a similar way as for standard descriptor systems in the work of Kaczorek (2015b). ■

Theorem 4. *Let the decomposition (4) of the system be possible for a monomial matrix $Q(i) \in \mathbb{R}_+^{n \times n}, i \in \mathbb{Z}_+$. The system (1) for $0 < \alpha < 1$ is positive if and only if*

1. the conditions (22) are satisfied,
2. (21) holds,

3. $C(i) \in \mathbb{R}_+^{p \times n}$ for $i \in \mathbb{Z}_+$.

Proof. By Theorems 3 and 2 the solutions (6a) and (6b) are positive if and only if the conditions (21) and (22) are met. From (1b) and (5), we have

$$y_i = C(i)Q(i)Q^{-1}(i)x_i = \bar{C}(i)\bar{x}_i, \quad i \in \mathbb{Z}_+, \quad (24a)$$

where

$$\bar{C}(i) = C(i)Q(i). \quad (24b)$$

For a monomial matrix $Q(i) \in \mathbb{R}_+^{n \times n}$, from (22) we have

$$\begin{aligned} \bar{C}(i) \in \mathbb{R}_+^{p \times n}, \quad i \in \mathbb{Z}_+ \\ \Leftrightarrow C(i) \in \mathbb{R}_+^{p \times n}, \quad i \in \mathbb{Z}_+, \end{aligned} \quad (25)$$

and

$$y_i \in \mathbb{R}_+^p, \quad i \in \mathbb{Z}_+, \Leftrightarrow C(i) \in \mathbb{R}_+^{p \times n}, \quad i \in \mathbb{Z}_+. \quad (26)$$

■

Example 2. Consider now the fractional descriptor time-varying system described by Eqn. (1) with the matrices

$$E(i) = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2\sin(i)+4} \\ -\cos(i)-1 & \frac{1}{\cos(i)+2} & 0 & -\frac{e^{-i}+2}{2\sin(i)+4} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B(i) = \begin{bmatrix} -\frac{1}{\sin(i)+2} \\ e^{-i} + \frac{e^{-i}+2}{\sin(i)+2} \\ 0 \\ 0 \\ 0 \\ -(\cos(i)+1)(e^{-i} + \sin(i) + 2) \\ e^{-i} + \sin(i) + 2 \\ -1 \end{bmatrix}, \quad (27)$$

$$C(i) = \begin{bmatrix} 0 & \frac{1}{\cos(i)+2} & 0 & 0.5 \\ \frac{i+2}{i+1} & 0 & \frac{e^{-i}}{e^{-i}+1} & 0 \end{bmatrix},$$

$$A(i) = \begin{bmatrix} 0 & 0 & a_{13}(i) & 0 \\ a_{21}(i) & a_{22}(i) & a_{23}(i) & a_{24}(i) \\ a_{31}(i) & 0 & 0 & a_{34}(i) \\ 0 & 0 & 0 & a_{44}(i) \end{bmatrix},$$

where

$$\begin{aligned}
 a_{13}(i) &= \frac{1}{(\sin(i) + 2)(e^{-i} + 1)}, \\
 a_{21}(i) &= -e^{-i} - \cos(i) - \sin(i) - e^{-i} \cos(i), \\
 a_{22}(i) &= \frac{i + 1}{(i + 2)(\cos(i) + 2)}, \\
 a_{23}(i) &= -\frac{e^{-i} + 2}{(\sin(i) + 2)(e^{-i} + 1)}, \\
 a_{24}(i) &= \frac{(i + 2)(\cos(i) + 1)(e^{-i} + 1)}{2(i + 1)}, \\
 a_{31}(i) &= e^{-i} + 1, \\
 a_{34}(i) &= -\frac{(i + 2)(e^{-i} + 1)}{2(i + 1)}, \\
 a_{44}(i) &= \frac{i + 2}{2(i + 1)}.
 \end{aligned}$$

The condition (2) is satisfied since

$$\begin{aligned}
 &\det[E(i)\lambda - A(i)] \\
 &= \frac{(e^{-i} - \lambda + 1)(i - 2\lambda - \lambda i + 1)}{2(i + 1)(\cos(i) + 2)(\sin(i) + 2)(e^{-i} + 1)} \quad (28) \\
 &\neq 0.
 \end{aligned}$$

