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Some Strongly Almost Summable Sequence Spaces

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Abstract: In the present paper we introduce some strongly almost summable sequence spaces using ideal convergence and Musielak-Orlicz function $\mathcal{M} = (M_k)$ in n-normed spaces. We examine some topological properties of the resulting sequence spaces.

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1. Introduction and preliminaries

Mursaleen and Noman [18] introduced the notion of λ-convergent and λ-bounded sequences as follows:
Let $\lambda = (\lambda_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity i.e.

$$0 < \lambda_0 < \lambda_1 < \cdots \text{ and } \lambda_k \to \infty \text{ as } k \to \infty$$

and said that a sequence $x = (x_k) \in w$ is λ-convergent to the number L, called the λ-limit of x if $\Lambda_m(x) \longrightarrow L$ as $m \to \infty$, where

$$\lambda_m(x) = \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) x_k.$$

The sequence $x = (x_k) \in w$ is λ-bounded if $\sup_m |\Lambda_m(x)| < \infty$. It is well known [18] that if $\lim_m x_m = a$ in the ordinary sense of convergence, then

$$\lim_m \left(\frac{1}{\lambda_m} \left(\sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) |x_k - a| \right) \right) = 0.$$

This implies that

$$\lim_m |\Lambda_m(x) - a| = \lim_m |\frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1})(x_k - a)| = 0,$$

which yields that $\lim_m \Lambda_m(x) = a$ and hence $x = (x_k) \in w$ is λ-convergent to a.
The concept of 2-normed spaces was initially developed by Gähler [6] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [15]. Since then, many others have studied this concept and obtained various results, see Gunawan ([8, 9]) and Gunawan and Mashadi [10] and references therein. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

1. $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \; ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and
4. $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called a n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$.
For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm $||x_1, x_2, \cdots, x_n||_E =$ the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.
A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k \to \infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{\substack{k \to \infty \\ p \to \infty}} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.
An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [13] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\Big\},$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

Also, it was shown in [13] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p(p \geq 1)$. The Δ_2- condition is equivalent to $M(Lx) \leq LM(x)$, for all L with $0 < L < 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt,$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.
A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function see ([14, 23]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\},\ k = 1, 2, \ldots$$

is called the complementary function of the Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$ for all $x \in X$,
2. $p(-x) = p(x)$ for all $x \in X$,
3. $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [30], Theorem 10.4.2, p. 183). For more details about sequence spaces (see [16, 17, 19, 20, 21, 22, 24, 25, 26, 27, 29]) and reference therein.
A sequence space E is said to be solid (or normal) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ and for all $k \in \mathbb{N}$.
The notion of ideal convergence was introduced first by P. Kostyrko [11] as a generalization of statistical convergence which was further studied in topological spaces (see [2]). More applications of ideals can be seen in [2, 3].
A linear functional $\mathcal{L}$ on ℓ_∞ is said to be a Banach limit see [1] if it has the properties:

1. $\mathcal{L}(x) \geq 0$ if $x \geq 0$ (i.e. $x_n \geq 0$ for all n),
2. $\mathcal{L}(e) = 1$, where $e = (1, 1, \ldots)$,
3. $\mathcal{L}(Dx) = \mathcal{L}(x)$,

where the shift operator D is defined by $(Dx_n) = (x_{n+1})$.
Let $\mathfrak{B}$ be the set of all Banach limits on ℓ_∞. A sequence x is said to be almost convergent to a number L if $\mathcal{L}(x) = L$ for all $\mathcal{L} \in \mathfrak{B}$. Lorentz [12] has shown that x is almost convergent to L if and only if

$$t_{km} = t_{km}(x) = \frac{x_m + x_{m+1} + \cdots + x_{m+k}}{k+1} \to L \text{ as } k \to \infty, \text{ uniformly in } m.$$

Recently a lot of activities have started to study sumability, sequence spaces and related topics in these non linear spaces see [4, 28]. In particular Sahiner [28] combined these two concepts and investigated ideal sumability in these spaces and introduced certain sequence spaces using 2-norm.
We continue in this direction and by using Musielak-Orlicz function, generalized sequences and also ideals we introduce I-convergence of generalized sequences with respect to Musielak-Orlicz function in n-normed spaces.

