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Higher order efficiency and duality for multiobjective variational problem*

by

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Abstract: In this paper, multiobjective variational programming problem is considered. Mond-Weir type higher order duality results are established by using the efficiency of higher order as the optimization tool. In order to prove these results, we propose the notion of generalized higher order $(F, \rho, \theta, m, h)-$invexity.

Keywords: higher order duality, variational problem, efficiency of higher order

1. Introduction

One of the most important aspects in the development of numerical algorithms for solving optimization problems is duality. Higher order duality has even greater significance than the first order duality, since it provides tighter bounds for the value of the objective function when approximations are used, because it involves more parameters. Higher order duality for the variational problem has been studied by Sharma (see Sharma, 2015) and Padhan and Nahak (see Padhan and Nahak, 2013), whereas first order duality was studied by Stancu-Minasian and Mititelu (see Stancu-Minasian and Mititelu, 2008). They used optimality as solution concept for the single objective problem and efficiency for the multiobjective problem to prove duality results. In this paper, we use the concept of efficiency of higher order to study the multiobjective variational problem. A higher order dual of Mond-Weir type is proposed. The class of generalized higher order $(F, \rho, \theta, m, h)-$invex functionals is also introduced in this paper, which facilitates obtaining the duality results for the proposed dual. The paper is organized as follows: in Section 2 some basic definitions and preliminaries are given. Mond-Weir type higher order dual of the multiobjective variational problem is proposed in Section 3 and duality results are obtained under the assumptions of generalized higher order $(F, \rho, \theta, m, h)-$invexity.

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2. Definitions and preliminaries

Let n, r and p be positive integers. Here $\mathbb{R}^n$ denotes an n-dimensional Euclidean space.
For any $x = (x^1, x^2, \dots, x^n)^T$, $y = (y^1, y^2, \dots, y^n)^T \in \mathbb{R}^n$.
(i) $x = y \Leftrightarrow x^i = y^i$ for all $i = 1, 2, \dots, n$.
(ii) $x < y \Leftrightarrow x^i < y^i$ for all $i = 1, 2, \dots, n$.
(iii) $x \leqq y \Leftrightarrow x^i \leq y^i$ for all $i = 1, 2, \dots, n$.
(iv) $x \leq y \Leftrightarrow x \leqq y$ and $x \neq y$.

Let $\mathbb{R}^n_+$ and $\operatorname{int}\mathbb{R}^n_+$ denote the non negative and positive orthant of $\mathbb{R}^n$, respectively. For a given real interval $I = [a, b]$, let $x : I \to \mathbb{R}^n$ be a piecewise smooth state function with its derivative $\dot{x}$. For notational convenience $x(t)$ and $\dot{x}(t)$ will be written as x and $\dot{x}$, respectively. Let $f^i : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $i \in K = \{1, 2, \dots, k\}$ and $g^i : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $i \in M = \{1, 2, \dots, m\}$ be continuously differentiable functions with respect to each of its argument. Let us denote the partial derivative of $f^i, i \in K$, with respect to t, x and $\dot{x}$ by f^i_t, f^i_x and $f^i_{\dot{x}}$, respectively. Analogously, we write the partial derivative of $g^i, i \in M$.

Let X be the space of piecewise smooth state functions $x : I \to \mathbb{R}^n$, equipped with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differential operator D is given by $u = Dx \Leftrightarrow x(t) = x(a) + \int_a^t u(s)ds$.

Therefore, $D = \frac{d}{dt}$ except at discontinuities.

The Multi-Objective Variational Problem (P) is defined as follows:

$$\text{(P) Minimize } \int_a^b f(t, x, \dot{x})dt = \left(\int_a^b f^1(t, x, \dot{x})dt, \dots, \int_a^b f^k(t, x, \dot{x})dt \right)$$

subject to

$$g(t, x, \dot{x}) = (g^1(t, x, \dot{x}), \dots, g^p(t, x, \dot{x})) \leqq 0, \; t \in I, \tag{1}$$

$$x(a) = \alpha, x(b) = \beta, \alpha, \beta \in \mathbb{R}^n. \tag{2}$$

Let $X_0 = \{x \in X \mid g(t, x, \dot{x}) \leqq 0, \; t \in I, x(a) = \alpha, \; x(b) = \beta\}$ be the set of all feasible solutions of (P).

