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Empirical methods of reducing the observations in geodetic networks

Roman Kadaj

Rzeszów University of Technology
Department of Geodesy and Geotechnics
12 Powstańców Warszawy St., 35-959 Rzeszów
e-mail: geonet@geonet.net.pl

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Abstract: The paper presents empirical methodology of reducing various kinds of observations in geodetic network. A special case of reducing the observation concerns cartographic mapping. For numerical illustration and comparison of methods an application of the conformal *Gauss-Krüger* mapping was used. Empirical methods are an alternative to the classic differential and multi- stages methods. Numerical benefits concern in particular very long geodesics, created for example by GNSS vectors. In conventional methods the numerical errors of reduction values are significantly dependent on the length of the geodesic. The proposed empirical methods do not have this unfavorable characteristics. Reduction value is determined as a difference (or especially scaled difference) of the corresponding measures of geometric elements (distances, angles), wherein these measures are approximated independently in two spaces based on the known and corresponding approximate coordinates of the network points. Since in the iterative process of the network adjustment, coordinates of the points are systematically improved, approximated reductions also converge to certain optimal values.

Keywords: cartographic mapping, reducing of geodetic observations, empirical methods of reducing the observations, mapping of long geodesics

1. Introduction and formulation of the issues

The problem of reducing observations in geodetic networks can be treated as a conversion of observations from a physical measuring space to certain mathematical space, where a network adjustment or computation should be realized. Mathematical space may be represented, for example, by geodetic (ellipsoidal) coordinates system (*B, L, H*), three-dimensional *Cartesian* geocentric system (*X, Y, Z*) or *Cartesian* system (*x, y*) on the plane of a cartographic mapping. Formulas for all kinds of observational reductions (corrections to direct observations) are usually derived as differential form of elementary conversions, defined with a numerical approximation (see about classic ellipsoidal reductions e.g. in Czarnecki, 1994; Szpunar, 1982; Warchałowski, 1952; Zakatow, 1976 or classic mapping reductions e.g. in Balcerzak, 1994, 1995; Gajderowicz, 2009; Kadaj, 2001). Unfortunately, numerical accuracy of these formulas is not always enough in contemporary observation systems, where, next to terrestrial observations, there are very long GNSS vectors or pseudo-observations (for example, the length and azimuth of the geodesic, ellipsoidal heights differences) related to them. Some extreme examples are shown in Section 5.

Empirical methods are presented in this paper as an alternative to conventional ways of reducing the observations as they guarantee the highest precision, irrespective to the spatial location of points and their mutual distances. The idea of empirical methods is that the measures of reductions (observational corrections) are obtained as differences between measures of network elements computed in two coordinates systems by using approximate coordinates of network points and relationship between two systems, known a priori. Furthermore, unlike conventional methods where reductions of observations consist of many components (for example for measured distances: reduction to a level, reduction on the ellipsoid and on the mapping plane), in the empirical methods reduction is a direct (one-step) conversion from the observational space to a specific mathematical space.

A special case of observational reductions are mapping reductions, used for example in the *Gauss-Krüger* (transverse *Mercator*) projection. Explicit forms of this type of observational reductions are inseparable elements of mapping procedures. Unfortunately, they are usually limited in terms of numerical error (truncation of Taylor series), thus not suitable for very long geodesics, created from GNSS vectors. The correspond examples are discussed in Section 5.

Empirical algorithms, exclusively for the mapping reductions, were constructed in 2001 (Kadaj, 2001, in Section 5.11) with the use of the procedures of calculating length and azimuth of geodesics. The similar ideas and algorithms can be found in the new editions of the books: Leick, 2004 – in Section 9.2, Gajderowicz, 2009 – in Section 10.9.

This paper presents a generalized approach to empirical methodology of observational reductions in geodetic networks, relating to different types of geometrical or physical reductions, not only for cartographic mapping. An important part of the work was dedicated to the practical verification of the proposed methods and their comparison with classic methods (e.g. Balcerzak, 1994, 1995; Czarnecki, 1994; Gajderowicz, 2009; Szpunar, 1982; Warchałowski, 1952; Zakatow, 1976) .

