Article citation info:
HU L, SU P, PENG R, ZHANG Z. Fuzzy Availability Assessment for Discrete Time Multi-State System under Minor Failures and Repairs by Using Fuzzy $L_z$-transform. Eksploatacja i Niezawodnosc – Maintenance and Reliability 2017; 19 (2): 179–190, http://dx.doi.org/10.17531/ein.2017.2.5.

Linmin HU
Peng SU
Rui PENG
Zhilan ZHANG

# FUZZY AVAILABILITY ASSESSMENT FOR DISCRETE TIME MULTI-STATE SYSTEM UNDER MINOR FAILURES AND REPAIRS BY USING FUZZY $L_Z$-TRANSFORM

# WYKORZYSTANIE ROZMYTEJ TRANSFORMATY DO OCENY ROZMYTEJ GOTOWOŚCI EKSPLOATACYJNEJ DYSKRETNEGO W CZASIE SYSTEMU WIELOSTANOWEGO DZIAŁAJĄCEGO W TRYBIE DROBNYCH USZKODZEŃ I NAPRAW

*This paper studies assessment approach of dynamic fuzzy availability for a discrete time multi-state system under minor failures and repairs. Traditionally, it was assumed that the exact reliability data of a component/system with discrete time are given in reliability analysis. In practical engineering, it is difficult to obtain precise data to evaluate the characteristics of a component/system. To overcome the problem, fuzzy set theory is employed to deal with dynamic availability assessment for a discrete time multi-state system in this paper. A fuzzy discrete time Markov model with fuzzy transition probability matrix is proposed to analyze the fuzzy state probability of each component at any discrete time. The fuzzy $L_z$-transform of the discrete-state discrete-time fuzzy Markov chain is developed to extend the $L_z$-transform of the discrete-state continuous-time Markov model with crisp sets. Based on the α-cut approach and the fuzzy $L_z$-transform, the dynamic fuzzy availability of the system is computed by using parametric programming technique. To illustrate the proposed method, a flow transmission system is analyzed as a numerical example.*

**Keywords**: *discrete time, Markov model, fuzzy $L_z$-transform, multi-state system, availability.*

*W niniejszej pracy badano metodę oceny dynamicznej, rozmytej gotowości eksploatacyjnej (dostępności) dyskretnego w czasie systemu wielostanowego pracującego w trybie drobnych uszkodzeń i napraw. Tradycyjnie zwykło się zakładać, że analiza niezawodności dostarcza dokładnych danych niezawodnościowych na temat danego dyskretnego w czasie komponentu/systemu. W praktyce inżynieryjnej jednak trudno jest uzyskać dokładne dane do oceny właściwości komponentu/systemu. W niniejszej pracy zaproponowano jak problem ten można rozwiązać wykorzystując do oceny dynamicznej gotowości dyskretnego w czasie systemu wielostanowego, teorię zbiorów rozmytych. Rozmyty model Markowa z dyskretnym czasem i rozmytą macierzą prawdopodobieństw przejść zastosowano do analizy rozmytego prawdopodobieństwa stanu każdego elementu w dowolnym czasie dyskretnym. Opracowano rozmytą transformatę $L_z$ rozmytego, dyskretnego w stanach i czasie łańcucha Markowa, która pozwala poszerzyć transformatę $L_z$ modelu Markowa dyskretnego w stanach i ciągłego w czasie o zbiory ostre. W oparciu o metodę alfa przekrojów oraz rozmytą transformatę $L_z$, obliczono dynamiczną rozmytą gotowość eksploatacyjną systemu, wykorzystując do tego celu technikę programowania parametrycznego. Zastosowanie proponowanej metody zilustrowano na przykładzie liczbowym analizując układ przesyłu.*

**Słowa kluczowe**: *czas dyskretny, model Markowa, rozmyta transformata, system wielostanowy, dostępność.*

## 1. Introduction

The availability, as a performance measure, is one of the most important indicators for characterizing a repairable system and its components. For repairable multi-state systems with various performance levels, the availability is more meaningful than reliability to measure the effectiveness of the system to satisfy consumer demand.

In the past few years, a variety of methods are available in the literature for analyzing the availability of repairable multi-state systems. Some of them are Monte Carlo simulation [41, 42], stochastic Petri nets [19, 21], universal generating function [24, 32], Markov models [6, 37] and the combinations of the above methods [9]. However, conventional availability analysis methods for repairable multi-state systems are based on the continuous time models. In some engineering circumstances, it is sometimes impossible or inconvenient to measure the lifetimes and repair times length of some systems (components) on a continuous scale. In such situations, the lifetimes and repair times can be expressed in terms of the number of working and repairing periods (cycles), respectively. Thus, it is essential to construct discrete time reliability models for repairable multi-state systems.

The discrete time reliability has drawn continuous attention in both model analysis and problem solution. Bracquemond and Gaudoin [5] presented a good overview of discrete probability distributions used in reliability for modeling discrete lifetimes of non-repairable systems. The discrete time reliability modeling for general binary systems can be found in [4, 22, 34]. Eryilmaz [14], Guerry [16] and Sadek and Limnios [35], presented the discrete time reliability models for Markov multi-state systems. The discrete time reliability models for semi-Markov multi-state systems were investigated in [1, 4, 10]. However, most of the reported works mainly focus on the issues of discrete time systems with the exact reliability data. As stated in Garg [15], the complicated system has the massive fuzzy uncertainty due to which it is difficult to get the exact probability of the events. Thus, it is of large practical value to investigate the availability assessment for discrete time repairable system with fuzzy uncertainty.

Fuzzy reliability theory, which is based on the fuzzy set theory introduced by Zadeh [39,40], is becoming a useful tool for dealing with the imprecision and uncertainty problems of reliability evaluation for many industrial systems. The basic concept and theory of the fuzzy reliability have been introduced and developed by several authors [7, 18, 27, 38]. More recently, fuzzy reliability research has focused on reliability/availability evaluation of fuzzy multi-state system according to various analysis methods. Ding and Lisnianski [12] firstly provided the basic definition of the fuzzy multi-state system model, and then investigated the system reliability based on the proposed fuzzy universal generating function technique. The concepts of relevancy, coherency, equivalence, and performance evaluation algorithms for the fuzzy multi-state systems were given by Ding, Zuo, Lisnianski and Tian [13]. Liu and Huang [33] proposed a fuzzy Markov model to establish dynamic state probabilities of fuzzy multi-state elements, and investigated a dynamic fuzzy reliability assessment method for fuzzy multi-state systems. Li, Chen, Yi and Tao [25] developed interval universal generating function to analyze the reliability of multi-state systems when the available data of components are insufficient. Bamrungsetthapong and Pongpullponsak [3] studied the fuzzy system reliability for a non-repairable multi-state series-parallel system by using fuzzy Bayesian inference based on prior interval probabilities. Hu, Yue and Tian [17] provided a special assessment approach for evaluating the fuzzy steady-state availability of a repairable multi-state series-parallel system based on fuzzy universal generating function and parametric programming technique.

However, all the reported works for fuzzy multi-state system reliability mainly focus on the issues of dynamic reliability/availability assessment for continuous time multi-state systems or steady-state availability assessment for repairable multi-state systems. The fuzzy reliability/availability assessment for a discrete-time multi-state system has been seldom discussed in the literature. Moreover, availability has a wider scope than reliability as it takes into account maintenance time analysis in addition to failure time analysis [8]. Therefore, the main objective of our work is to present an analytical technique of dynamic fuzzy availability assessment for a discrete time repairable multi-state system (DTRMSS) with fuzzy consumer demand. The technique called the fuzzy $L_z$-transform is based on the combination of fuzzy universal generating function technique and discrete time Markov process method. In the presented paper, the fuzzy $L_z$-transform for discrete-state discrete-time Markov process is developed to extend the $L_z$-transform for discrete-state continuous-time Markov process [28] with crisp sets. Minor failures and repairs of components [30] (that transitions can only occur between adjacent states) with fuzzy state transition probabilities are considered. Fuzzy discrete-state discrete-time Markov model is proposed to perform fuzzy state probability analysis for each component in dynamic modes. The dynamic fuzzy state probability, dynamic fuzzy performance level and dynamic fuzzy availability of the system are evaluated by the proposed fuzzy$L_z$-transform method, and the $\alpha$-cut of dynamic fuzzy availability is computed according to parametric programming technique.

The rest of this paper is organized as follows. The discrete time fuzzy Markov model for a repairable multi-state component is presented in Section 2. Section 3 describes the definition of fuzzy $L_z$-transform. The fuzzy dynamic availability assessment method for the DTRMSS is given in Section 4. The analytical technique and assessment method are illustrated in Section 5 via a flow transmission system. Conclusions are given in Section 6.

## 2. Discrete time repairable multi-state component

### 2.1. Markov model for repairable multi-state component

Consider a discrete time repairable multi-state component with $m$ different possible states $i$ $\left(i=1,2,\ldots,m\right)$, where 1 and $m$ represent perfect function and complete failure states, respectively. Assume that these states of the component correspond to different performance levels $x_1,x_2,\ldots,x_m$, where $x_i$ is the performance level associated with the state $i$. Let $x(k)$ denote the performance level of the component at the end of the $k$th time period (such as hour, day, month, etc.), $x(k)$ takes values from $\{x_1,x_2,\ldots,x_m \,|\, x_i \ge 0, i=1,2,\ldots,m\}$ : $x\left(k\right)\in \{x_1,x_2,\ldots,x_m \,|\, x_i \ge 0, i=1,2,\ldots,m\}$, $k=0,1,\ldots$ . Thus, the performance level $x(k)$ is a discrete-state discrete-time stochastic process.