In the sequel, we shall define efficient solution and efficient solution of higher order for (P).

Definition 2.1 *$\bar{x} \in X_0$ is said to be an efficient solution for (P) if there is no other $x \in X_0$ such that*

$$\int_a^b f^i(t, x, \dot{x})dt \leq \int_a^b f^i(t, \bar{x}, \dot{\bar{x}})dt, \text{ for all } i \in K \text{ and,}$$

$$\int_a^b f^j(t,x,\dot{x})dt < \int_a^b f^j(t,\bar{x},\dot{\bar{x}})dt, \textit{for at least one } j \in K.$$

Let $m \geq 1$ be an integer and θ be a piecewise smooth vector valued function on $I \times \mathbb{R}^n \times \mathbb{R}^n$.

Definition 2.2 *$\bar{x} \in X_0$ is said to be an efficient solution of order m for (P) with respect to θ if there exists $c = (c^1, c^2, ..., c^k) \in int\mathbb{R}^k_+$ such that for no other $x \in X_0$*

$$\int_a^b f^i(t,x,\dot{x})dt \leq \int_a^b \{f^i(t,\bar{x},\dot{\bar{x}}) + c^i \|\theta(t,x,\bar{x})\|^m\}dt, \;\; \textit{for all } i \in K \textit{ and}$$

$$\int_a^b f^j(t,x,\dot{x})dt < \int_a^b \{f^j(t,\bar{x},\dot{\bar{x}}) + c^j \|\theta(t,x,\bar{x})\|^m\}dt, \textit{ for at least one } j \in K.$$

To facilitate the establishment of higher order duality for this solution concept, we introduce the following class of functionals. Let $\Phi : X \to \mathbb{R}$ defined by $\Phi(x) = \int_a^b \phi(t,x,\dot{x})dt$, be Fréchet differentiable, where $\phi(t,x,\dot{x})$ is a scalar function with continuous derivatives up to and including second order with respect to each of its arguments.

Let there exist a real number ρ and a functional $F : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\cdot)$ is sub-linear on $\mathbb{R}^n$, i.e. for any $x(t), \bar{x}(t) \in \mathbb{R}^n$

$$F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\xi_1+\xi_2) \leqq F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\xi_1) + F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\xi_2), \quad \text{for any } \xi_1, \xi_2 \in \mathbb{R}^n, \tag{3}$$

$$F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\gamma\xi) = \gamma F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\xi), \text{ for any } \gamma \in \mathbb{R}_+ \text{ and } \xi \in \mathbb{R}^n. \tag{4}$$

It is quite evident from (4) that

$$F(t,x,\dot{x},\bar{x},\dot{\bar{x}},0) = 0. \tag{5}$$

For the sake of notational convenience, let $\phi_x(t)$ represent $\phi_x(t,x(t),\dot{x}(t))$ and $\phi_{\dot{x}}(t)$ represent $\phi_{\dot{x}}(t,x(t),\dot{x}(t))$. Let $p \in \mathbb{R}^n$ and $h : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a differentiable function.

Definition 2.3 *A functional $\Phi(x)$ is said to be higher order $(F,\rho,\theta,m,h)-$invex at $\bar{x} \in X$ if*

$$\Phi(x) - \Phi(\bar{x}) - \int_a^b \{h(t,\bar{x},\dot{\bar{x}},p) - p^T \bigtriangledown_p h(t,\bar{x},\dot{\bar{x}},p)\}dt$$
$$\geq \int_a^b \{F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\bigtriangledown_p h(t,\bar{x},\dot{\bar{x}},p)) + \rho\|\theta(t,x,\bar{x})\|^m\}dt, \textit{ for all } x \in X.$$