2. Theoretical grounds of observational reduction in empirical methodology

2.1. General principle

A geodetic network, in general, can be a set of different kinds of observations or pseudo-observations (for example, processed from the GNSS vectors), which, through appropriate analytical compounds, including reference conditions, stochastic models and adjustment methods, define the position of points in a conventional frame of coordinates or in a mathematical space. In specific cases it may be, for instance, geodetic (ellipsoidal) coordinate system (B, L, h), *Cartesian* geocentric coordinates system (X, Y, Z) or *Cartesian* system (x, y) based on a cartographic mapping. Direct observations defining a geodetic network come from the measurement space, different to the defined mathematical space, not only in terms of geometry or its dimension and metric, but also in terms of deformation under the influence of physical fields of the Earth, such as gravity field or atmospheric refraction. This is with geodesy the universal knowledge – see e.g. in Czarnecki (1994). Therefore, the problem to reduce (mapping, transformation) the observations from the physical measurement space to a specific mathematical space occurs. In a conventional approach of this issue, measurements (observations) are subject to many kinds of observational reductions, for instance, a measured slant distance is reduced to horizontal position and then to the ellipsoid (reduction due to the ellipsoidal height) and finally put on the model of cartographic mapping (see e.g. Fig. 1, 2).

In comparison to the classic methodology, in the proposed empirical methodology of observational reductions, a direct (one-step) conversion of observation measures between two spaces can be realized. For this purpose, the approximate coordinates of points are used as they allow to obtain (approximate) independent observation measures in two spaces. Based on the difference of approximated measures in both spaces and after some non essential scaling, an empirical reduction as a correction of original observation measures is finally obtained.

Assuming that Θ_I is an elementary observation (e.g. a distance, an angle) in a measured space of geodetic network, we will reduce the observation to a measure Θ_{II} in the mathematical space where an adjustment and calculation of the network should be carried out. For this purpose, we need the quantity $\delta\Theta_{I\text{-}II}$ as a reduction of an observation value between two spaces (in here named symbolically: I, II), $\Theta_{II} = \Theta_I + \delta\Theta_{I\text{-}II.}$ The reduction $\delta\Theta_{I\text{-}II}$ can be calculated using classic explicit form (see also in Section 3.1 and 6.1). In general, it is the sum of any components: $\delta\Theta_{I\text{-}II}{}^{(\text{class})} = \delta_1 + \delta_2 + \ldots \cong \delta\Theta_{I\text{-}II.}$ The proposed empirical methodology leads to the direct approximation of a full quantity $\delta\Theta_{I\text{–}II}$ with use of approximate coordinates of network points corresponding in both spaces. Empirical reduction approximates corresponding theoretical quantity $\delta\Theta_{I\text{-}II}{}^{(\text{emp})} \cong \delta\Theta_{I\text{-}II}$ and it is particularly defined as follows.

Let $\boldsymbol{X}^{(0)}$ be the vector containing coordinates of points representing the approximate geometric model of a given observation in a network. For example, if the observation is a slant distance, the vector $\boldsymbol{X}^{(0)}$ includes approximate coordinates of two points in any three-dimensional space. It may be for example a geocentric *Cartesian* or three-dimensional topocentric system. The measure $\Theta_I{}^{(0)}$ of a geometric network element, corresponding to the approximate coordinates, as a model of observation Θ_I in the measurement space, is expressed as a certain (known a priori) function f_I :

$$\Theta_I{}^{(0)} = f_I(\boldsymbol{X}^{(0)};\ c_1, c_2, \ldots) \tag{1}$$

where c_1, c_2, ... parameters representing physical parameters of measurement space relative to an adopted coordinate system (e.g. local components of vertical deviations). The counterpart of the vector $\boldsymbol{X}^{(0)}$ in the mathematical space, such as a mapping plane, is a vector $\boldsymbol{x}^{(0)}$ obtained by a known a priori transformation (mapping) function $\boldsymbol{F}$:

$$\boldsymbol{x}^{(0)} = \boldsymbol{F}(\boldsymbol{X}^{(0)}) \tag{2}$$

Now, the measure of the corresponding geometric elements in the defined mathematical space is denoted as $\Theta_{II}^{(0)}$, using a known function f_{II}:

$$\Theta_{II}^{(0)} = f_{II}(\boldsymbol{x}^{(0)}) \tag{3}$$

Designating $\delta\Theta_{I\text{-}II}{}^{(0)} = \Theta_{II}{}^{(0)} - \Theta_I{}^{(0)}$ we define the empirical reduction in two cases:

$$\delta\Theta_{I\text{-}II}{}^{(\mathrm{emp})} = \begin{cases} \delta\Theta_{I\text{-}II}{}^{(0)} & \text{for azimuths, angles and directions} \quad (4) \\ (\delta\Theta_{I\text{-}II}{}^{(0)} / \Theta_I{}^{(0)}) \cdot \Theta_I & \text{for distances} \quad (5) \end{cases}$$