Let $(X, ||.||)$ be a normed space. Recall that a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of X is called statistically convergent to $x \in X$ if the set $A(\epsilon) = \Big\{n \in \mathbb{N} : ||x_n - x|| \geq \epsilon\Big\}$ has natural density zero for each $\epsilon > 0$.
A family $\mathcal{I} \subset 2^Y$ of subsets of a non empty set Y is said to be an ideal in Yif

1. $\phi \in \mathcal{I}$;
2. $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$;
3. $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$ (see [6]).

Given $\mathcal{I} \subset 2^{\mathbb{N}}$ be a non trivial ideal in $\mathbb{N}$. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be I-convergent to $x \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \Big\{n \in \mathbb{N} : ||x_n - x|| \geq \epsilon\Big\}$ belongs to $\mathcal{I}$ (see [11]).
Let I be an admissible ideal of $\mathbb{N}$, $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space. Let $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. By $S(n-X)$ we denote the space of all sequences defined over $(X, ||\cdot, \cdots, \cdot||)$. We define the following sequence spaces in this paper:
$\hat{w}^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||) =$

$$\Big\{x = (x_k) \in S(n-X) : \forall\, \epsilon > 0,\ \Big\{n \in \mathbb{N} :$$

$$\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x) - L)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I$$

$$\text{for some } \ \rho > 0,\ \ L \in X \ \text{ and } \ \ z_1, \cdots, z_{n-1} \in X\Big\},$$

$\hat{w}_0^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||) =$

$$\Big\{x = (x_k) \in S(n-X) : \forall\, \epsilon > 0,\ \Big\{n \in \mathbb{N} :$$

$$\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I$$

$$\text{for some } \ \rho > 0,\ \text{ and } \ \ z_1, \cdots, z_{n-1} \in X\Big\},$$

$\hat{w}_\infty(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||) =$

$$\Big\{x = (x_k) \in S(n-X) : \exists\ K > 0 \ \text{ such that}$$

$$\sup_{n\in\mathbb{N}}\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq K$$

$$\text{for some } \ \rho > 0,\ \text{ and } \ z_1, \cdots, z_{n-1} \in X\Big\},$$

$\hat{w}_\infty^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||) =$

$$\Big\{x = (x_k) \in S(n-X) : \exists\ K > 0 \ \text{ such that}$$

$$\Big\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \geq K\Big\} \in I$$

$$\text{for some } \ \rho > 0, \ \text{ and } \ z_1, \cdots, z_{n-1} \in X\Big\}.$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $D = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.
The main aim of this paper is to study some topological properties and inclusion relations between the above defined sequence spaces.

2. Main results

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and I be an admissible ideal of $\mathbb{N}$. Then $\hat{w}^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||)$, $\hat{w}_0^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||)$, $\hat{w}_\infty(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||)$ and $\hat{w}_\infty^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||)$ are linear spaces.*

Proof. Let $x, y \in \hat{w}^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||)$ and $\alpha, \beta \in \mathbb{C}$. So

$$\Big\{\Big\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x) - L)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I \quad \text{for some } \ \rho_1 > 0,$$

$$L \in X \text{ and } \quad z_1, \cdots, z_{n-1} \in X\Big\}$$

and

$$\Big\{\Big\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(y) - L)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I \ \text{ for some } \ \rho_2 > 0,$$

$$L \in X \text{ and } \quad z_1, \cdots, z_{n-1} \in X\Big\}.$$

Since $||\cdot, \cdots, \cdot||$ is a n-norm, $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and so by using inequality (1.1), we have