Remark 2.1 *(a) If* $h(t,\bar{x},\dot{\bar{x}},p) = p^T[\phi_{\bar{x}}(t) - \frac{d}{dt}(\phi_{\dot{\bar{x}}}(t))]$*, then Definition* 2.3 *reduces to the definition of* $(F,\rho)-$*invexity of higher order (see Kumar and Sharma, 2016).*
(b) If $h(t,\bar{x},\dot{\bar{x}},p) = p^T[\phi_{\bar{x}}(t) - \frac{d}{dt}(\phi_{\dot{\bar{x}}}(t))] + \frac{1}{2}p^T[\phi_{\bar{x}\bar{x}}(t) - 2\frac{d}{dt}\phi_{\bar{x}\dot{\bar{x}}}(t) + \frac{d^2}{dt^2}\phi_{\dot{\bar{x}}\dot{\bar{x}}}(t)]p$*, then Definition* 2.3 *reduces to the definition of second order* $(F,\alpha,\rho,\theta)-$*convexity with* $\alpha(x,\bar{x}) = 1, \theta\colon I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},\ \beta(t) = p,\ t \in I$ *(see Jayswal, Stancu-Minasian and Choudhury, 2015).*

Definition 2.4 *A functional* $\Phi(x)$ *is said to be higher order* (F,ρ,θ,m,h)*-quasiinvex type 1 at* $\bar{x} \in X$ *if*

$$\Phi(x) \le \Phi(\bar{x}) + \int_a^b \{h(t,\bar{x},\dot{\bar{x}},p) - p^T \bigtriangledown_p h(t,\bar{x},\dot{\bar{x}},p)\}dt$$

$$\Rightarrow \int_a^b \{F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\bigtriangledown_p h(t,\bar{x},\dot{\bar{x}},p)) + \rho\|\theta(t,x,\bar{x})\|^m\}dt \le 0,\ \textit{for all}\ x \in X.$$

Definition 2.5 *A functional* $\Phi(x)$ *is said to be higher order* (F,ρ,θ,m,h)*-quasiinvex type 2 at* $\bar{x} \in X$ *if*

$$\Phi(x) \le \Phi(\bar{x}) + \int_a^b \{h(t,\bar{x},\dot{\bar{x}},p) - p^T \bigtriangledown_p h(t,\bar{x},\dot{\bar{x}},p) + \rho\|\theta(t,x,\bar{x})\|^m\}dt$$

$$\Rightarrow \int_a^b \{F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\bigtriangledown_p h(t,\bar{x},\dot{\bar{x}},p))\}dt \le 0,\ \textit{for all}\ \ x \in X.$$

Definition 2.6 *A functional* $\Phi(x)$ *is said to be higher order* (F,ρ,θ,m,h)*-pseudoinvex type 1 at* $\bar{x} \in X$ *if*

$$\int_a^b \{F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\bigtriangledown_p h(t,\bar{x},\dot{\bar{x}},p)) + \rho\|\theta(t,x,\bar{x})\|^m\}dt \ge 0$$

$$\Rightarrow \Phi(x) \ge \Phi(\bar{x}) + \int_a^b \{h(t,\bar{x},\dot{\bar{x}},p) - p^T \bigtriangledown_p h(t,\bar{x},\dot{\bar{x}},p)\}dt$$

for all $x \in X$.

Definition 2.7 *A functional* $\Phi(x)$ *is said to be higher order* (F,ρ,θ,m,h)*-pseudoinvex type 2 at* $\bar{x} \in X$ *if*

$$\int_a^b \{F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\bigtriangledown_p h(t,\bar{x},\dot{\bar{x}},p)\}dt \ge 0$$

$$\Rightarrow \Phi(x) \ge \Phi(\bar{x}) + \int_a^b \{h(t,\bar{x},\dot{\bar{x}},p) - p^T \bigtriangledown_p h(t,\bar{x},\dot{\bar{x}},p) + \rho\|\theta(t,x,\bar{x})\|^m\}dt$$

for all $x \in X$.

DEFINITION 2.8 *A functional $\Phi(x)$ is said to be higher order (F, ρ, θ, m, h)-strictly pseudoinvex type 2 at $\bar{x} \in X$ if*

$$\int_a^b \{F(t, x, \dot{x}, \bar{x}, \dot{\bar{x}}, \bigtriangledown_p h(t, \bar{x}, \dot{\bar{x}}, p)\}dt \geq 0$$

$$\Rightarrow \Phi(x) > \Phi(\bar{x}) + \int_a^b \{h(t, \bar{x}, \dot{\bar{x}}, p) - p^T \bigtriangledown_p h(t, \bar{x}, \dot{\bar{x}}, p) + \rho\|\theta(t, x, \bar{x})\|^m\}dt$$

for all $x \in X - \{\bar{x}\}$.