Where: Θ_I (without upper index) is an initial observation measure and $\Theta_I{}^{(0)}$ corresponding measure computed with approximated point coordinates. Why has the form (5) differed since (4)? In case of angles (azimuths, directions) the quantity $\delta\Theta_{I\text{-}II}{}^{(0)}$ is a relative measure (e.g. in radian, if $\alpha = s/r = arc\ /\ radius$ the differential is $\delta\alpha = \delta s/r$), however in case of distances, the analogue relative measure will be $\delta\Theta_{I\text{-}II}{}^{(0)}/\Theta_I{}^{(0)}$. Yet, significant inequality between values (5) and (4) can be observed in case of short distances and big errors of approximated coordinates. In turn, in case of long distances and bounded errors of coordinates, the difference between (4) and (5) should be, in principle, numerically not significant, especially if we assume that the approximate coordinates of the network points are successively improved in the nonlinear iterative process of network computation. For the nonlinear adjustment problem of geodetic networks is usually the *Gauss – Newton* iterative method implemented. The theoretical basis to the *Gauss-Newton* method can be found e.g. in Deutsch (1965), Sections 6.3, 6.4 and 7.4. The general theorems for the nonlinear optimization problems can be found e.g. in Zangwill (1969).

Of course, natural question about the accuracy of such approximation in terms of its maximum error e: $|\delta\Theta_{I\text{-}II}{}^{(\mathrm{emp})} - \delta\Theta_{I\text{-}II}| \leq e$, dependent on the accuracy of the approximate coordinates arises. However, it is known that the adjustment of observation and computation of coordinates of network points is a nonlinear least squares problem, solved with iterative procedures. In the properly defined task, the iterative process should be converged to the unknowns estimator $\boldsymbol{x}\hat{}$ of : $lim(\boldsymbol{x}^{(k)}) = \boldsymbol{x}\hat{}$, where $\boldsymbol{x}^{(k)}$ is the vector of coordinates of network points in k-th cycle of iterative process (*Gauss-Newton* procedure, characterized by the convergence of square type). Consider now the relationship (2). In typical tasks of observational reductions, the function $\boldsymbol{F}$ is at least conditionally invertible (invertible under special conditions for the components of the vector coordinates), therefore similar convergence for the coordinates in the measurement space *lim* $(\boldsymbol{X}^{(k)}) = \boldsymbol{X}$^ should occur, where $\boldsymbol{X}^{(k)}$, $\boldsymbol{X}$^ are, corresponding to $\boldsymbol{x}^{(k)}$, $\boldsymbol{x}$^, vectors in measurement space. Hence, the empirical measurements of reductions defined by (4), (5) and (1), (3) can be computed as functions of optimal coordinates (vectors $\boldsymbol{x}$^, $\boldsymbol{X}$^).

2.2. Empirical reduction limited to the task of a mapping

A special task of an observational reduction is related to the same stage of a cartographic mapping. It is, therefore, a situation where the observations in the network are already reduced to the ellipsoid and the adjustment of the observations should be realized on the mapping plane. Reducing original observations to the ellipsoid means that the corresponding geometric elements on the ellipsoid are determined by geodesics: lengths of vectors (e.g. as lengths of GNSS vectors) are reduced to the lengths of geodesics segments, directional angles of GNSS vectors are mapped in the geodetic azimuths, measure angles are defined as the angle between the geodesics. We assume that for a given mapping of an ellipsoid, coordinate transformation formulas with the inverse task are known (are available in the form of practical procedures):

$$(x, y) = \boldsymbol{F}(B, L), (B, L) = \boldsymbol{F}^{-1}(x, y). \tag{6}$$

For numerical examples we will use the application of *Gauss-Krüger* (transverse *Mercator*) mapping of GRS80 ellipsoid, defining Polish cartographic system PL-1992 (Balcerzak, 1994, 1995) .

Line in a general task, we assume that the approximate coordinates of points on the ellipsoid are known. In this case, these are geodetic coordinates $(B_i^{(0)}, L_i^{(0)})$ (i – conventional index of network points), which, by the formula (6) of the mapping, provide the appropriate coordinates in a mapping plane $(x_i^{(0)}, y_i^{(0)}) = \boldsymbol{F}(B_i^{(0)}, L_i^{(0)})$. Let Θ_e mean the measure of observation reduced to the ellipsoid. Then, in particular to (1),

$$\Theta_e^{(0)} = f_e(\ldots, (B_i^{(0)}, L_i^{(0)}), \ldots) \tag{7}$$

is an approximation measure Θ of a network element, based on the approximate coordinates, and also in particular to (3),

$$\Theta_m^{(0)} = f_m(\ldots, (x_i^{(0)}, y_i^{(0)}), \ldots) \tag{8}$$

is an approximation of an observation measure, reduced to the mapping plane.