For the repairable multi-state component, we assume that minor failures and repairs [30] are considered. Minor failures are failures causing component transition from state $i$ to the adjacent state $i+1$, and minor repairs are repairs causing component transition from state $i$ to the adjacent state $i-1$. In some components development process, the life of the components need to test. The geometric distribution has an important application in testing the life of the components. Sarhan, Guess and Usher [36] stated that the geometric distribution is a common discrete distribution used to model the lifetime of a device. Eryilmaz [14] investigated the mean residual life of discrete time multi-state systems based on the geometric distribution. In our work, it is assumed that the time between transitions from one state to another has geometric distribution with constant mean values $1/q_{i,i+1}$ and $1/p_{i,i-1}$. Let $T_{c_i}$ and $T_{r_i}$ denote the time between transitions from the state $i$ to the state $i+1$ and from the state $i$ to the state $i-1$, respectively. We have:

$$P\left\{T_{c_i}=k\right\}=q_{i,i+1}\left(1-q_{i,i+1}\right)^{k-1}, i=1,2,\ldots,m-1, k=1,2,\ldots$$

and:

$$P\left\{T_{r_i}=k\right\}=p_{i,i-1}\left(1-p_{i,i-1}\right)^{k-1},\ i=2,3,\ldots,m,\quad k=1,2,\ldots$$

It is obvious that the process $x(k)$ has the property of being memory-less. Furthermore, for the repairable multi-state component, its performance level $x(k)$ ( $k=0,1,\ldots$ ) is a discrete-state discrete-time Markov chain with the following one-step transition probability matrix:

$$\mathbf{P}=\begin{bmatrix} P_{11} & P_{12} & 0 & \cdots & 0 & 0 & 0 \\ P_{21} & P_{22} & P_{23} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & P_{(m-1)(m-2)} & P_{(m-1)(m-1)} & P_{(m-1)m} \\ 0 & 0 & 0 & \cdots & 0 & P_{m(m-1)} & P_{mm} \end{bmatrix}$$

That is, the one-step transition probability $P_{ij}$ from component state $i$ to component state $j$ is determined by:

$$P_{ij}=P\left\{x\left(k+1\right)=j\,\middle|\,x\left(k\right)=i\right\}=\begin{cases} q_{i,i+1}, & j=i+1,\ i=1,2,\ldots,m-1 \\ p_{i,i-1}, & j=i-1,\ i=2,3,\ldots,m \\ 1-q_{1,2}, & j=i=1 \\ 1-p_{m,m-1}, & j=i=m \\ 1-p_{i,i-1}-q_{i,i+1}, & j=i=2,3,\ldots,m-1 \\ 0, & \text{otherwise} \end{cases}$$

(1)

where $0 \le q_{i,i+1} \le 1$ $(i=1,2,\dots,m-1)$ , $0 \le p_{i,i-1} \le 1$ $(i=2,3,\dots,m)$ and $0 \le q_{i,i+1}+p_{i,i-1} \le 1$ $(i=2,3,\dots,m-1)$ .

Let $P_i(k)$ denote the probability that the component is in state $i$ $(i=1,2,\dots,m)$ at time $k$ , the state probabilities of the component at any discrete time can be calculated by the matrix equation:

$$\left[P_1(k),P_2(k),\cdots,P_m(k)\right]=\left[P_1(k-1),P_2(k-1),\cdots,P_m(k-1)\right]\cdot\mathbf{P} \quad (2)$$

with the initial conditions:

$$P_1(0)=1,\ P_2(0)=P_3(0)=\cdots=P_m(0)=0 \quad (3)$$

By solving the matrix equation (2) under the initial condition (3), we can determine the following matrix equation:

$$\left[P_1(k),P_2(k),\cdots,P_m(k)\right]=\left[P_1(0),P_2(0),\cdots,P_m(0)\right]\cdot\mathbf{P}^k \quad (4)$$

The matrix equation (4) can be written as:

$$\mathbf{P}_k=\mathbf{P}_0\cdot\mathbf{P}^k \quad (5)$$

where $\mathbf{P}_k=\left[P_1(k),P_2(k),\cdots,P_m(k)\right]$ is the row vector of the state probabilities of the component at time $k$ and $\mathbf{P}_0=\left[P_1(0),P_2(0),\cdots,P_m(0)\right]=[1,0,\cdots,0]$ is the row vector of the state probabilities of the component at time 0.

## 2.2. Fuzzy Markov model for repairable multi-state component

In this subsection, we define the fuzzy discrete time repairable multi-state component and propose a discrete time fuzzy Markov model to evaluate the dynamic fuzzy state probability for the discrete time repairable multi-state component.

### 2.2.1. Definition and assumption

The fuzzy discrete time repairable multi-state component is defined as the component in which the different state performance levels, the corresponding state probabilities or one-step transition probabilities between each pair of adjacent states are represented as fuzzy values. The general assumptions of a fuzzy discrete time repairable multi-state component are given as follows:

(1) State index of the component is a crisp value taking integer values only, and the state space is $\{1,2,,\dots,m\}$.

(2) State performance level $x(k)$ of the component at time $k$ $(k=0,1,\dots)$ can be measured as fuzzy value. We substitute $\tilde{x}(k)$ for $x(k)$ in the subsection 2.1 to denote the fuzzy performance level [12, 17, 33] of the component at time $k$ , and $\tilde{x}_i$ for $x_i$ in the subsection 2.1 to denote the fuzzy performance level associated with the state $i$ , $i=1,2,\dots,m$ . The fuzzy performance level $\tilde{x}(k)$ takes values from $\{\tilde{x}_1,\tilde{x}_2,\dots,\tilde{x}_m\}$. Because the $x_i$ value is presented as fuzzy number in the model, we have $x_i \in (\tilde{x}_i)_\alpha$ with $x_i \ge 0$ , where $(\tilde{x}_i)_\alpha$ is $\alpha$-cut of the fuzzy number $\tilde{x}_i$ for $0 \le \alpha \le 1$ .

(3) One-step transition probability of the component from one state to another state (that transitions can only occur between adjacent states) can be measured as fuzzy value. We substitute $\tilde{P}_{ij}$ for $P_{ij}$ in Eq. (1) to denote the fuzzy one-step transition probability [2, 23, 26] from the component state $i$ to the component state $j$ ( $j=i-1,\ i,\ i+1$). The $q_{i,i+1}$ $(i=1,2,\dots,m-1)$ and $p_{i,i-1}$ $(i=2,3,\dots,m)$ values are fuzzy values and we substitute $\tilde{q}_{i,i+1}$ for $q_{i,i+1}$ and $\tilde{p}_{i,i-1}$ for $p_{i,i-1}$ in Eq. (1). $\tilde{P}_{ij}$ can be determined by the fuzzy values $\tilde{q}_{i,i+1}$ and $\tilde{p}_{i,i-1}$ based on Eq. (1). Because the $q_{i,i+1}$ and $p_{i,i-1}$ values are presented as fuzzy numbers in the model, we put the following restrictions on the $\tilde{q}_{i,i+1}$ and $\tilde{p}_{i,i-1}$ values: $q_{i,i+1} \in (\tilde{q}_{i,i+1})_\alpha$ and $p_{i,i-1} \in (\tilde{p}_{i,i-1})_\alpha$ with $0 \le q_{i,i+1} \le 1$, $0 \le p_{i,i-1} \le 1$ and $0 \le q_{i,i+1}+p_{i,i-1} \le 1$ , where $(\tilde{q}_{i,i+1})_\alpha$ and $(\tilde{p}_{i,i-1})_\alpha$ are $\alpha$-cuts of the fuzzy numbers $\tilde{q}_{i,i+1}$ and $\tilde{p}_{i,i-1}$ for $0 \le \alpha \le 1$ , respectively.

### 2.2.2. Fuzzy Markov model

According to definition and assumptions of the fuzzy discrete time repairable multi-state component, the fuzzy performance level $\tilde{x}(k)$ ( $k=0,1,\dots$ ) forms a discrete-state discrete-time fuzzy Markov chain [2, 23, 26] with the following fuzzy one-step transition probability matrix:

$$\tilde{\mathbf{P}}=\begin{bmatrix} 1-\tilde{q}_{1,2} & \tilde{q}_{1,2} & 0 & \cdots & 0 & 0 & 0 \\ \tilde{p}_{2,1} & 1-\tilde{p}_{2,1}-\tilde{q}_{2,3} & \tilde{q}_{2,3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \tilde{p}_{m-1,m-2} & 1-\tilde{p}_{m-1,m-2}-\tilde{q}_{m-1,m} & \tilde{q}_{m-1,m} \\ 0 & 0 & 0 & \cdots & 0 & \tilde{p}_{m,m-1} & 1-\tilde{p}_{m,m-1} \end{bmatrix}$$

where the fuzzy uncertainty is on the state transition probabilities of the component from the state $i$ to the state $i+1$ and from the state $i$ to the state $i-1$ , but not on the fact that every row must add to 1. The fuzzy values $\tilde{q}_{i,i+1}$ and $\tilde{p}_{i,i-1}$ are restricted by $q_{i,i+1} \in (\tilde{q}_{i,i+1})_\alpha$ and $p_{i,i-1} \in (\tilde{p}_{i,i-1})_\alpha$ with $0 \le q_{i,i+1}+p_{i,i-1} \le 1$ .

With the fuzzy state transition probabilities, the state probability of the component in the state $i$ at time $k$ must also be a fuzzy value denoted as $\tilde{P}_i(k)$ . The dynamic fuzzy state probability $\tilde{P}_i(k)$ can be determined by:

$$\left[\tilde{P}_1(k),\tilde{P}_2(k),\cdots,\tilde{P}_m(k)\right]=\left[\tilde{P}_1(0),\tilde{P}_2(0),\cdots,\tilde{P}_m(0)\right]\cdot\tilde{\mathbf{P}}^k \quad (6)$$

The matrix equation (6) can be written as:

$$\tilde{\mathbf{P}}_k=\tilde{\mathbf{P}}_0\cdot\tilde{\mathbf{P}}^k \quad (7)$$

where $\tilde{\mathbf{P}}_k=\left[\tilde{P}_1(k),\tilde{P}_2(k),\cdots,\tilde{P}_m(k)\right]$ and $\tilde{\mathbf{P}}_0=[1,0,\cdots,0]$. By solving the matrix equations (6) or (7), the dynamic fuzzy state probability $\tilde{P}_i(k)$ at time $k$ can be given as function of fuzzy variables $\tilde{q}_{i,i+1}(i=1,2,\dots,m-1)$ and $\tilde{p}_{i,i-1}(i=2,3,\dots,m)$ , and then $\tilde{P}_i(k)$ can be written as $\tilde{P}_i(k)=\tilde{P}_i\left(k;\tilde{q}_{1,2},\tilde{q}_{2,3},\dots,\tilde{q}_{m-1,m};\tilde{p}_{2,1},\tilde{p}_{3,2},\dots,\tilde{p}_{m,m-1}\right)$. If the fuzzy state transition probabilities are represented by fuzzy vectors $\tilde{\mathbf{q}}=\{\tilde{q}_{1,2},\tilde{q}_{2,3},\dots,\tilde{q}_{m-1,m}\}$ and $\tilde{\mathbf{p}}=\{\tilde{p}_{2,1},\tilde{p}_{3,2},\dots,\tilde{p}_{m,m-1}\}$, we have $\tilde{P}_i(k)=\tilde{P}_i\left(k,\tilde{\mathbf{q}},\tilde{\mathbf{p}}\right)$.