In this case,

$$\begin{aligned}
 P &= \begin{bmatrix} 2 + e^{-i} & 1 & 1 + \cos(i) & 0 \\ 0 & 0 & 1 & 1 + e^{-i} \\ 2 + \sin(i) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i+1}{i+2} \end{bmatrix}, \\
 Q &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 + \cos(i) & 0 & 0 & 0 \\ 0 & 0 & 1 + e^{-i} & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad (29)
 \end{aligned}$$

and

$$\begin{aligned}
 \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} &= P(i)E(i)Q(i) \\
 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},
 \end{aligned}$$

$$\begin{aligned}
 \begin{bmatrix} A_1(i) & 0 \\ 0 & I_{n_2} \end{bmatrix} &= P(i)A(i)Q(i) \\
 &= \begin{bmatrix} \frac{i+1}{i+2} & 1 - \sin(i) & 0 & 0 \\ 0 & 1 + e^{-i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
 \end{aligned}$$

$$\begin{aligned}
 \begin{bmatrix} B_1(i) \\ B_2(i) \end{bmatrix} &= P(i)B(i) = \begin{bmatrix} e^{-i} & 0 \\ 0 & 1 + \sin(i) \\ -1 & 0 \\ 0 & -\frac{i+1}{i+2} \end{bmatrix}, \\
 \bar{C}(i) &= C(i)Q(i) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \frac{i+2}{i+1} & e^{-i} & 0 \end{bmatrix}.
 \end{aligned}$$

The descriptor system is positive since the tree conditions of Theorem 3 are satisfied. The matrix $Q(i)$ defined by (29) is monomial, the conditions (21) and (22) are met,

$$\begin{aligned}
 -B_2(i) &= \begin{bmatrix} 1 & 0 \\ 0 & \frac{i+1}{i+2} \end{bmatrix} \in \mathbb{R}_+^{2 \times 2}, \\
 A_1(i) &= \begin{bmatrix} \frac{i+1}{i+2} & 1 - \sin(i) \\ 0 & 1 + e^{-i} \end{bmatrix} \in \mathbb{R}_+^{2 \times 2},
 \end{aligned}$$

and

$$B_1(i) = \begin{bmatrix} e^{-i} & 0 \\ 0 & 1 + \sin(i) \end{bmatrix} \in \mathbb{R}_+^{2 \times 2}, \quad i \in \mathbb{Z}_+$$

and

$$C(i) = \begin{bmatrix} 0 & \frac{1}{\cos(i)+2} & 0 & 0.5 \\ \frac{i+2}{i+1} & 0 & \frac{e^{-i}}{e^{-i}+1} & 0 \end{bmatrix} \in \mathbb{R}_+^{2 \times 4}$$

for \mathbb{Z}_+ .

The solution to Eqn. (1) with the matrices $E(i)$, $A(i)$, $B(i)$ given by (27) can be found in a similar way as in Example 1. \blacklozenge

4. Stability of fractional positive descriptor systems

First we shall recall the basic definition and tests on stability of positive time-varying linear systems described by the equation (Kaczorek, 2015b)

$$\begin{aligned}
 x_{i+1} &= A(i)x_i, \\
 A(i) &\in \mathbb{R}_+^{n \times n}, \quad i \in \mathbb{Z}_+ = \{0, 1, \dots\}. \quad (30)
 \end{aligned}$$

Definition 2. The positive system (30) is called *asymptotically stable* if the norm $\|x_1\|$ of the state vector $x_i \in \mathbb{R}_+^n$, $i \in \mathbb{Z}_+$, satisfies the condition

$$\lim_{i \rightarrow \infty} \|x_i\| = 0 \quad (31)$$

for any finite $x_0 \in \mathbb{R}_+^n$.

Theorem 5. The positive system (30) is asymptotically stable if the norm $\|A(i)\|$ of the matrix $A(i)$, $i \in \mathbb{Z}_+$, satisfies the condition

$$\|A\| < 1 \quad \text{for } i \in \mathbb{Z}_+, \quad (32a)$$

where

$$\|A\| \geq \max_{0 \leq i \leq \infty} \|A(i)\| \quad \text{for } i \in \mathbb{Z}_+. \quad (32b)$$

The proof is given by Kaczorek (2015b).

Theorem 6. *The positive system (30) is asymptotically stable if its system matrix $A(i) = [a_{jk}(i)] \in \mathbb{R}_+^{n \times n}$ satisfies the condition*

$$\max_{0 \leq j \leq n} \sum_{k=1}^n a_{jk}(i) < 1 \quad \text{for } i \in \mathbb{Z}_+ \quad (33a)$$

or

$$\max_{0 \leq k \leq n} \sum_{j=1}^n a_{jk}(i) < 1 \quad \text{for } i \in \mathbb{Z}_+. \quad (33b)$$

The proof is given by Kaczorek (2015b).