Finally, the mapping reductions, according to (4) and (5) express the form:

$$\delta\Theta_{e\text{-}m}^{\ (emp)} = \begin{cases} \delta\Theta_{e\text{-}m}^{(0)} & \text{for azimuths, angles and directions} \quad (9) \\ (\delta\Theta_{e\text{-}m}^{(0)} / \Theta_e^{(0)}) \cdot \Theta_e & \text{for distances} \quad (10) \end{cases}$$

where Θ_e is an observation measure reduced to the ellipsoid (see in: Kadaj, 2001, p. 49-51 with table in p. 51). In Leick (2004) only the absolute difference has been applied for all types of observation, the same as in (9). Obviously, during iterative process of network adjustment, the approximate coordinates will be converged to some optimal values. This implies also the corresponding corrections for values of mapping reductions (9), (10). Based on properties of used cartographic mapping, elementary distortion values (in local scale and convergence) change so slowly that even a significant point shift on the mapping plane (coordinate errors) does not cause measurable changes in distortion parameters that determine value of the observational reduction (see e.g. in Doskocz, 2007).

3. Reducing measured distances

3.1. Reduction components in the known classic methodology

As an example we will take the reduction of the measured slant distance to the length of the corresponding section of a certain mapping plane (Fig. 1). In classic terms, the transformation of $d_s \Rightarrow d_m$, is composed of several conversions:

• leveling the slant distance at the mean height of section ends, including reduction due to atmospheric refraction, meaning transformation $d_s \Rightarrow d_o$ (d_o – the leveled distance) realized by adding appropriate corrections (reduction) $\delta d_{s\text{-}o} \leq 0$: $d_o = d_s + \delta d_{s\text{-}o}$,

• conversion of the leveled distance on the length of the ellipsoid chord: $d_o \Rightarrow d_c$, taking into account the relevant correction (reduction) $\delta d_{o\text{-}c}$: $d_c = d_o + \delta d_{o\text{-}c}$,

• conversion of the chord length d_c to the length of the corresponding geodesic segment $d_c \Rightarrow d_{e,}$ by adding the correction (reduction) $\delta d_{c\text{-}e} > 0$: $d_e = d_c + \delta d_{c\text{-}e}$,

• transforming geodesic segment d_e to the length of the section on the mapping plane $d_e \Rightarrow d_m$, by adding the appropriate corrections (reduction) $\delta d_{e\text{-}m}$: $d_m = d_e + \delta d_{e\text{-}m}$.