Let $\eta_{\tilde{q}_{i,i+1}}$ and $\eta_{\tilde{p}_{i,i-1}}$ denote the membership functions of $\tilde{q}_{i,i+1}$ and $\tilde{p}_{i,i-1}$, respectively. The $\alpha$-cuts of $\tilde{q}_{i,i+1}$ and $\tilde{p}_{i,i-1}$ can be determined as crisp intervals:

$$\begin{aligned}\left(\tilde{q}_{i,i+1}\right)_\alpha &= \left[\min_{q_{i,i+1}\in Q_{i,i+1}}\left\{q_{i,i+1}\left|\eta_{\tilde{q}_{i,i+1}}\left(q_{i,i+1}\right)\ge\alpha\right.\right\},\max_{q_{i,i+1}\in Q_{i,i+1}}\left\{q_{i,i+1}\left|\eta_{\tilde{q}_{i,i+1}}\left(q_{i,i+1}\right)\ge\alpha\right.\right\}\right]\\ &=\left[\left(q_{i,i+1}\right)_\alpha^L,\left(q_{i,i+1}\right)_\alpha^U\right]\end{aligned} \tag{8}$$

$$\begin{aligned}\left(\tilde{p}_{i,i-1}\right)_\alpha &= \left[\min_{p_{i,i-1}\in P_{i,i-1}}\left\{p_{i,i-1}\left|\eta_{\tilde{p}_{i,i-1}}\left(p_{i,i-1}\right)\ge\alpha\right.\right\},\max_{p_{i,i-1}\in P_{i,i-1}}\left\{p_{i,i-1}\left|\eta_{\tilde{p}_{i,i-1}}\left(p_{i,i-1}\right)\ge\alpha\right.\right\}\right]\\ &=\left[\left(p_{i,i-1}\right)_\alpha^L,\left(p_{i,i-1}\right)_\alpha^U\right]\end{aligned} \tag{9}$$

where $Q_{i,i+1}$ and $P_{i,i-1}$ are the crisp universal sets of the state transition probabilities for the component from the state $i$ to the state $i+1$ and from the state $i$ to the state $i-1$, respectively. According to parametric programming technique [20, 33], the lower bound $\left(P_i(k)\right)_\alpha^L$ and upper bound $\left(P_i(k)\right)_\alpha^U$ of the $\alpha$-cut of $\tilde{P}_i\left(k,\tilde{\mathbf{q}},\tilde{\mathbf{p}}\right)$ can be computed as:

$$\begin{aligned}&\left(P_i(k)\right)_\alpha^L = \min P_i\left(k,\mathbf{q},\mathbf{p}\right) \quad \left(k=1,2,\ldots,0\le\alpha\le1\right)\\ &\quad \text{s.t.}\ \left(q_{i,i+1}\right)_\alpha^L \le q_{i,i+1} \le \left(q_{i,i+1}\right)_\alpha^U,\quad i=1,2,\ldots,m-1\\ &\quad\quad \left(p_{i,i-1}\right)_\alpha^L \le p_{i,i-1} \le \left(p_{i,i-1}\right)_\alpha^U,\quad i=2,3,\ldots,m\\ &\quad\quad 0\le q_{i,i+1}+p_{i,i-1}\le 1,\quad i=2,3,\ldots,m-1\end{aligned} \tag{10}$$

and:

$$\begin{aligned}&\left(P_i(k)\right)_\alpha^U = \max P_i\left(k,\mathbf{q},\mathbf{p}\right) \quad \left(k=1,2,\ldots,0\le\alpha\le1\right)\\ &\quad \text{s.t.}\ \left(q_{i,i+1}\right)_\alpha^L \le q_{i,i+1} \le \left(q_{i,i+1}\right)_\alpha^U,\quad i=1,2,\ldots,m-1\\ &\quad\quad \left(p_{i,i-1}\right)_\alpha^L \le p_{i,i-1} \le \left(p_{i,i-1}\right)_\alpha^U,\quad i=2,3,\ldots,m\\ &\quad\quad 0\le q_{i,i+1}+p_{i,i-1}\le 1,\quad i=2,3,\ldots,m-1\end{aligned} \tag{11}$$

where $\mathbf{q}=\left\{q_{1,2},q_{2,3},\ldots,q_{m-1,m}\right\}$ and $\mathbf{p}=\left\{p_{2,1},p_{3,2},\ldots,p_{m,m-1}\right\}$ are crisp state transition probability vectors. Then $\alpha$-cut of $\tilde{P}_i(k)$ can be denoted as $\left(\tilde{P}_i(k)\right)_\alpha=\left[\left(P_i(k)\right)_\alpha^L,\left(P_i(k)\right)_\alpha^U\right]$.

## 3. Fuzzy $L_z$-transform

Consider the discrete-state discrete-time fuzzy Markov chain $\tilde{x}(k)(k=0,1,\ldots)$ in subsection 2.2, the fuzzy Markov chain can be completely determined by set of possible fuzzy performance levels $\tilde{\mathbf{x}}=\left\{\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_m\right\}$, fuzzy transition probability matrix $\tilde{\mathbf{P}}$ and the initial state probability distribution $\tilde{\mathbf{P}}_0$. The discrete-state discrete-time fuzzy Markov chain can be notated by using triplet:

$$\tilde{x}(k)=\left\langle\tilde{\mathbf{x}},\tilde{\mathbf{P}},\tilde{\mathbf{P}}_0\right\rangle \tag{12}$$

Based on the fuzzy universal generating function [12,33] and the $L_z$-transform of a discrete-state continuous-time Markov process [28,29,31], fuzzy $L_z$-transform of a discrete-state discrete-time fuzzy Markov chain $\tilde{x}(k)=\left\langle\tilde{\mathbf{x}},\tilde{\mathbf{P}},\tilde{\mathbf{P}}_0\right\rangle$ is defined as:

$$\tilde{L}Z\left\{\tilde{x}(k)\right\}=\tilde{u}\left(z,k,\tilde{\mathbf{P}}_0\right)=\sum_{i=1}^{m}\tilde{P}_i(k)\cdot z^{\tilde{x}_i} \tag{13}$$

where $\tilde{P}_i(k)$ is the fuzzy state probability that the fuzzy Markov chain is in the state $i$ at time $k\left(k=0,1,\ldots\right)$ for any given initial state probability distribution $\tilde{\mathbf{P}}_0$, and $z$ is a complex variable in the general case. Under given initial state probability distribution $\tilde{\mathbf{P}}_0$, the discrete-state discrete-time fuzzy Markov chain has one and only one fuzzy $L_z$-transform.

For example, consider a simple component which has only two different fuzzy performance levels $\tilde{x}_1=\tilde{\chi}$ and $\tilde{x}_2=0$. It means that $\tilde{x}_1$ is the performance level associated with nominal working state and $\tilde{x}_2$ is the performance level associated with complete failure state. The working time and the repair time of the component have geometric distributions with fuzzy mean values $1/\tilde{q}$ and $1/\tilde{p}$. The fuzzy Markov chain $\tilde{x}(k)\in\left\{\tilde{\chi},0\right\}$ $(k=0,1,\ldots)$ for the considered example is denoted by $\tilde{x}(k)=\left\langle\tilde{\mathbf{x}},\tilde{\mathbf{P}},\tilde{\mathbf{P}}_0\right\rangle$, where $\tilde{\mathbf{x}}$, $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{P}}_0$ are given respectively as:

- Set of the possible fuzzy performance levels $\tilde{\mathbf{x}}=\left\{\tilde{x}_1,\tilde{x}_2\right\}=\left\{\tilde{\chi},0\right\}$;
- Fuzzy transition probability matrix $\tilde{\mathbf{P}}=\begin{bmatrix}1-\tilde{q} & \tilde{q}\\ \tilde{p} & 1-\tilde{p}\end{bmatrix}$;
- Initial state probability distribution $\tilde{\mathbf{P}}_0=[1,0]$.

The fuzzy dynamic state probabilities of the fuzzy Markov chain $\tilde{x}(k)$ at time $k(k=0,1,\ldots)$ can be calculated by:

$$\left[\tilde{P}_1(k),\tilde{P}_2(k)\right]=[1,0]\cdot\begin{bmatrix}1-\tilde{q} & \tilde{q}\\ \tilde{p} & 1-\tilde{p}\end{bmatrix}^k \tag{14}$$

Solving (14), we can obtain:

$$\tilde{P}_1(k)=\frac{\tilde{p}+\tilde{q}\left(1-\tilde{q}-\tilde{p}\right)^k}{\tilde{q}+\tilde{p}},\ \tilde{P}_2(k)=\frac{\tilde{q}-\tilde{q}\left(1-\tilde{q}-\tilde{p}\right)^k}{\tilde{q}+\tilde{p}}$$

For the component with a number of fuzzy performance levels, the forms of closed-form solutions for the dynamic fuzzy state probabilities $\tilde{P}_i(k)$ $(i=1,2,\ldots,m)$ are very complicated. A numerical technique can be used to obtain these solutions.