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(\alpha x+\beta y)-L)}{|\alpha|\rho_1+|\beta|\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\quad\leq\; D\frac{1}{n}\sum_{k=1}^{n}\Big[\frac{|\alpha|}{(|\alpha|\rho_1+|\beta|\rho_2)}M_k\Big(||\frac{t_{km}(\Lambda_k(x)-L)}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\quad+\; D\frac{1}{n}\sum_{k=1}^{n}\Big[\frac{|\beta|}{(|\alpha|\rho_1+|\beta|\rho_2)}M_k\Big(||\frac{t_{km}(\Lambda_k(y)-L)}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\quad\leq\; DF\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x)-L)}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\quad+\; DF\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(y)-L)}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k},
\end{aligned}$$

where $F=\max\Big[1,\Big(\frac{|\alpha|}{(|\alpha|\rho_1+|\beta|\rho_2)}\Big)^H,\Big(\frac{|\beta|}{(|\alpha|\rho_1+|\beta|\rho_2)}\Big)^H\Big]$. From the above inequality, we get

$$\begin{aligned}
&\Big\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(\alpha x+\beta y)-L)}{|\alpha|\rho_1+|\beta|\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\geq\epsilon\Big\}\\
&\subseteq\;\Big\{n\in\mathbb{N}:DF\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x)-L)}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\geq\frac{\epsilon}{2}\Big\}\\
&\cup\;\Big\{n\in\mathbb{N}:DF\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(y)-L)}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\geq\frac{\epsilon}{2}\Big\}.
\end{aligned}$$

Two sets on the right hand side belong to I and this completes the proof. Similarly, we can prove that $\hat{w}_0^I(\mathcal{M},\Lambda,p,||\cdot,\cdots,\cdot||)$, $\hat{w}_\infty(\mathcal{M},\Lambda,p,||\cdot,\cdots,\cdot||)$ and $\hat{w}_\infty^I(\mathcal{M},\Lambda,p,||\cdot,\cdots,\cdot||)$ are linear spaces.

Theorem 2.2. *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function, $p=(p_k)$ be a bounded sequence of positive real numbers. For any fixed $n\in\mathbb{N}$, $\hat{w}_\infty(\mathcal{M},\Lambda,p,||\cdot,\cdots,\cdot||)$ is a paranormed space with the paranorm defined by*

$$g(x)=\inf\Big\{\rho^{\frac{p_n}{H}}:\rho>0\quad\text{is such that}\;\sup_k\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\leq 1,\quad\forall z_1,\cdots,z_{n-1}\in X\Big\}.$$

Proof. It is clear that $g(x)=g(-x)$. Since $M_k(0)=0$, we get $\inf\{\rho^{\frac{p_n}{H}}\}=0$ for $x=0$ therefore, $g(0)=0$. Let us take $x,y\in\hat{w}_\infty(\mathcal{M},\Lambda,p,||\cdot,\cdots,\cdot||)$. Let

$$B(x)=\Big\{\rho^{\frac{p_n}{H}}:\rho>0,\sup_k\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\leq 1,$$

$$\forall z_1, \cdots, z_{n-1} \in X\Big\},$$

$$B(y) = \Big\{\rho^{\frac{p_n}{H}} : \rho > 0, : \sup_k \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(y))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq 1,$$

$$\forall z_1, \cdots, z_{n-1} \in X\Big\}.$$

Let $\rho_1 \in B(x)$ and $\rho_2 \in B(y)$. If $\rho = \rho_1 + \rho_2$, then we have
$\sup_k \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x+y))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]$

$$\begin{aligned} &\leq \frac{\rho_1}{\rho_1+\rho_2}\sup_k \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big] \\ &+ \frac{\rho_2}{\rho_1+\rho_2}\sup_k \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(y))}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]. \end{aligned}$$