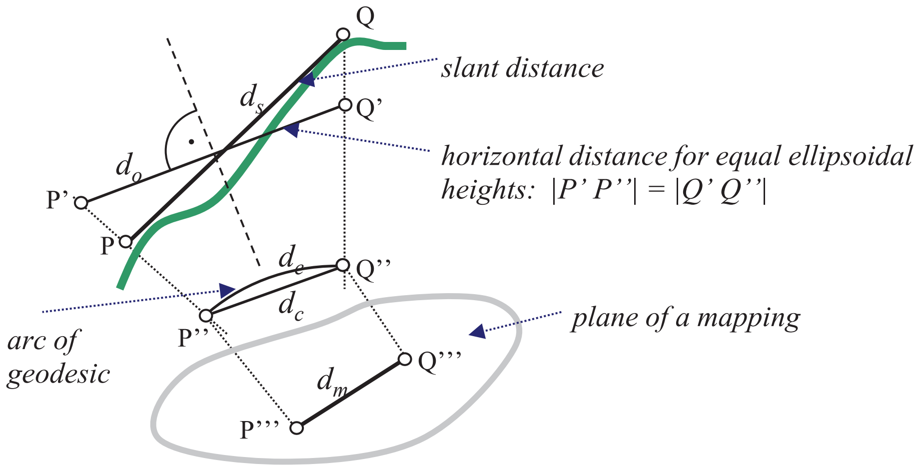


Fig. 1. Steps of the distance reduction

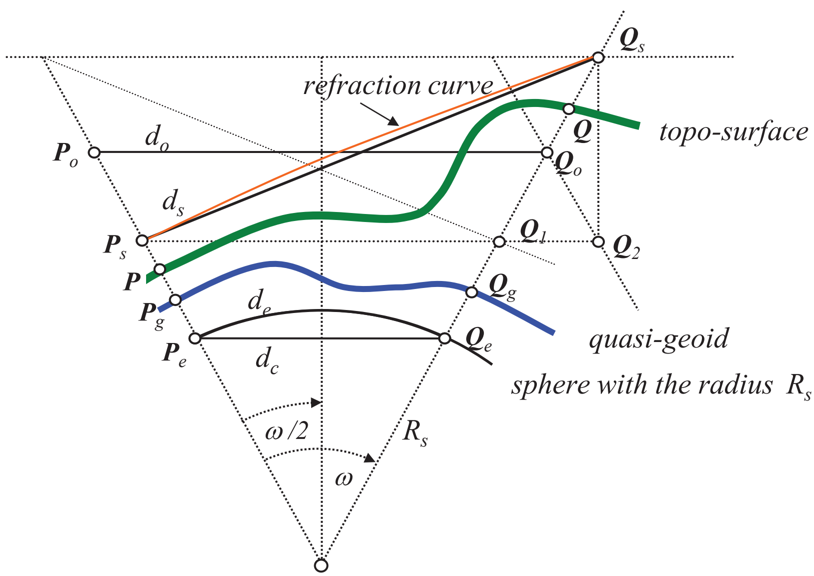


Fig. 2. Elements of the distance reduction by the assumption of local approximation of the ellipsoid surfacein reference to the sphere model of an average radius of curvature of the ellipsoid

More in detail, the following designations and relationships are related to Fig. 2:
$d_s = |\boldsymbol{P_s}\,\boldsymbol{Q_s}|$ – observation,
i_P, s_Q – offsets of measurement points $\boldsymbol{P_s}$, $\boldsymbol{Q_s}$ in relation to marked points $\boldsymbol{P}$,$\boldsymbol{Q}$,
$i_P = |\,\boldsymbol{P_s}\,\boldsymbol{P}\,|$ – instrument height, $s_Q = |\boldsymbol{Q_s}\,\boldsymbol{Q}|$ – height of the target point,

$H_P = |\boldsymbol{P}\,\boldsymbol{P_g}|$, $H_B = |\boldsymbol{Q}\,\boldsymbol{Q_g}|$ – normal heights of the marked points,
$\zeta_P = |\boldsymbol{P_g}\,\boldsymbol{P_e}|$, $\zeta_B = |\boldsymbol{Q_g}\,\boldsymbol{Q_e}|$ – height anomalies (quasi-geoid heights),
$h_{Ps} = |\boldsymbol{P_s}\,\boldsymbol{P_e}| = H_P + \zeta_P + i_P$, $h_{Qs} = |\boldsymbol{Q_s}\,\boldsymbol{Q_e}| = H_Q + \zeta_Q + s_Q$ – geodetic (ellipsoidal) heights,
$\Delta h_{Ps\text{-}Qs} = h_{Qs} - h_{Ps} = sign(h_{Qs} - h_{Ps}) \cdot |\boldsymbol{Q_1}\,\boldsymbol{Q_s}|$ – difference of geodetic heights,
$(½) \cdot |\Delta h_{Ps\text{-}Qs}| = |\boldsymbol{Q_1}\,\boldsymbol{Q_o}| = |\boldsymbol{P_s}\,\boldsymbol{P_o}|$,
$\Delta w = |\boldsymbol{Q_2}\,\boldsymbol{Q_s}|$ (the distance of the target point $\boldsymbol{Q_s}$ from leveled reference plane with the points $\boldsymbol{P_s}$, $\boldsymbol{Q_1}$),
$d_o = |\boldsymbol{P_o}\,\boldsymbol{Q_o}| = |\boldsymbol{P_s}\,\boldsymbol{Q_2}|$– leveled distance at the bisecting line of the angle ω and the average height, d_c, d_e – length of the chord and arc on the reference surface between the projections $\boldsymbol{P_e}$, $\boldsymbol{Q_e}$ of points $\boldsymbol{P}$, $\boldsymbol{Q}$.

Let's focus on the influence of the atmospheric refraction on the measured length, assuming a rough estimation of its influence on trigonometric leveling of territory of Poland. As it is known, refraction effect by the height difference determines the value of $\delta_r = 0.13 \cdot \delta_R$, where δ_R is the influence of Earth curvature, $\delta_R = d_0^{\,2} / (2 \cdot R_s)$, R_s – average radius of curvature of the ellipsoid in the middle of measured distance. As a consequence, the radius of curvature r - curve of refraction is approximately the value of $r = R_s / 0.13 \approx 49\,000\,000$ m. The difference in the arc of 10 km and the corresponding chord will be approx. 0.000017 m, or less than 0.02 mm, which means that the value is basically irrelevant. That is why we adopted the length of straight line section as an observed value. At present, using the modern measuring instruments of type the total-station, the refractions influences can be automatically eliminated, in function of temperature and atmospheric pressure.