The fuzzy $L_z$-transform of the fuzzy Markov chain for the binary component can be obtained as follows:

$$\tilde{L}Z\left\{\tilde{x}(k)\right\}=\tilde{u}\left(z,k,\tilde{\mathbf{P}}_0\right)=\sum_{i=1}^{2}\tilde{P}_i(k)\cdot z^{\tilde{x}_i}$$
$$=\frac{\tilde{p}+\tilde{q}\left(1-\tilde{q}-\tilde{p}\right)^k}{\tilde{q}+\tilde{p}}\cdot z^{\tilde{\chi}}+\frac{\tilde{q}-\tilde{q}\left(1-\tilde{q}-\tilde{p}\right)^k}{\tilde{q}+\tilde{p}}\cdot z^0$$

Assume that the $\alpha$-cuts of $\tilde{q}$ and $\tilde{p}$ are $\left[(q)_\alpha^L,(q)_\alpha^U\right]$ and $\left[(p)_\alpha^L,(p)_\alpha^U\right]$ respectively, we can obtain the $\alpha$-cuts of the fuzzy dynamic state probabilities $\tilde{P}_1(k)$ and $\tilde{P}_2(k)$ for the fuzzy Markov chain according to (10) and (11).

## 4. Fuzzy dynamic availability assessment for DTRMSS

The fuzzy dynamic availability assessment method for the DTRMSS under minor failures and repairs, which is based on using the proposed fuzzy $L_z$ -transform. It is assumed that the behavior of any component $l$ ($l\in\{1,2,\ldots,n\}$) in the DTRMSS with $n$ components can be characterized by the discrete-state discrete-time fuzzy Markov chain $\tilde{G}_l(k)$, which has $m_l$ different states that correspond to different fuzzy performance levels represented by the ordered fuzzy set $\tilde{\mathbf{g}}_l=\left\{\tilde{g}_{l1},\tilde{g}_{l2},\cdots,\tilde{g}_{lm_l}\right\}$ , where $\tilde{g}_{li_l}$ is the fuzzy performance level of component $l$ in its state $i_l$ ($i_l=1,2,\ldots,m_l$) .

Let $\tilde{\mathbf{P}}_{lk}=\left[\tilde{P}_{l1}(k),\tilde{P}_{l2}(k),\cdots,\tilde{P}_{lm_l}(k)\right]$ denote the fuzzy dynamic state probabilities associated with different states for the component $l$ at time $k$ , that is:

$$\tilde{P}_{li_l}(k)=\Pr\left\{\tilde{G}_l(k)=\tilde{g}_{li_l}\right\},\ i_l=1,2,\ldots,m_l\,,\ l\in\{1,2,\ldots,n\}\tag{15}$$

The fuzzy dynamic state probabilities $\tilde{P}_{li_l}(k)$ for each of $m_l$ states can be obtained by writing and solving a corresponding fuzzy matrix equation (7) with the given initial conditions $\tilde{\mathbf{P}}_{l0}$ .

Based on the fuzzy $L_z$ -transform method, each discrete-state discrete-time fuzzy Markov chain $\tilde{G}_l(k)$ associated with the fuzzy output Markov process of the component $l$ $\left(l\in\{1,2,\ldots,n\}\right)$ should be expressed as:

$$\tilde{L}Z\left\{\tilde{G}_l(k)\right\}=\tilde{u}\left(z,k,\tilde{\mathbf{P}}_{l0}\right)=\sum_{i_l=1}^{m_l}\tilde{P}_{li_l}(k)\cdot z^{\tilde{g}_{li_l}}\ ,\ l=1,2,\ldots,n\tag{16}$$

For the entire DTRMSS, its states are separated through its fuzzy performance levels, which are unambiguously determined by the fuzzy performance levels of components and its structure. Assume that the DTRMSS has $M$ different states and $\tilde{g}_j$ is the fuzzy performance level of the system in state $j$ ($j=1,2,\ldots,M$) . Let $\tilde{G}(k)$ denote the fuzzy performance level of the DTRMSS at time $k$, then $\tilde{G}(k)$ is a fuzzy stochastic process that takes fuzzy values from the set $\tilde{\mathbf{G}}=\left(\tilde{g}_1,\tilde{g}_2,\ldots,\tilde{g}_M\right)$ . Based on the fuzzy performance stochastic processes $\tilde{G}_l(k)$ $\left(l\in\{1,2,\ldots,n\}\right)$of all components at time $k$ and the system structure, the fuzzy stochastic process $\tilde{G}(k)$ can be given by:

$$\tilde{G}(k)=\phi\left(\tilde{G}_1(k),\tilde{G}_2(k),\cdots,\tilde{G}_n(k)\right)\tag{17}$$

where $\phi(\bullet)$ is the system structure function. Let $\tilde{P}_j(k)$ denote the fuzzy dynamic state probability of the DTRMSS in state $j$ at time $k$ , then the fuzzy dynamic state probabilities associated with different states for the system at time $k$ can be denoted by:

$$\tilde{\mathbf{P}}(k)=\left[\tilde{P}_1(k),\tilde{P}_2(k),\cdots,\tilde{P}_M(k)\right]\tag{18}$$

Based on the property of $L_z$ -transform [28] and the general fuzzy composition operator $\tilde{\Omega}_\phi$ [12], we can find the fuzzy $L_z$ -transform of the discrete-state discrete-time fuzzy stochastic process $\tilde{G}(k)$, which is a fuzzy single-valued function of $n$ independent discrete-state discrete-time fuzzy Markov chains $\tilde{G}_l(k)$ $\left(l\in\{1,2,\ldots,n\}\right)$. Appling the fuzzy composition operator $\tilde{\Omega}_\phi$ to all individual fuzzy $L_z$ -transforms $\tilde{L}Z\left\{\tilde{G}_l(k)\right\}$ $\left(l\in\{1,2,\ldots,n\}\right)$ over any discrete time $k$ $\left(k=0,1,\ldots\right)$ , we have:

$$\tilde{L}Z\left\{\tilde{G}(k)\right\}=\tilde{\Omega}_\phi\left\{\tilde{L}Z\left[\tilde{G}_1(k)\right],\tilde{L}Z\left[\tilde{G}_2(k)\right],\cdots,\tilde{L}Z\left[\tilde{G}_n(k)\right]\right\}\tag{19}$$

According to (16) and (19), the fuzzy $L_z$ -transform of the discrete-state discrete-time fuzzy stochastic process $\tilde{G}(k)$ can be written as:

$$\begin{aligned}\tilde{L}Z\left\{\tilde{G}(k)\right\}&=\tilde{\Omega}_\phi\left\{\tilde{u}\left(z,k,\tilde{\mathbf{P}}_{10}\right),\tilde{u}\left(z,k,\tilde{\mathbf{P}}_{20}\right),\cdots,\tilde{u}\left(z,k,\tilde{\mathbf{P}}_{n0}\right)\right\}\\&=\tilde{\Omega}_\phi\left\{\sum_{i_1=1}^{m_1}\tilde{P}_{1i_1}(k)\cdot z^{\tilde{g}_{1i_1}},\sum_{i_2=1}^{m_2}\tilde{P}_{2i_2}(k)\cdot z^{\tilde{g}_{2i_2}},\cdots,\sum_{i_n=1}^{m_n}\tilde{P}_{ni_n}(k)\cdot z^{\tilde{g}_{n\,i_n}}\right\}\\&=\sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_n=1}^{m_n}\left(\prod_{l=1}^{n}\tilde{P}_{li_l}(k)\bullet z^{\phi\left(\tilde{g}_{1i_1},\tilde{g}_{2i_2},\ldots,\tilde{g}_{n\,i_n}\right)}\right)\\&=\sum_{j=1}^{M}\tilde{P}_j(k)\cdot z^{\tilde{g}_j}\end{aligned}\tag{20}$$

where $\tilde{P}_j(k)=\prod_{l=1}^{n}\tilde{P}_{li_l}(k)$ and $\tilde{g}_j=\phi\left(\tilde{g}_{1i_1},\tilde{g}_{2i_2},\ldots,\tilde{g}_{n\,i_n}\right)$ . The lower and upper bounds of the $\alpha$-cuts of the fuzzy dynamic state probability $\tilde{P}_j(k)$ and the fuzzy performance level $\tilde{g}_j$ of the DTRMSS in state $j$ at time $k$ can be determined according to parametric programming technique.

The dynamic fuzzy availability of the entire DTRMSS for the fuzzy consumer demand $\tilde{\omega}$ at time $k$ $\left(k=0,1,\ldots\right)$ is defined as:

$$\tilde{A}\left(\tilde{\omega},k\right)=\sum_{j=1}^{M}\tilde{P}_j(k)\cdot P\left(\tilde{g}_j\geq\tilde{\omega}\right)\tag{21}$$

where $P\left(\tilde{g}_j\geq\tilde{\omega}\right)$ denotes the possibility of $\tilde{g}_j\geq\tilde{\boldsymbol{\omega}}$ . Because the $\tilde{g}_j$ and $\tilde{\omega}$ values are fuzzy numbers, we give a method for

$P\left(\tilde{g}_j \geq \tilde{\omega}\right)$ based on method of interval number ranking in accordance with possibility [11] and the $\alpha$-cuts of $\tilde{g}_j$ and $\tilde{\omega}$. We define the possibility of $\tilde{g}_j \geq \tilde{\omega}$ as:

$$p\left(\tilde{g}_j \geq \tilde{\omega}\right)_\alpha = \frac{\max\left\{0, \left(g_j\right)_\alpha^U - \left(g_j\right)_\alpha^L + (\omega)_\alpha^U - (\omega)_\alpha^L - \max\left\{0, (\omega)_\alpha^U - \left(g_j\right)_\alpha^L\right\}\right\}}{\left(g_j\right)_\alpha^U - \left(g_j\right)_\alpha^L + (\omega)_\alpha^U - (\omega)_\alpha^L} \tag{22}$$

where $\left(g_j\right)_\alpha^L$ and $\left(g_j\right)_\alpha^U$ are the lower bound and upper bound of the fuzzy performance level $\alpha$-cut of $\tilde{g}_j$, and $(\omega)_\alpha^L$ and $(\omega)_\alpha^U$ are the lower bound and upper bound of the $\alpha$-cut of the fuzzy consumer demand $\tilde{\omega}$, respectively.

In order to find the $\alpha$-cut of the dynamic fuzzy availability (21), the following procedures can be adopted.