Thus $\sup_k \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x+y))}{\rho_1+\rho_2}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \leq 1$ and

$$\begin{aligned} g(x+y) &\leq \inf\Big\{(\rho_1+\rho_2)^{\frac{p_n}{H}} : \rho_1 \in B(x),\ \rho_2 \in B(y)\Big\} \\ &\leq \inf\Big\{\rho_1^{\frac{p_n}{H}} : \rho_1 \in B(x)\Big\} + \inf\Big\{\rho_2^{\frac{p_n}{H}} : \rho_2 \in B(y)\Big\} \\ &= g(x) + g(y). \end{aligned}$$

Let $\sigma^m \to \sigma$ where $\sigma, \sigma^m \in \mathbb{C}$ and let $g(x^m - x) \to 0$ as $m \to \infty$. We have to show that $g(\sigma^m x^m - \sigma_x) \to 0$ as $m \to \infty$. Let

$$B(x^m) = \Big\{\rho_m^{\frac{p_n}{H}} : \rho_m > 0, \sup_k \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x^m))}{\rho_m}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq 1,$$

$$\forall z_1, \cdots, z_{n-1} \in X\Big\},$$

$$B(x^m - x) = \Big\{\rho_m'^{\frac{p_n}{H}} : \rho_m' > 0, \sup_k \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x^m - x))}{\rho_m'}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq 1,$$

$$\forall z_1, \cdots, z_{n-1} \in X\Big\}.$$

If $\rho_m \in B(x^m)$ and $\rho_m' \in B(x^m - x)$ then we observe that

$$
\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}\Big[M_k||\frac{t_{km}(\sigma^m\Lambda_k(x^m)-\sigma\Lambda_k(x))}{\rho_m|\sigma^m-\sigma|+\rho_m^{'}|\sigma|},z_1,\cdots,z_{n-1}||\\
&\leq \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\sigma^m\Lambda_k(x^m)-\sigma\Lambda_k(x^m))}{\rho_m|\sigma^m-\sigma|+\rho_m^{'}|\sigma|},z_1,\cdots,z_{n-1}||\\
&+ ||\frac{t_{km}(\sigma\Lambda_k(x^m)-\sigma\Lambda_k(x))}{\rho_m|\sigma^m-\sigma|+\rho_m^{'}|\sigma|},z_1,\cdots,z_{n-1}||\Big)\Big]\\
&\leq \frac{|\sigma^m-\sigma|\rho_m}{\rho_m|\sigma^m-\sigma|+\rho_m^{'}|\sigma|}\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x^m))}{\rho_m},z_1,\cdots,z_{n-1}||\Big)\Big]\\
&+ \frac{|\sigma|\rho_m^{'}}{\rho_m|\sigma^m-\sigma|+\rho_m^{'}|\sigma|}\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x^m)-\Lambda_k(x))}{\rho_m^{'}},z_1,\cdots,z_{n-1}||\Big)\Big].
\end{aligned}
$$

From the above inequality, it follows that

$$\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\sigma^m\Lambda_k(x^m)-\sigma\Lambda_k(x))}{\rho_m|\sigma^m-\sigma|+\rho_m^{'}|\sigma|},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\leq 1$$

and consequently,

$$
\begin{aligned}
g(\sigma^m x^m-\sigma x) &\leq \inf\Big\{\Big(\rho_m|\sigma^m-\sigma|+\rho_m^{'}|\sigma|\Big)^{\frac{p_n}{H}}:\rho_m\in B(x^m),\rho_m^{'}\in B(x^m-x)\Big\}\\
&\leq (|\sigma^m-\sigma|)^{\frac{p_n}{H}}\inf\Big\{\rho^{\frac{p_n}{H}}:\rho_m\in B(x^m)\Big\}\\
&+ (|\sigma|)^{\frac{p_n}{H}}\inf\Big\{(\rho_m^{'})^{\frac{p_n}{H}}:\rho_m^{'}\in B(x^m-x)\Big\}\longrightarrow 0 \text{ as } m\longrightarrow\infty.
\end{aligned}
$$

This completes the proof.