First of all we consider the distance reduction on the reference surface (ellipsoid) using classic designs, which take into account already calculated ellipsoid heights of points, and hence the heights difference instead of zenith angle. In classic formulation, the ellipsoid is replaced by the sphere of an average radius of curvature defined with Euler's formula at the midpoint of the line section with the azimuth α: (see e.g. in Czarnecki (1994), Section 2.1.3.):

$$R_s = [\, R_M^{-1} \cdot cos^2(\alpha) + R_N^{-1} \cdot sin^2(\alpha)]^{-1} \tag{11}$$

where R_M, R_N – a principal radii of curvature:

$$R_M = a\,(1 - e^2) / [1 - e^2 \cdot sin^2(B)]^{3/2},\ R_N = a / [1 - e^2 \cdot sin^2(B)]^{1/2}\,, \tag{12}$$

$e^2 = (a^2 - b^2) / a^2$ (first eccentricity squared),
a, b – semi-axes of a reference ellipsoid, $B = (B_P + B_Q)/2$ (average geodetic latitude),
α – geodetic azimuth.

Full reduction is distributed into elementary components: leveling $d_s \Rightarrow d_o$, what explains precisely Fig. 2, then conversion on the chord and on the arc of sphere: $d_o \Rightarrow d_c \Rightarrow d_e$ (it is also possible the slant distance reducing to geodesic without leveling, see e.g. in Leick, (2004, Section 9.1); Zakatow (1976, Section 80). The first reduction (leveling) is expressed in elementary formulas (it results easily with Fig. 2):

$$\begin{aligned}
d_o &= d_s \cdot [1-(\Delta w/d_s)^2]^{1/2} = d_s \cdot [1-(\Delta h/d_s)^2]^{1/2} + \delta d_R = \\
&= d_s - d_s \cdot \{1-[1-(\Delta h/d_s)^2]^{1/2}\} + \delta d_R = d_s + \delta d_{s\text{-}o} \qquad (13)\\
\Delta w &= \Delta h \cdot cos(\omega/2); \\
\delta d_{s\text{-}o} &= -d_s \cdot (\Delta h/d_s)^2 \cdot \{1+[1-(\Delta h/d_s)^2]^{1/2}\}^{-1} + \delta d_R \\
\delta d_R &= d_s \cdot \{[1-(\Delta w/d_s)^2]^{1/2} - [1-(\Delta h/d_s)^2]^{1/2}\} \approx d_s \cdot [(\Delta h^2 - \Delta w^2)/(2 \cdot d_s^2)] \\
&= [\Delta h^2/(2 \cdot d_s)] \cdot (1 - cos^2(\omega/2)) = [\Delta h^2/(2 \cdot d_s)] \cdot sin^2(\omega/2) = \\
&\approx [\Delta h^2/(2 \cdot d_s)] \cdot (d_s^2/4)/R_s^2 = \Delta h^2 \cdot d_s/(8 \cdot R_s^2)
\end{aligned}$$

where:
ω – sphere central angle as shown in Fig. 2,
δd_R – correction resulting from the difference between Δw (the heights difference in case of the horizontal reference plane) and Δh (the difference of the ellipsoidal heights).
The correction $\delta d_{R,}$ even for very extreme geometrical conditions, e.g. d_s = 10 km, Δh = 500 m, does not exceed the value of 0.01 mm, and therefore, with respect to the possible accuracy of the measurement, is insignificant.

We find the corresponding distances: the chord d_c and the arc d_e of the sphere as the reference surface:

$$d_c = d_o + \delta d_{o\text{-}c}\ ;\ \delta d_{o\text{-}c} = -d_o \cdot h_{sr}/(R_s + h_{sr}); \qquad (14)$$

where: R_s – average radius of curvature of the ellipsoid (according to (11)), h_{sr} – the average height of the ellipsoidal distance measured points), $h_{sr} = h_{Ps} + 0.5 \cdot \Delta h_{Ps\text{-}Qs}$.