Theorem 7. *The positive system (31) is asymptotically stable if its system matrix*

$$A(i) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_0(i) & a_1(i) & a_2(i) & \dots & a_{n-1}(i) \end{bmatrix} \quad (34)$$

$\in \mathbb{R}_+^{n \times n}$

satisfies the condition

$$\sum_{k=0}^{n-1} a_k(i) < 1 \quad \text{for } i \in \mathbb{Z}_+. \quad (35)$$

The proof is given by Kaczorek (2015b).

Consider the fractional descriptor system (1) for $B(i)u_i = 0, i \in \mathbb{Z}_+$,

$$E(i)\Delta^\alpha x_{i+1} = A(i)x_i, i \in \mathbb{Z}_+ = \{0, 1, \dots\}. \quad (36)$$

If the assumption (2) is satisfied, then the system can be decomposed into the subsystems

$$\bar{x}_{1,i+1} = A_{1\alpha}(i)\bar{x}_{1,i} + \sum_{j=2}^{i+1} c_j \bar{x}_{1,i-j+1}, \quad (37)$$

$$N\bar{x}_{2,i+1} = (N_\alpha + I_{n_2})\bar{x}_{2,i} + \sum_{j=2}^{i+1} c_j N\bar{x}_{2,i-j+1}, \quad (38)$$

where $\bar{x}_{1,i}$, $\bar{x}_{2,i}$ and $A_{1\alpha}(i)$ are defined by (5) and (6c), respectively.

Note that $\bar{x}_{2,i} = 0$ for $i = 1, 2, \dots$, and the stability of the fractional descriptor system (36) depends only on that stability of the subsystem (37). Therefore, the following theorem has been proved.

Theorem 8. *The positive system (36) is asymptotically stable if its system matrix*

$$\|A_{1\alpha}\| < 1, \quad (39a)$$

where

$$\|A_{1\alpha}\| \geq \max_{0 \leq i \leq \infty} \|A_1(i)\| \quad \text{for } i \in \mathbb{Z}_+. \quad (39b)$$

Proof. By Theorem 8 and Definition 2, the fractional descriptor positive system (36) is asymptotically stable if and only if

$$\begin{aligned} \lim_{i \rightarrow \infty} \|\bar{x}_{1,i}\| &= \lim_{i \rightarrow \infty} \|\Phi_1(i, 0)\bar{x}_{10}\| \\ &= \lim_{i \rightarrow \infty} \|\Phi_1(i, 0)\| = 0 \end{aligned} \quad (40)$$

for any finite \bar{x}_{10} .

From (40), it follows that

$$\lim_{i \rightarrow \infty} \|\Phi_1(i, 0)\| = 0$$

if (39a) holds since

$$c_j < \frac{1}{a^j}, \quad j = 2, 3, \dots, \quad (41a)$$

and

$$\begin{aligned} \|\Phi_1(i+1, 0)\| &\leq \|A_{1\alpha}(i)\| \|\Phi_1(i, 0)\| \\ &+ \sum_{j=2}^{i+1} \frac{1}{a^j} \|\Phi_1(i-j+1, 0)\|. \end{aligned} \quad (41b)$$

■

To check the asymptotic stability of the fractional descriptor positive system (36), Theorems 5–7 can be used in the condition (39a).

Example 3. Consider now the fractional descriptor time-varying system (36) with the matrices

$$E(i) = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2 \sin(i)+4} \\ -\cos(i) - 1 & \frac{1}{\cos(i)+2} & 0 & -\frac{e^{-i}+2}{2 \sin(i)+4} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A(i) = \begin{bmatrix} 0 & 0 & a_{13}(i) & 0 \\ a_{21}(i) & a_{22}(i) & a_{23}(i) & a_{24}(i) \\ a_{31}(i) & 0 & 0 & a_{34}(i) \\ 0 & 0 & 0 & a_{44}(i) \end{bmatrix}, \quad (42)$$

where

$$\begin{aligned} a_{13}(i) &= \frac{1}{(\sin(i) + 2)(e^{-i} + 1)}, \\ a_{21}(i) &= 0.1 - 0.1 \cos(i) - 0.2 \sin(i) \\ &\quad - 0.03e^{-i} \cos(i) - 0.03e^{-i}, \\ a_{22}(i) &= \frac{0.1(i+1)}{(i+2)(\cos(i) + 2)}, \end{aligned}$$