Theorem 2.3. *Let $\mathcal{M},\mathcal{M}',\mathcal{M}''$ are Musielak-Orlicz functions. Then we have*
(i) $\hat{w}_0^I(\mathcal{M}',\Lambda,p,||\cdot,\cdots,\cdot||)\subseteq\hat{w}_0^I(\mathcal{M}\circ\mathcal{M}',\Lambda,p,||\cdot,\cdots,\cdot||)$ *provided (p_k) is such that*

$$H_0=\inf p_k>0.$$

(ii) $\hat{w}_0^I(\mathcal{M}',\Lambda,p,||\cdot,\cdots,\cdot||)\cap\hat{w}_0^I(\mathcal{M}'',\Lambda,p,||\cdot,\cdots,\cdot||)\subseteq\hat{w}_0^I(\mathcal{M}'+\mathcal{M}'',\Lambda,p,||\cdot,\cdots,\cdot||)$.

Proof. (i) For given $\epsilon>0$, first choose $\epsilon_0>0$ such that $\max\{\epsilon_0^H,\epsilon_0^{H_0}\}<\epsilon$. Since (M_k) is continuous, choose $0<\delta<1$ such that $0<t<\delta$, this implies that $M_k(t)<\epsilon_0$. Let $x\in\hat{w}_0^I(\mathcal{M},\Lambda,p,||\cdot,\cdots,\cdot||)$. Now from the definition

$$B(\delta)=\Big\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}\Big[M_k'\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\geq\delta^H\Big\}\in I.$$

Thus if $n\notin B(\delta)$ then

$$\frac{1}{n}\sum_{k=1}^{n}\Big[M_k'\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}<\delta^H$$

$$\Longrightarrow \sum_{k=1}^{n}\Big[M'_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} < n\delta^H$$

$$\Longrightarrow \Big[M'_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} < \delta^H \text{ for all } k, m = 1, 2, 3, \ldots$$

$$\Longrightarrow \Big[M'_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] < \delta \text{ for all } k, m = 1, 2, 3, \ldots.$$

Hence from above and using the continuity of $\mathcal{M} = (M_k)$, we have

$$\Big[M_k\Big(M'_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big] < \epsilon_0 \ \forall \ k, m = 1, 2, 3, \ldots ,$$

which consequently implies that

$$\sum_{k=1}^{n}\Big[M_k\Big(M'_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} < \max\{\epsilon_0^H, \epsilon_0^{H_0}\} < \epsilon.$$

$$\text{Thus } \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(M'_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} < \epsilon.$$

This shows that

$$\Big\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(M'_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \geq \epsilon\Big\} \subset B(\delta)$$

and so belongs to I. This proves the result.

(ii) Let $(x_k) \in \hat{w}_0^I(\mathcal{M}', \Lambda, p, ||\cdot, \cdots, \cdot||) \cap \hat{w}_0^I(\mathcal{M}'', \Lambda, p, ||\cdot, \cdots, \cdot||)$. Then the fact

$$\frac{1}{n}\Big[(M'_k + M''_k)\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq$$

$$D\frac{1}{n}\Big[M'_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} + D\frac{1}{n}\Big[M''_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$$

gives the result.

Theorem 2.4. *The sequence spaces* $\hat{w}_0^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||)$ *and* $\hat{w}_\infty^I(\mathcal{M}', \Lambda, p, ||\cdot, \cdots, \cdot||)$ *are solid.*

Proof. Let $x \in \hat{w}_0^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||)$, let (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then we have

$$\Big\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(\alpha_k x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big\} \subset$$

$$\Big\{n \in \mathbb{N} : \frac{C}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I,$$

where $C = \max\{1, |\alpha_k|^H\}$. Hence $(\alpha_k x) \in \hat{w}_0^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||)$ for all sequences of scalars α_k with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$ whenever $(x_k) \in \hat{w}_0^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||)$. Similarly, we can prove that $\hat{w}_\infty^I(\mathcal{M}', \Lambda, p, ||\cdot, \cdots, \cdot||)$ is also solid.

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