The correction (reduction) to the length of the arc is expressed as follows (see e.g. in Zakatow, 1976, Section 80):

$$\begin{aligned}
&\delta d_{c\text{-}e} = d_e - d_c = R_s \cdot \omega - 2 \cdot R_s \cdot sin(\omega/2) = R_s \cdot [\omega - 2 \cdot sin(\omega/2)] = \\
&\approx R_s \cdot \{\omega - 2 \cdot [(\omega/2) - (\omega/2)^3/3! + \ldots]\} \approx R_s \cdot (\omega^3/24) \approx d_o^3/(24 \cdot R_s^2) \qquad (15)\\
&(\text{for } \omega \approx d_o/R_s)
\end{aligned}$$

This reduction, for various lengths, is presented in Table 1.

Table 1. Reduction of the chord to the length of the arc of the reference surface

d [km]	$\delta d_{c\text{-}e}$ [m]
5	0.0001
10	0.0010
20	0.0082
40	0.0657

The last of the classicly considered distance reductions applies to the conformal mapping, e.g. in the Polish national cartographic system PL-1992. For this purpose, we use formulas specified for the wide *Gauss-Krüger* mapping area (Balcerzak, 1994, 1995).

3.2. Reducing distances in empirical methodology

We can consider, of course, different cases of distance reductions using empirical methods. We can group them as follows:

A) $d_s \Rightarrow d_m$: General option, containing all elementary reductions, from slant distance to the segment length on the mapping plane.

B) $d_o \Rightarrow d_m$: We assume that the initial reduction (leveling slant distance) is executed independently, while empirical reducing includes all elementary reductions, up to conversion on the mapping plane.

C) $d_e \Rightarrow d_m$: Includes only the reduction step from the ellipsoid to the mapping plane.

D) $d_s \Rightarrow d_e$: Reducing of the slant distance to the length of the geodesic segment on the ellipsoid. Formally includes three elementary reductions: leveling resulting in a length d_o, height reducing resulting in the chord length of the ellipsoid $d_o \Rightarrow d_c$, and the reduction of the chord to the length of the geodesic on the ellipsoid $d_c \Rightarrow d_e$. The described variant may relate to network adjustment on the ellipsoid. Then, final coordinates other than geodetic (e.g. mapping system coordinates) are obtained by the transformation.

E) $d_o \Rightarrow d_e$: this case differs from the previous one with the fact that the leveled distance is present from the beginning.

Empirical methods allow to realize direct conversion $d_s \Rightarrow d_m$. Let us assume that the approximate coordinates of the points ***P, Q*** are available. Without limiting the generality of issues, we can assume geodetic coordinates, ***P*** $(B_P^{(0)}, L_P^{(0)}, h_P^{(0)})$, ***Q*** $(B_Q^{(0)}, L_Q^{(0)}, h_Q^{(0)})$. If we have the heights of another type (e.g. normal heights), they should be transformed into ellipsoidal heights using local model of geoid (quasi-geoid). Calculations are made sequentially according to the following steps (in distance and reduction signs, network point names and index of the coordinate iteration are added – here in initial state):

1. We transform the approximated geodetic coordinates $(B^{(0)}, L^{(0)}, h^{(0)})$ of network points on the corresponding *Cartesian* coordinates $(X^{(0)}, Y^{(0)}, Z^{(0)})$,

$$(B_P^{(0)}, L_P^{(0)}, h_P^{(0)}) \Rightarrow (X_P^{(0)}, Y_P^{(0)}, Z_P^{(0)})$$
$$\text{and } (B_Q^{(0)}, L_Q^{(0)}, h_Q^{(0)}) \Rightarrow (X_Q^{(0)}, Y_Q^{(0)}, Z_Q^{(0)}) \tag{16}$$

and then calculate the slant distance

$$(d_s)_{PQ}^{(0)} = [(X_Q^{(0)} - X_P^{(0)})^2 + (Y_Q^{(0)} - Y_P^{(0)})^2 + (Z_Q^{(0)} - Z_P^{(0)})^2]^{1/2} \tag{17}$$

2. In accordance with known rules of mapping, we find images of points ***P, Q*** on the plane of mapping

$$(B_P^{(0)}, L_P^{(0)}) \Rightarrow (x_P^{(0)}, y_P^{(0)}), (B_Q^{(0)}, L_Q^{(0)}) \Rightarrow (x_Q^{(0)}, y_Q^{(0)}) \tag{18}$$

and then we find a flat mapping distance

$$(d_m)_{PQ}{}^{(0)} = [\ (x_Q^{(0)} - x_P^{(0)})^2 + (y_Q^{(0)} - y_P^{(0)})]^{1/2} \tag{19}$$