(1) For each repairable multi-state component $l$ $\left(l \in \{1,2,\ldots,n\}\right)$ under minor failures and repairs, the $\alpha$-cuts $\left(0 \leq \alpha \leq 1\right)$ of the fuzzy state transition probabilities $\tilde{q}_{i_l,i_l+1}$ $(i_l = 1,2,\ldots,m_l - 1)$ from the state $i_l$ to the state $i_l+1$, $\tilde{p}_{i_l,i_l-1}$ $(i_l = 2,3,\ldots,m_l)$ from the state $i_l$ to the state $i_l - 1$, and the fuzzy performance level $\tilde{g}_{li_l}$ in state $i_l$ $(i_l = 1,2,\ldots,m_l)$ can be determined according to individual membership functions as follows:

$$\left(\tilde{q}_{i_l,i_l+1}\right)_\alpha = \left[\left(q_{i_l,i_l+1}\right)_\alpha^L, \left(q_{i_l,i_l+1}\right)_\alpha^U\right] \tag{23}$$

$$\left(\tilde{p}_{i_l,i_l-1}\right)_\alpha = \left[\left(p_{i_l,i_l-1}\right)_\alpha^L, \left(p_{i_l,i_l-1}\right)_\alpha^U\right] \tag{24}$$

$$\left(\tilde{g}_{li_l}\right)_\alpha = \left[\left(g_{li_l}\right)_\alpha^L, \left(g_{li_l}\right)_\alpha^U\right] \tag{25}$$

(2) Based on the proposed fuzzy Markov model in the subsection 2.2.2, the fuzzy dynamic state probabilities $\tilde{P}_{li_l}(k)$ ( $i_l = 1,2,\ldots,m_l$, $l \in \{1,2,\ldots,n\}$ ) for each component at time $k$ can be determined. According to the general fuzzy composition operator $\tilde{\Omega}_\phi$ [12] and the individual fuzzy $L_z$-transforms of all components at time $k$, the fuzzy $L_z$-transform $\tilde{L}Z\left\{\tilde{G}(k)\right\}$ of the entire DTRMSS can be obtained.

(3) According to parametric programming technique, the lower bound $\left(P_j(k)\right)_\alpha^L$ and upper bound $\left(P_j(k)\right)_\alpha^U$ of the α-cut of the fuzzy dynamic state probability $\tilde{P}_j(k)$ of the DTRMSS in state $j$ at time $k$ can be computed as:

$$\begin{aligned}
\left(P_j(k)\right)_\alpha^L = \ & \min\ P_j(k) = \prod_{l=1}^{n} P_{li_l}(k) \quad \left(k = 1,2,\ldots, 0 \leq \alpha \leq 1\right) \\
& \text{s.t.}\ \left(q_{i_l,i_l+1}\right)_\alpha^L \leq q_{i_l,i_l+1} \leq \left(q_{i_l,i_l+1}\right)_\alpha^U, \quad i_l = 1,2,\ldots,m_l - 1 \\
& \left(p_{i_l,i_l-1}\right)_\alpha^L \leq p_{i_l,i_l-1} \leq \left(p_{i_l,i_l-1}\right)_\alpha^U, \quad i_l = 2,3,\ldots,m_l \\
& 0 \leq q_{i_l,i_l+1} + p_{i_l,i_l-1} \leq 1, \quad i = 2,3,\ldots,m_l - 1 \\
& l = 1,2,\ldots,n
\end{aligned} \tag{26}$$

and:

$$\begin{aligned}
\left(P_j(k)\right)_\alpha^U = \ & \max\ P_j(k) = \prod_{l=1}^{n} P_{li_l}(k) \quad \left(k = 1,2,\ldots, 0 \leq \alpha \leq 1\right) \\
& \text{s.t.}\ \left(q_{i_l,i_l+1}\right)_\alpha^L \leq q_{i_l,i_l+1} \leq \left(q_{i_l,i_l+1}\right)_\alpha^U, \quad i_l = 1,2,\ldots,m_l - 1 \\
& \left(p_{i_l,i_l-1}\right)_\alpha^L \leq p_{i_l,i_l-1} \leq \left(p_{i_l,i_l-1}\right)_\alpha^U, \quad i_l = 2,3,\ldots,m_l \\
& 0 \leq q_{i_l,i_l+1} + p_{i_l,i_l-1} \leq 1, \quad i = 2,3,\ldots,m_l - 1 \\
& l = 1,2,\ldots,n
\end{aligned} \tag{27}$$

where $P_{li_l}(k)$ is the function of $q_{i_l,i_l+1}$ and $p_{i_l,i_l-1}$.

The lower bound $\left(g_j\right)_\alpha^L$ and upper bound $\left(g_j\right)_\alpha^U$ of the $\alpha$-cut of the fuzzy performance level $\tilde{g}_j$ of the DTRMSS in state $j$ can be computed as:

$$\begin{aligned}
\left(g_j\right)_\alpha^L = \ & \min\ g_j = \phi\left(g_{1i_1}, g_{2i_2}, \ldots, g_{n\,i_n}\right) \quad \left(0 \leq \alpha \leq 1\right) \\
& \text{s.t.}\ \left(g_{li_l}\right)_\alpha^L \leq g_{li_l} \leq \left(g_{li_l}\right)_\alpha^U, \quad i_l = 1,2,\ldots,m_l \\
& l = 1,2,\ldots,n
\end{aligned} \tag{28}$$

and:

$$\begin{aligned}
\left(g_j\right)_\alpha^U = \ & \max\ g_j = \phi\left(g_{1i_1}, g_{2i_2}, \ldots, g_{n\,i_n}\right) \quad \left(0 \leq \alpha \leq 1\right) \\
& \text{s.t.}\ \left(g_{li_l}\right)_\alpha^L \leq g_{li_l} \leq \left(g_{li_l}\right)_\alpha^U, \quad i_l = 1,2,\ldots,m_l \\
& l = 1,2,\ldots,n
\end{aligned} \tag{29}$$

(4) Based on the defined possibility of $\tilde{g}_j \geq \tilde{\omega}$ in Eq. (22) and the defined availability $\tilde{A}\left(\tilde{\omega}, k\right)$ in Eq. (21), the lower bound $\left(A(\omega,k)\right)_\alpha^L$ and upper bound $\left(A(\omega,k)\right)_\alpha^U$ of the $\alpha$-cut of the dynamic system fuzzy availability $\tilde{A}\left(\tilde{\omega}, k\right)$ for the fuzzy consumer demand $\tilde{\omega}$ at time $k$ can be computed as:

$$\left(A(\omega,k)\right)_{\alpha}^{L} = \min \sum_{j=1}^{M} P_j(k)\cdot P\left(\tilde{g}_j \geq \tilde{\omega}\right)_{\alpha} \quad (k=1,2,\ldots,0\leq\alpha\leq 1)$$
$$\text{s.t. } \left(P_j(k)\right)_{\alpha}^{L} \leq P_j(k) \leq \left(P_j(k)\right)_{\alpha}^{U}, \quad j=1,2,\ldots,M$$
$$\sum_{j=1}^{M} P_j(k) = 1 \tag{30}$$

and:

$$\left(A(\omega,k)\right)_{\alpha}^{U} = \max \sum_{j=1}^{M} P_j(k)\cdot P\left(\tilde{g}_j \geq \tilde{\omega}\right)_{\alpha} \quad (k=1,2,\ldots,0\leq\alpha\leq 1)$$
$$\text{s.t. } \left(P_j(k)\right)_{\alpha}^{L} \leq P_j(k) \leq \left(P_j(k)\right)_{\alpha}^{U}, \quad j=1,2,\ldots,M$$
$$\sum_{j=1}^{M} P_j(k) = 1 \tag{31}$$

## 5. Numerical example

Consider a repairable multi-state flow transmission system with three subsystems (LSS subsystem, HSS subsystem and CP subsystem), where the structure of the system and the state-transition diagrams for its components are presented in Fig. 1. It can be seen from Fig. 1 that only components of the LSS subsystem have two states, whereas components of the HSS subsystem and components of the CP subsystem are multi-state components.

The fuzzy state transition probabilities and fuzzy performance level for each component are treated as trapezoidal fuzzy numbers. To reduce the computational complexity, the following operations of the trapezoidal fuzzy numbers are used to determine the fuzzy performance levels of the each parallel subsystem and the entire series-parallel system.

Assume that $\tilde{a}=(a_1,a_2,a_3,a_4)$ and $\tilde{b}=(b_1,b_2,b_3,b_4)$ are two trapezoidal fuzzy numbers, then:

$$\tilde{a}+\tilde{b}=(a_1+b_1,a_2+b_2,a_3+b_3,a_4+b_4) \tag{32}$$

$$\min\{\tilde{a},\tilde{b}\}=\left(\min\{a_1,b_1\},\min\{a_2,b_2\},\min\{a_3,b_3\},\min\{a_4,b_4\}\right) \tag{33}$$

The LSS subsystem consists of two different components connected in parallel, and their respective performance is allowed in only one of two principal states: perfect performance level (state 1) and zero performance level (state 2). The fuzzy state transition probabilities of each component from state 1 to state 2 are $\tilde{q}^{LSS_1}=(0.020,0.025,0.030,0.035)$ and $\tilde{q}^{LSS_2}=(0.05,0.06,0.07,0.08)$, respectively. The fuzzy state transition probabilities of each component from state 2 to state 1 are $\tilde{p}^{LSS_1}=(0.30,0.35,0.40,0.45)$ and $\tilde{p}^{LSS_2}=(0.40,0.50,0.60,0.70)$, respectively. The fuzzy performance levels of the first component are $\tilde{g}_{11}^{LSS}=(4,5,6,7)$ (perfect performance) and $\tilde{g}_{12}^{LSS}=0$ (zero performance). The fuzzy performance levels of the second component are $\tilde{g}_{21}^{LSS}=(8,10,12,14)$ (perfect performance) and $\tilde{g}_{22}^{LSS}=0$ (zero performance).

The HSS subsystem has two identical parallel components with three different states: perfect operation state (corresponding to the fuzzy performance level $\tilde{g}_{l1}^{HSS}=(18,20,22,24)$), partial operation state (corresponding to the fuzzy performance level $\tilde{g}_{l2}^{HSS}=(10,12,14,16)$), and complete failure state (corresponding to the performance level $\tilde{g}_{l3}^{HSS}=0$), $l=1,2$. The fuzzy state transition probabilities of the two components are $\tilde{q}^{HSS_1}=\tilde{q}^{HSS_3}=(0.04,0.05,0.06,0.07)$, $\tilde{q}^{HSS_2}=\tilde{q}^{HSS_4}=(0.06,0.07,0.08,0.09)$, $\tilde{p}^{HSS_1}=\tilde{p}^{HSS_3}=(0.4,0.5,0.6,0.7)$ and $\tilde{p}^{HSS_2}=\tilde{p}^{HSS_4}=(0.5,0.6,0.7,0.8)$.