3. Higher order duality

Consider the following problem:

$$\text{(D) maximize} \left(\int_a^b \{f^1(t, \bar{x}, \dot{\bar{x}}) + h^1(t, \bar{x}, \dot{\bar{x}}, p) - p^T \bigtriangledown_p h^1(t, \bar{x}, \dot{\bar{x}}, p)\}dt, \ldots, \right.$$

$$\left. \ldots, \int_a^b \{f^k(t, \bar{x}, \dot{\bar{x}}) + h^k(t, \bar{x}, \dot{\bar{x}}, p) - p^T \bigtriangledown_p h^k(t, \bar{x}, \dot{\bar{x}}, p)\}dt \right)$$

subject to

$$\sum_{i=1}^{k} \bar{\lambda}^i [\bigtriangledown_p h^i(t, \bar{x}, \dot{\bar{x}}, p)] + \sum_{j=1}^{m} \bar{y}^j(t) \bigtriangledown_p k^j(t, \bar{x}, \dot{\bar{x}}, p) = 0, \; t \in I, \tag{6}$$

$$\int_a^b \{\sum_{j=1}^{m} \bar{y}^j(t)(g^j(t, \bar{x}, \dot{\bar{x}}) + k^j(t, \bar{x}, \dot{\bar{x}}, p)) - p^T(\sum_{j=1}^{m} \bar{y}^j(t) \bigtriangledown_p k^j(t, \bar{x}, \dot{\bar{x}}, p))\}dt \geqq 0. \tag{7}$$

$$\bar{y}^j(t) \geq 0, t \in I, j \in M, \bar{x} \in X. \tag{8}$$

$$\bar{x}(a) = 0, \bar{x}(b) = 0, \tag{9}$$

where $h^i : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, i \in K$ and $k^i : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, i \in M$ are differentiable functions. Let Y_0 be the set of all feasible solutions of (D).

THEOREM 3.1 *(Weak duality) Let $x \in X_0$, $(\bar{x}, \bar{\lambda}, \bar{y}, p) \in Y_0$. Let*
(i) $\Phi^i(x) = \int_a^b \{f^i(t, x, \dot{x})\}dt, i \in K$ be higher order $(F, \rho^i, \theta, m, h^i)-$strictly pseudoinvex type 2 at $\bar{x}$.
(ii) $\Psi(x) = \int_a^b \{\sum_{i \in M} \bar{y}^i(t) g^i(t, x, \dot{x})\}dt$ be higher order $(F, \rho', \theta, m, l)-$quasiinvex type 1 at $\bar{x}$ where $l(t, x, \dot{x}, p) = \sum_{j=1}^{m} \bar{y}^j(t)(k^j(t, x, \dot{x}, p))$.

(iii) $(\rho^1, \rho^2, ..., \rho^k) \in int\mathbb{R}^k_+$ *and* $\rho' > 0$.
Then there exists $c = (c^1, \ldots, c^k) \in int\mathbb{R}^k_+$ *such that the following cannot hold:*

$$\int_a^b f^i(t, x, \dot{x})dt \leq \int_a^b \{f^i(t, \bar{x}, \dot{\bar{x}}) + h^i(t, \bar{x}, \dot{\bar{x}}, p) - p^T \bigtriangledown_p h^i(t, \bar{x}, \dot{\bar{x}}, p) + c^i \|\theta(t, x, \bar{x})\|^m\}dt,$$
for all $i \in K$ (10)

and

$$\int_a^b f^j(t, x, \dot{x})dt < \int_a^b \{f^j(t, \bar{x}, \dot{\bar{x}}) + h^j(t, \bar{x}, \dot{\bar{x}}, p) - p^T \bigtriangledown_p h^j(t, \bar{x}, \dot{\bar{x}}, p) + c^j \|\theta(t, x, \bar{x})\|^m\}dt,$$
for at least one $j \in K$. (11)