$$\begin{aligned}
 a_{23}(i) &= -\frac{e^{-i} + 2}{(\sin(i) + 2)(e^{-i} + 1)}, \\
 a_{24}(i) &= \frac{(i + 2)(\cos(i) + 1)(e^{-i} + 1)}{2(i + 1)}, \\
 a_{31}(i) &= 0.03e^{-i} + 0.1, \\
 a_{34}(i) &= -\frac{(i + 2)(e^{-i} + 1)}{2(i + 1)}, \\
 a_{44}(i) &= \frac{i + 2}{2(i + 1)},
 \end{aligned}$$

and $\alpha = 0.5$.

The condition (2) is satisfied since

$$\begin{aligned}
 &\det E(i) \\
 &= \begin{vmatrix} 0 & 0 & 0 & \frac{1}{2\sin(i)+4} \\ -\cos(i) - 1 & \frac{1}{\cos(i)+2} & 0 & -\frac{e^{-i}+2}{2\sin(i)+4} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \\
 &= 0
 \end{aligned} \tag{43a}$$

and

$$\begin{aligned}
 &\det[E(i)\lambda - A(i)] \\
 &= \frac{-10(i + 2)\lambda^2 + [0.6e^{-i} + i(0.3e^{-i} + 2) + 3]\lambda}{(e^{-i} + 1)(i + 1)(10\sin(2i) + 40\cos(i) + 40\sin(i) + 80)} \\
 &\quad - \frac{[0.03e^{-i} + i(0.03e^{-i} + 0.1) + 0.1]}{(e^{-i} + 1)(i + 1)(10\sin(2i) + 40\cos(i) + 40\sin(i) + 80)} \\
 &\neq 0.
 \end{aligned} \tag{43b}$$

In this case,

$$\begin{aligned}
 P &= \begin{bmatrix} 2 + e^{-i} & 1 & 1 + \cos(i) & 0 \\ 0 & 0 & 1 & 1 + e^{-i} \\ 2 + \sin(i) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i+1}{i+2} \end{bmatrix}, \\
 Q &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 + \cos(i) & 0 & 0 & 0 \\ 0 & 0 & 1 + e^{-i} & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}
 \end{aligned} \tag{44}$$

and

$$\begin{aligned}
 \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} &= P(i)E(i)Q(i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
 \begin{bmatrix} A_1(i) & 0 \\ 0 & I_{n_2} \end{bmatrix} &= P(i)A(i)Q(i) \\
 &= \begin{bmatrix} 0.1\frac{i+1}{i+2} & 0.2(1 - \sin(i)) & 0 & 0 \\ 0 & 0.1(1 + 0.3e^{-i}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
 \end{aligned} \tag{45}$$

Note that the matrix $Q(i)$ is monomial and the matrix

$$A_1(i) = \begin{bmatrix} 0.1\frac{i+1}{i+2} & 0.2(1 - \sin(i)) \\ 0 & 0.1(1 + 0.3e^{-i}) \end{bmatrix} \in \mathbb{R}_+^{2 \times 2} \tag{46}$$

for $i \in \mathbb{Z}_+$ satisfies the condition (39a) since, by (33a), we have

$$\begin{aligned}
 &\|A_{1\alpha}\| \\
 &= \|A_1(i) + I_{n_1}\alpha\| \\
 &= \left\| \begin{bmatrix} 0.1\frac{i+1}{i+2} + 0.5 & 0.2(1 - \sin(i)) \\ 0 & 0.1(1 + 0.3e^{-i}) + 0.5 \end{bmatrix} \right\| \\
 &< 1 \quad \text{for } i \in \mathbb{Z}_+.
 \end{aligned} \tag{47}$$

Therefore, the fractional descriptor system is asymptotically stable.

5. Concluding remarks

The Weierstrass–Kronecker theorem on the decomposition of the regular pencil were extended to fractional descriptor time-varying discrete-time linear systems. A method for computing solutions of fractional systems were proposed. Necessary and sufficient conditions for positivity were established. Sufficient conditions for asymptotic stability and some simple tests for checking stability were proposed. The discussion was illustrated by examples of fractional descriptor positive systems. The findings can be extended to fractional descriptor time-varying continuous-time linear systems.

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