The CP subsystem has only one component with four states: two perfect operation states (corresponding to the fuzzy performance level $\tilde{g}_1^{CP}=\tilde{g}_2^{CP}=(14,16,18,20)$), a partial operation state (corresponding to the fuzzy performance level $\tilde{g}_3^{CP}=(7,8,9,10)$) and a complete failure state (corresponding to the performance level $\tilde{g}_4^{CP}=0$). The fuzzy state transition probabilities of the component are $\tilde{q}^{CP_1}=(0.03,0.04,0.05,0.06)$, $\tilde{q}^{CP_2}=(0.05,0.06,0.07,0.08)$, $\tilde{q}^{CP_3}=(0.09,0.1,0.11,0.12)$, $\tilde{p}^{CP_1}=(0.2,0.3,0.4,0.5)$, $\tilde{p}^{CP_2}=(0.4,0.5,0.6,0.7)$ and $\tilde{p}^{CP_3}=(0.5,0.6,0.7,0.8)$.

According to the method and procedures provided in Section 4, we can calculate the $\alpha$-cuts of the dynamic fuzzy availability for the system with different fuzzy consumer demand levels: $\tilde{\omega}_1=(12,15,18,21)$ and $\tilde{\omega}_2=(8,10,12,14)$.

Based on different fuzzy consumer demand levels $\tilde{\omega}_1$ and $\tilde{\omega}_2$, Tables 1-2 presents the $\alpha$-cuts of the dynamic fuzzy availability at 11 distinct $\alpha$ values: $0,0.1,\ldots,1.0$ for the mission time $k=2$, 4, 8, 12. Suppose that the reliability standard of the repairable multi-state flow transmission system at the confidence level $\alpha=0.9$ for the mission time $k=2$ requires the system performance must satisfy a fuzzy consumer demand level ($\tilde{\omega}_2=(8,10,12,14)$) of the system availability, which is set as 0.4. It can be seen from Table 2 that the result of the availability assessment ($[0.4026,0.6353]$) can satisfy the availability requirement of the repairable multi-state flow transmission system, which ensures the system relatively reliable working. In the same manner, one can explain the rest of the results shown in Tables 1-2.

Figs. 2-3 illustrate the membership functions of the dynamic fuzzy availabilities $\tilde{A}(\tilde{\omega}_1,k)$ and $\tilde{A}(\tilde{\omega}_2,k)$ when $k=2$, 4, 8, 12. The dynamic fuzzy availability curves of the system under different fuzzy consumer demand levels, shown in Figs. 4-5, represent the variation in availability of the system with time at the confidence levels $\alpha$=0, $\alpha=0.5$ and $\alpha$=*1*. It can be seen from Figs. 4-5 that the system dynamic fuzzy availability decreases as time $k$ increases, and the decrease is very rapid when $k<2$ and tends vanish when $k$ becomes large. This means that the system availability has relative stability in a certain range when $k\geq 2$.

These results in the numerical example will be beneficial for system analyst or design engineer. Based on these results, the system manager can carry out design modifications and make maintenance (repair and replacement actions) decision for the discrete time repairable multi-state systems under imprecise and fuzzy environment.

*Table 1. The α-cuts of the fuzzy availability $\tilde{A}(\tilde{\omega}_1, k)$ at 11 distinct α values when $k = 2,4,8,12$*

| $\alpha$ | $\left[\left(A(\omega_1,k)\right)_\alpha^L,\left(A(\omega_1,k)\right)_\alpha^U\right]$ | | | |
|---|---|---|---|---|
| | $\left[\left(A(\omega_1,2)\right)_\alpha^L,\left(A(\omega_1,2)\right)_\alpha^U\right]$ | $\left[\left(A(\omega_1,4)\right)_\alpha^L,\left(A(\omega_1,4)\right)_\alpha^U\right]$ | $\left[\left(A(\omega_1,8)\right)_\alpha^L,\left(A(\omega_1,8)\right)_\alpha^U\right]$ | $\left[\left(A(\omega_1,12)\right)_\alpha^L,\left(A(\omega_1,12)\right)_\alpha^U\right]$ |
| 0.0 | [0.2521,0.8005] | [0.2342,0.7823] | [0.2260,0.7783] | [0.2245,0.7781] |
| 0.1 | [0.2654,0.7771] | [0.2468,0.7589] | [0.2390,0.7546] | [0.2377,0.7544] |
| 0.2 | [0.2790,0.7542] | [0.2604,0.7358] | [0.2523,0.7313] | [0.2512,0.7311] |
| 0.3 | [0.2929,0.7314] | [0.2744,0.7132] | [0.2661,0.7083] | [0.2651,0.7081] |
| 0.4 | [0.3073,0.7093] | [0.2879,0.6909] | [0.2802,0.6851] | [0.2793,0.6855] |
| 0.5 | [0.3220,0.6871] | [0.3021,0.6677] | [0.2952,0.6635] | [0.2939,0.6633] |
| 0.6 | [0.3370,0.6663] | [0.3171,0.6461] | [0.3101,0.6417] | [0.3089,0.6413] |
| 0.7 | [0.3524,0.6442] | [0.3326,0.6250] | [0.3253,0.6202] | [0.3242,0.6198] |
| 0.8 | [0.3681,0.6240] | [0.3477,0.6041] | [0.3410,0.5989] | [0.3399,0.5986] |
| 0.9 | [0.3831,0.6024] | [0.3641,0.5837] | [0.3569,0.5782] | [0.3559,0.5779] |
| 1.0 | [0.4002,0.5827] | [0.3800,0.5626] | [0.3733,0.5577] | [0.3724,0.5574] |

*Table 2. The α-cuts of the fuzzy availability $\tilde{A}(\tilde{\omega}_2, k)$ at 11 distinct α values when $k = 2,4,8,12$*

| $\alpha$ | $\left[\left(A(\omega_2,k)\right)_\alpha^L,\left(A(\omega_2,k)\right)_\alpha^U\right]$ | | | |
|---|---|---|---|---|
| | $\left[\left(A(\omega_2,2)\right)_\alpha^L,\left(A(\omega_2,2)\right)_\alpha^U\right]$ | $\left[\left(A(\omega_2,4)\right)_\alpha^L,\left(A(\omega_2,4)\right)_\alpha^U\right]$ | $\left[\left(A(\omega_2,8)\right)_\alpha^L,\left(A(\omega_2,8)\right)_\alpha^U\right]$ | $\left[\left(A(\omega_2,12)\right)_\alpha^L,\left(A(\omega_2,12)\right)_\alpha^U\right]$ |
| 0.0 | [0.2662,0.8416] | [0.2538,0.8343] | [0.2495,0.8341] | [0.2488,0.8342] |
| 0.1 | [0.2801,0.8173] | [0.2671,0.8099] | [0.2633,0.8093] | [0.2593,0.8095] |
| 0.2 | [0.2943,0.7934] | [0.2815,0.7857] | [0.2773,0.7852] | [0.2735,0.7853] |
| 0.3 | [0.3087,0.7697] | [0.2963,0.7621] | [0.2919,0.7613] | [0.2881,0.7614] |
| 0.4 | [0.3237,0.7466] | [0.3105,0.7388] | [0.3068,0.7377] | [0.3029,0.7378] |
| 0.5 | [0.3390,0.7234] | [0.3254,0.7146] | [0.3226,0.7146] | [0.3182,0.7147] |
| 0.6 | [0.3546,0.7018] | [0.3412,0.6919] | [0.3383,0.6918] | [0.3338,0.6918] |
| 0.7 | [0.3707,0.6787] | [0.3576,0.6698] | [0.3543,0.6694] | [0.3498,0.6693] |
| 0.8 | [0.3870,0.6576] | [0.3734,0.6479] | [0.3708,0.6471] | [0.3662,0.6472] |
| 0.9 | [0.4026,0.6353] | [0.3906,0.6265] | [0.3875,0.6254] | [0.3828,0.6256] |
| 1.0 | [0.4204,0.6145] | [0.4073,0.6043] | [0.4047,0.6040] | [0.4000,0.6042] |

## 6. Conclusions

The dynamic fuzzy availability assessment technique for a DTRMSS under minor failures and repairs is investigated in the paper. The technique is based on the Fuzzy $L_z$-transform of the discrete-state discrete-time Markov process, the $\alpha$-cut approach and the parametric programming algorithm. A discrete-state discrete-time fuzzy Markov model is proposed to analyze the dynamic fuzzy state probability for each multi-state component at any discrete time. The analytical procedure is provided to calculate the $\alpha$-cut of the dynamic fuzzy availability of the system. A flow transmission system with five components is considered, and it illustrates the performance of the proposed technique. This technique is suitable and effective for the dynamic fuzzy availability assessment of the DTRMSS with fuzzy uncertainty. In future, we will concern the fuzzy $L_z$ -transform of the discrete-state discrete-time Markov process and its application to the fuzzy multi-state system reliability, and the dynamic fuzzy availability optimization design for the DTRMSS.