Proof. Suppose (3.1) and (3.1) hold for all $c = (c^1, \ldots, c^k) \in \mathrm{int}\,\mathbb{R}^k_+$, in particular, by taking $c^i = \rho^i,\ i \in K$ in (3.1) and (3.1), Hypothesis (i) yields

$$\int_a^b \{F(t, x, \dot{x}, \bar{x}, \dot{\bar{x}}, \bigtriangledown_p h^i(t, \bar{x}, \dot{\bar{x}}, p))\}dt < 0,\ i \in K.$$

Multiplying the above inequalities by $\bar{\lambda}^i, i \in K$, and then adding, gives

$$\int_a^b \{\sum_{i=1}^k \bar{\lambda}^i F(t, x, \dot{x}, \bar{x}, \dot{\bar{x}}, \bigtriangledown_p h^i(t, \bar{x}, \dot{\bar{x}}, p))\}dt < 0.$$

Now, using (3) and (4), we get

$$\int_a^b \{F(t, x, \dot{x}, \bar{x}, \dot{\bar{x}}, \sum_{i=1}^k \bar{\lambda}^i \bigtriangledown_p h^i(t, \bar{x}, \dot{\bar{x}}, p))\}dt < 0. \tag{12}$$

Using feasibility of x and $(\bar{x}, \bar{\lambda}, \bar{y}, p)$, we get

$$\Psi(x) \leqq 0 \leqq \Psi(\bar{x}) + \int_a^b \{\sum_{j=1}^m \bar{y}^j(t)(k^j(t, \bar{x}, \dot{\bar{x}}, p)) - p^T(\sum_{j=1}^m \bar{y}^j(t) \bigtriangledown_p k^j(t, \bar{x}, \dot{\bar{x}}, p))\}dt.$$

Hypothesis (ii) yields

$$\int_a^b \{F(t, x, \dot{x}, \bar{x}, \dot{\bar{x}}, \sum_{j=1}^m \bar{y}^j(t) \bigtriangledown_p k^j(t, \bar{x}, \dot{\bar{x}}, p)) + \rho' \|\theta(t, x, \bar{x})\|^m\}dt \leq 0. \tag{13}$$

Upon using (3) and (6), after adding both sides of the inequalities (12) and (13), we get

$$\int_a^b \rho' \|\theta(t,x,\bar{x})\|^m dt < 0.$$

This is a contradiction, as $\rho' > 0, \|\theta(t,x,\bar{x})\|^m \geq 0$. Thereby, the proof is completed. ∎

THEOREM 3.2 *(Strong duality) Let $\bar{x}$ be an efficient solution of order m for (P) with respect to θ, which is normal and let $h^i(t,\bar{x},\dot{\bar{x}},0) = 0, i \in K, k^i(t,\bar{x},\dot{\bar{x}},0) = 0,\ i \in M$,*

$$\bigtriangledown_P h^i(t,\bar{x},\dot{\bar{x}},0) = f^i_{\bar{x}}(t) - \frac{d}{dt} f^i_{\dot{\bar{x}}}(t),\ i \in K,$$

$$\sum_{j=1}^m \bar{y}^j(t) \bigtriangledown_p k^j(t,\bar{x},\dot{\bar{x}},0) = \sum_{j=1}^m \bar{y}^j(t) g^i_{\bar{x}}(t) - \frac{d}{dt}[\sum_{j=1}^m \bar{y}^j(t) g^i_{\dot{\bar{x}}}(t)],$$

for any piecewise smooth functions $\bar{y} : I \to \mathbb{R}^p$ such that

$$\bar{y}(t) = (\bar{y}^1(t), \ldots, \bar{y}^p(t)), t \in I.$$

Then there exist $\bar{\lambda} = (\bar{\lambda}^1, \ldots, \bar{\lambda}^k) \in \mathbb{R}^k$, piecewise smooth function $\bar{y} : I \to \mathbb{R}^p$ and $\bar{p} \in \mathbb{R}^n$ such that $(\bar{x}, \bar{\lambda}, \bar{y}, \bar{p}) \in Y_0$. Further, if weak duality theorem holds and $\|\theta(t,x,\bar{x})\|^m = \|\theta(t,\bar{x},x)\|^m$, for all $x \in X$, then $(\bar{x}, \bar{\lambda}, \bar{y}, \bar{p})$ is an efficient solution of order m for (D) with respect to same θ.