**Acknowledgment**

*The authors are grateful to the reviewers for their insightful comments and suggestions. This paper is partially supported by the Natural Science Foundation of Hebei Province (No.A2014203096), the National Natural Science Foundation of China (No.11601469), the Science Research Project of Yanshan University (No.13LGA017) and the Science Research Project of Education Department of Hebei Province (No. QN2016016), People's Republic of China.*

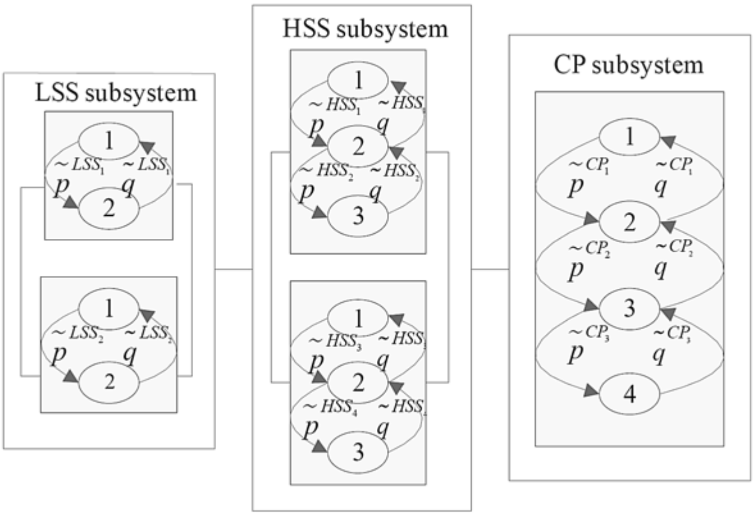


*Fig. 1. Structure and component state-transition diagrams of the repairable multi-state flow transmission system*

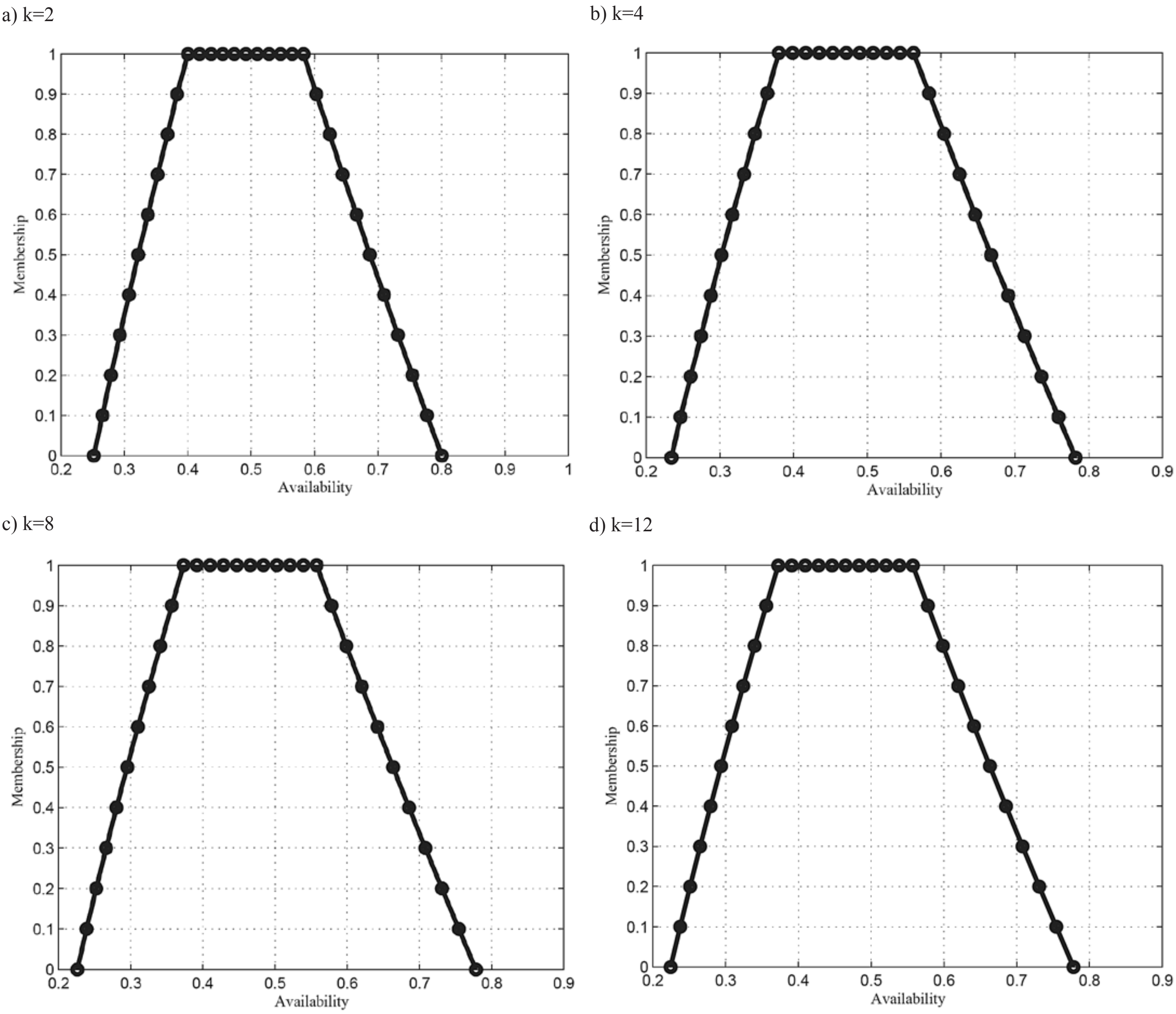


*Fig. 2. The membership functions of the fuzzy availability $\tilde{A}(\tilde{\omega}_1, k)$ at k=2,4,8,12*

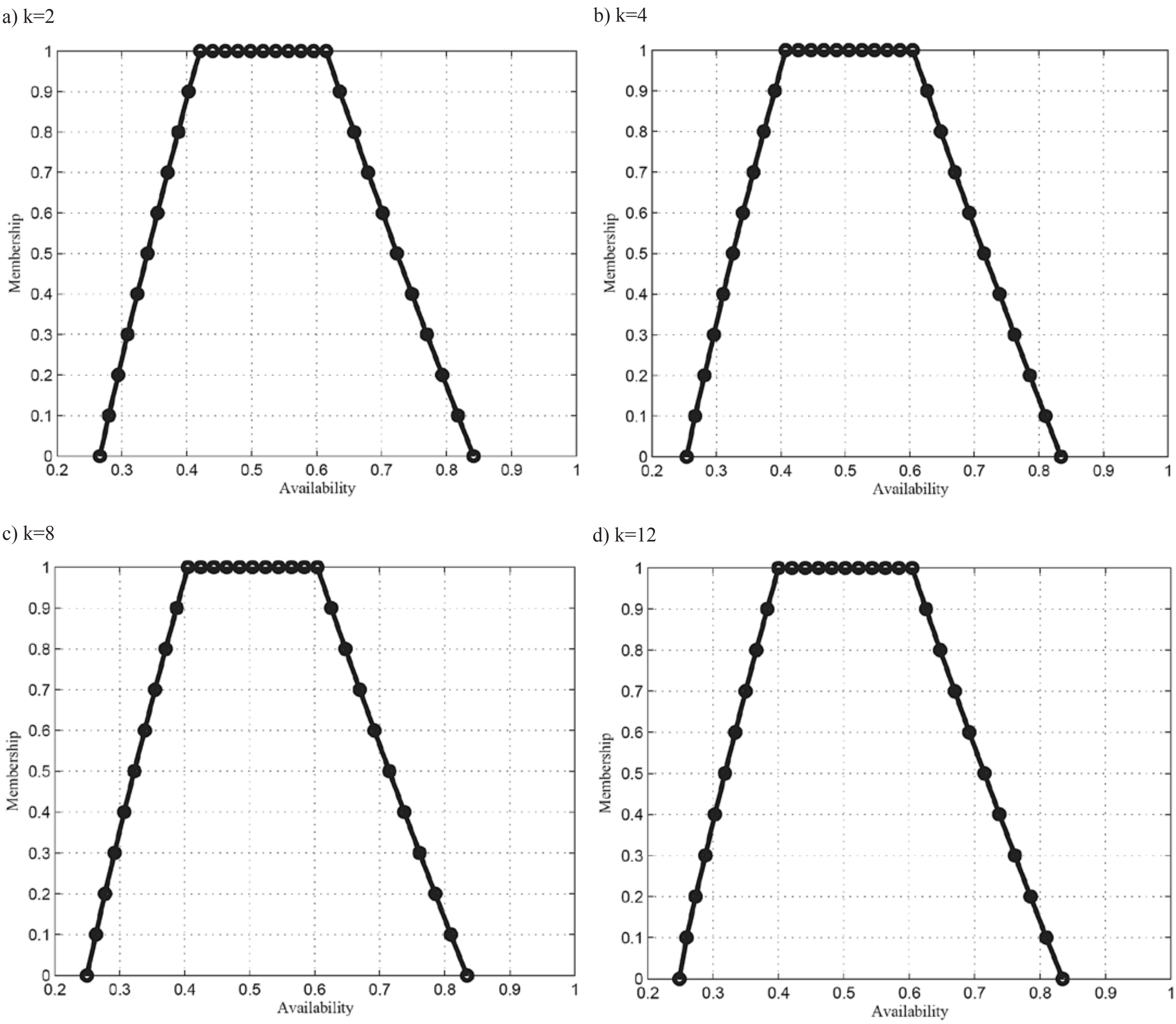


*Fig. 3. The membership functions of the fuzzy availability $\tilde{A}(\tilde{\omega}_2, k)$ at k=2,4,8,12*

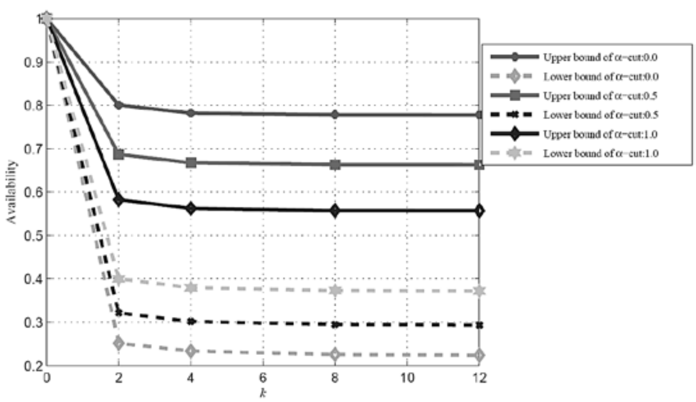


*Fig. 4. The fuzzy availability $\tilde{A}(\tilde{\omega}_1, k)$ with $\alpha$=0.0, 0.5, 1.0*

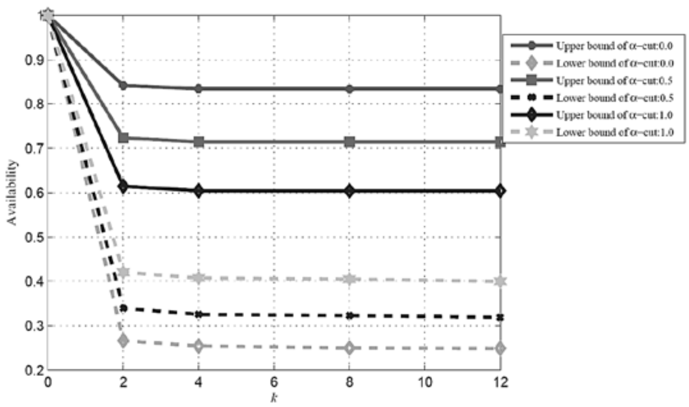


*Fig. 5. The fuzzy availability $\tilde{A}(\tilde{\omega}_2, k)$ with $\alpha$=0.0, 0.5, 1.0*

## References


1. Amico G. Single-use reliability computation of a semi-Markovian system. Applications of Mathematics 2014; 59(5): 571-588, https://doi.org/10.1007/s10492-014-0072-4.
2. Backley J. Fuzzy Probabilities: New Approach and Applications. Springer, 2005.
3. Bamrungsetthapong W, Pongpullponsak A. System reliability for non-repairable multi-state series–parallel system using fuzzy Bayesian inference based on prior interval probabilities. International Journal of General Systems 2015; 44(4): 442-456, https://doi.org/10.1080/03081079.2014.976215.
4. Barbu V, Limnios N. Empirical estimation for discrete time semi-Markov processes with applications in reliability. Journal of Nonparametric Statistics 2006; 18(7-8): 483-498, https://doi.org/10.1080/10485250701261913.
5. Bracquemond C, Gaudoin O. A survey on discrete lifetime distributions. International Journal on Reliability, Quality, and Safety Engineering 2003; 10(1), 69-98, https://doi.org/10.1142/S0218539303001007.
6. Brameret P, Raury A, Roussel J. Automated generation of partial Markov chain form high level descriptions. Reliability Engineering and System Safety 2015; 139: 179-187, https://doi.org/10.1016/j.ress.2015.02.009.
7. Cai K. Introduction to Fuzzy Reliability, Dordrecht: Kluwer Academic Publishers, 1996, https://doi.org/10.1007/978-1-4613-1403-5.
8. Castro H, Cavalca K. Maintenance resourced optimization applied to a manufacturing system. Reliability Engineering and System Safety 2006; 91(4): 413-420, https://doi.org/10.1016/j.ress.2005.02.004.
9. Chen Y L, Chang C C, Sheu D F. Optimum random and age replacement policies for customer-demand multi-state system reliability under imperfect maintenance. International Journal of Systems Science 2016; 47(5): 1130-1141, https://doi.org/10.1080/00207721.2014.915353.
10. Chryssaphinou O, Limnios N. Multi-state reliability systems under discrete time semi-Markovian hypothesis. IEEE Transactions on Reliability 2011; 60(1): 80-87, https://doi.org/10.1109/TR.2010.2104210.
11. Da Q, Liu X. Interval number linear programming and its satisfactory solution. Systems Engineering Theory and Practice 1999; 19(4): 3-7.
12. Ding Y, Lisnianski A. Fuzzy universal generating functions for multi-state system reliability assessment. Fuzzy Sets and Systems 2008; 159(3): 307-324, https://doi.org/10.1016/j.fss.2007.06.004.
13. Ding Y, Zuo M, Lisnianski A, Tian Z. Fuzzy multi-state systems: general definitions, and performance assessment. IEEE Transactions on Reliability 2008; 57(4): 589-594, https://doi.org/10.1109/TR.2008.2006078.
14. Eryilmaz S. On Mean residual life of discrete time multi-state systems. Quality Technology and Quantitative Management 2013; 10(2): 241-249, https://doi.org/10.1080/16843703.2013.11673319.
15. Garg H. An approach for analyzing the reliability of industrial system using fuzzy Kolmogorov's differential equations. Arabian Journal for Science and Engineering 2015; 40: 975-987, https://doi.org/10.1007/s13369-015-1584-2.
16. Guerry M. On the embedding problem for discrete-time Markov chains. Journal of Applied Probability 2013; 50(4): 918-930, https://doi.org/10.1017/S002190020001370X.
17. Hu L, Yue D, Tian R. Fuzzy availability assessment for a repairable multi-state series-parallel system. Discrete Dynamics in Nature and Society 2015; Article ID 156059, https://doi.org/10.1155/2015/156059.
18. Huang H. Structural reliability analysis using fuzzy sets theory. Eksploatacja i Niezawodnosc– Maintenance and Reliability 2012; 14(4): 284-294.
19. Jin Q, Sugasawa Y. Representation and analysis of behavior for multiprocess systems by using stochastic Petri nets. Mathematical and Computer Modelling 1995; 22(10-12): 109-118, https://doi.org/10.1016/0895-7177(95)00187-7.
20. Ke J, Huang H, Lin C. Fuzzy analysis for steady state availability: a mathematical programming approach. Engineering Optimization 2006; 38(8): 909-921, https://doi.org/10.1080/03052150600795854.
21. Kumar G, Jain V, Gandhi O. Reliability and availability modeling of mechanical systems using stochastic Petri Net. Proceedings of the ASME 2011 Power Conference 2011; 211-220, Denver, Colorado, USA, https://doi.org/10.1115/POWER2011-55325.
22. Kumar S, Kumar B, Chaudhary A. Reliability analysis of a discrete life time model. International Journal of Computer Applications 2014; 86(10): 35-39, https://doi.org/10.5120/15024-3313.
23. Kurano M, Yasuda M, Nakagami J, Yoshida Y. A fuzzy approach to Markov decision processes with uncertain transition probabilities. Fuzzy Sets and Systems 2006; 157(19): 2674-2682, https://doi.org/10.1016/j.fss.2004.10.023.
24. Levitin G, Zhang T, Xie M. State probability of a series-parallel repairable system with two-types of failure states. International Journal of Systems Science 2006; 37(14): 1011-1020, https://doi.org/10.1080/00207720600903201.
25. Li C, Chen X, Yi X, Tao J. Interval-valued reliability analysis of multi-state systems. IEEE Transactions on Reliability 2011; 60(1): 323-330, https://doi.org/10.1109/TR.2010.2103670.
26. Li G, Xiu B. Fuzzy Markov chains based on the fuzzy transition probability. Chinese Control and Decision Conference 2014; 4351-4356, https://doi.org/10.1109/ccdc.2014.6852945.
27. Li Z, Kapur K C. Continuous-state reliability measures based on fuzzy sets. IIE Transactions 2012; 44: 1033-1044, https://doi.org/10.1080/0740817X.2011.588684.
28. Lisnianski A. Lz-transform for a discrete-state continuous-time Markov process and its application to multi-state system reliability. Multi-state Systems and Statistical Inference 2012; 79-96, Springer: London.
29. Lisnianski A, Ben-Haim H. Short-term reliability evaluation for power stations by using Lz-transform. Journal of Modern Power Systems and Clean Energy 2013; 1(2): 110-117, https://doi.org/10.1007/s40565-013-0021-3.
30. Lisnianski A, Ding Y. Redundancy analysis for repairable multi-state system by using combined stochastic processes methods and universal generating function technique. Reliability Engineering and System Safety 2009; 94(11): 1788-1795, https://doi.org/10.1016/j.ress.2009.05.006.
31. Lisnianski A, Frenkel I, Khvatskin L. On Birnbaum importance assessment for aging multi-state system under minimal repair by using the Lz-transform method. Reliability Engineering and System Safety 2015; 142: 258-266, https://doi.org/10.1016/j.ress.2015.05.006.
32. Lisnianski A, Levitin G. Multi-State System Reliability Assessment, Optimization, Applications. Singapore: World Scientific, 2003, https://doi.org/10.1142/5221.

33. Liu Y, Huang H. Reliability Assessment for fuzzy multi-state systems. International Journal of Systems Science 2010; 41(4): 365-379, https://doi.org/10.1080/00207720903042939.
34. Rocha-Martinez J, Shaked M. A discrete-time model of failure and repair. Applied Stochastic Models and Data Analysis 1995; 11(2): 167-180, https://doi.org/10.1002/asm.3150110207.
35. Sadek A, Limnios N. Asymptotic properties for maximum likelihood estimators for reliability and failure rates of Markov chains. Statistics-Theory and Methods 2002; 31(10): 1837-1861, https://doi.org/10.1081/STA-120014916.
36. Sarhan A, Guess F, Usher J. Estimators for reliability measures in geometric distribution model using dependent masked system life test data. IEEE Transactions on Reliability 2007; 56(2): 312-320, https://doi.org/10.1109/TR.2007.895300.
37. Shen J, Cui L. Reliability performance for dynamic systems with cycles of K regimes. IIE Transactions 2016; 48(4): 389-402, https://doi.org/10.1080/0740817X.2015.1110266.
38. Utkin L, Gurov S. A general formal approach for fuzzy reliability analysis in the possibility context. Fuzzy Sets and Systems 1996; 83(2): 203-213, https://doi.org/10.1016/0165-0114(95)00391-6.
39. Zadeh L. Fuzzy sets. Information Control 1965; 8: 338-353, https://doi.org/10.1016/S0019-9958(65)90241-X.
40. Zadeh L. Fuzzy sets as a basis for a theory of possibility. Fuzzy Sets and Systems 1978; 1(1): 3-28, https://doi.org/10.1016/0165-0114(78)90029-5.
41. Zio E, Baraldi P, Patelli E. Assessment of the availability of an offshore installation by Monte Carlo simulation. International Journal of Pressure Vessels and Piping 2006; 83(4): 312-320, https://doi.org/10.1016/j.ijpvp.2006.02.010.
42. Zio E, Marella M, Podofillini L. A Monte Carlo simulation approach to the availability assessment of multi-state systems with operational dependencies. Reliability Engineering and System Safety 2007; 92(7): 871-882, https://doi.org/10.1016/j.ress.2006.04.024.



**Linmin HU**
**Peng SU**
**Zhilan ZHANG**
College of Science
Yanshan University
Qinhuangdao 066004, P. R. China

**Rui PENG**
Donlinks School of Economics & Management
University of Science & Technology Beijing
Beijing 100083, P. R. China

E-mails: linminhu@ysu.edu.cn, 18713521817@163.com
pengrui1988@ustb.edu.cn, 1436440015@qq.com