PROOF. Since every efficient solution of order m for (P) with respect to θ is an efficient solution for (P), so $\bar{x}$ is an efficient solution for (P), which is normal. By Theorem 3.4. in Stancu-Minasian and Mititelu (2009), there exist $\bar{\lambda} = (\bar{\lambda}^1, ..., \bar{\lambda}^k) \in \mathbb{R}^k$, piecewise smooth function $\bar{y} : I \to \mathbb{R}^p$ and $\bar{p} = 0$ such that $(\bar{x}, \bar{\lambda}, \bar{y}, \bar{p}) \in Y_0$. Suppose weak duality theorem holds and $\|\theta(t,x,\bar{x})\|^m = \|\theta(t,\bar{x},x)\|^m$, for all $x \in X$. Suppose $(\bar{x}, \bar{\lambda}, \bar{y}, \bar{p})$ is not an efficient solution of order m with respect to θ for (D). Then, for any $c = (c^1, c^2, ..., c^k) \in \operatorname{int} \mathbb{R}^k_+$, there exist $(\hat{x}, \hat{\lambda}, \hat{y}, \hat{p}) \in Y_0$ such that

$$\int_a^b \{f^i(t,\bar{x},\dot{\bar{x}}) + h^i(t,\bar{x},\dot{\bar{x}},\bar{p}) - \bar{p}^T \bigtriangledown_{\bar{p}} h^i(t,\bar{x},\dot{\bar{x}},\bar{p}) dt$$
$$\leq \int_a^b \{f^i(t,\hat{x},\dot{\hat{x}}) + h^i(t,\hat{x},\dot{\hat{x}},\hat{p}) - \hat{p}^T \bigtriangledown_{\hat{p}} h^i(t,\hat{x},\dot{\hat{x}},\hat{p}) + c^i \|\theta(t,\bar{x},\hat{x})\|^m\} dt,$$

for all $i \in K$

and

$$\int_a^b \{f^j(t,\bar{x},\dot{\bar{x}}) + h^j(t,\bar{x},\dot{\bar{x}},\bar{p}) - \bar{p}^T \bigtriangledown_{\bar{p}} h^j(t,\bar{x},\dot{\bar{x}},\bar{p})dt$$
$$< \int_a^b \{f^j(t,\hat{x},\dot{\hat{x}}) + h^j(t,\hat{x},\dot{\hat{x}},\hat{p}) - \hat{p}^T \bigtriangledown_{\hat{p}} h^j(t,\hat{x},\dot{\hat{x}},\hat{p}) + c^j\|\theta(t,\bar{x},\hat{x})\|^m\}dt,$$
for at least one $j \in K$.

Using hypothesis, we get

$$\int_a^b \{f^i(t,\bar{x},\dot{\bar{x}})\}dt \leq \int_a^b \{f^i(t,\hat{x},\dot{\hat{x}}) + h^i(t,\hat{x},\dot{\hat{x}},\hat{p})$$
$$-\hat{p}^T \bigtriangledown_{\hat{p}} h^i(t,\hat{x},\dot{\hat{x}},\hat{p}) + c^i\|\theta(t,\bar{x},\hat{x})\|^m\}dt,$$
for all $i \in K$ and

$$\int_a^b \{f^j(t,\bar{x},\dot{\bar{x}})\}dt < \int_a^b \{f^j(t,\hat{x},\dot{\hat{x}}) + h^j(t,\hat{x},\dot{\hat{x}},\hat{p})$$
$$-\hat{p}^T \bigtriangledown_{\hat{p}} h^j(t,\hat{x},\dot{\hat{x}},\hat{p}) + c^j\|\theta(t,\bar{x},\hat{x})\|^m\}dt,$$
for at least one $j \in K$,

which is a contradiction to the weak duality theorem. ■

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