3. The expected empirical $\delta d_{s\text{-}m}{}^{(emp)}$ correction (reduction) to the measured distance d_s, $d_m = d_s + \delta d_{s\text{-}m}{}^{(emp)}$ (for simplicity we omit points signs):

$$\delta d_{s\text{-}m}{}^{(emp)} = \mathbf{[(}\ d_m^{(0)} - d_s{}^{(0)}\ ) / d_s{}^{(0)}\mathbf{]} \cdot d_s \tag{20}$$

The given example shows how different kinds of conventional methods can be replaced by only one empirical reduction. An interesting fact of the empirical methodology is that while creating measurement of length on the mapping plane we omit determination of the geodesic length on the ellipsoid. In contrast, in the classic methodology, we carry out independently all kinds of reductions. Indirectly, after corresponding reductions, we get the length of the geodesic segment on the ellipsoid. In turn, while performing certain mappings, we move to a straight line segment on the plane.

The essential difference between options A), B) and other variants where the initial or final coordinate system is geodetic coordinates (ellipsoidal) is that on the ellipsoid, length of the geodesic segment is usually considered as the distance between two points, while in other systems the *Euclidean* (*Pythagorean*) length determines the distance between points.

With option C, the empirical reduction (mapping reduction) is calculated using differences between the length on the mapping plane $(d_m)_{PQ}{}^{(0)}$ (compare (19)) and the length of the geodesic segment on the ellipsoid:

$$(d_e)_{PQ}{}^{(0)} = s_{PQ}{}^{(0)} = G_1\ (B_P^{(0)}, L_P^{(0)}, B_Q^{(0)}, L_Q^{(0)}), \tag{21}$$

$$\delta d_{e\text{-}m}{}^{(emp)} = \mathbf{[(}\ d_m^{(0)} - d_e{}^{(0)}\ ) / d_e{}^{(0)}\ \mathbf{]} \cdot d_e \tag{22}$$

(we omit here the point signs) where the function G_1 is one of the scalar functions for performing inverse primary task of higher geodesy, i.e. determine the length and azimuth of geodesic connecting two points on the ellipsoid with known geodetic coordinates. The function G_1 determine geodesic length and the second function (G_2) – the starting and ending azimuth of geodesic segment. The empirical reduction for option C is in general formed by (9), (10).

Options A), D), will transform elementary tasks of GNSS vectors on a plane mapping (A) or ellipsoids (D). We apprehend this issue separately in this study.

3.3. Numerical example

We are interested in comparing the classic and proposed (empirical) methods in various cases of distance reduction, for example, with certain extreme characteristics. In addition to the observational data, we assume location data necessary for distance reductions, including height anomalies (quasi-geoid height).

The measured slant distance $d_s = | \boldsymbol{P_s}\, \boldsymbol{Q_s} | = 13273.1496$ m between points $\boldsymbol{P_s}$, $\boldsymbol{Q_s}$ – Fig. 2. $\boldsymbol{P_s}$ is the mean point of an instrument, and $\boldsymbol{Q_s}$ – mean point of the target signal (stabilized points, named ***P, Q***). Geometrical quantities characterizing the height of points representing an observation are:

– normal height of marked (stabilized) points (calculated e.g. in the leveling trigonometric network with measurement of zenith angles): $H_{\boldsymbol{P}} = 422.334$ m, $H_{\boldsymbol{Q}} = 705.641$m,
– height of the instrument as a distance $| \boldsymbol{P_s}\, \boldsymbol{P} | = i = 1.420$ m,
– height of target signal as a distance $| \boldsymbol{Q_s}\, \boldsymbol{Q} | = s = 0.500$ m,
– height anomalies at the points of observation (quasi-geoid heights): ζ_P,$= 38.548$ m, $\zeta_Q = 37.714$ m.

Approximate coordinates (cut off to integer values) of the points ***P, Q*** in the coordinate system PL-1992 (application of *Gauss – Krüger* mapping, defined in Table 2, Balcerzak, 1995):

$$x_P = 183317 \text{ m},\ y_P = 644767 \text{ m},$$
$$x_Q = 194627 \text{ m},\ y_Q = 651695 \text{ m},$$

Based on these location data we calculate, corresponding to the above, approximate coordinates in other systems: geodetic (ellipsoidal) and *Cartesian* geocentric ones. Geodetic coordinates *B, L* are determined from inverse *Gauss – Krüger* mapping in PL-1992 application (Balcerzak, 1995), and ellipsoidal heights using the normal height and height anomalies for two points:

$$h_P = H_P + i + \zeta_P = 462.302 \text{ m},\ h_Q = H_Q + s + \zeta_Q = 743.855\text{m}.$$

Geodetic coordinates of points ***P, Q*** on the GRS80 ellipsoid are as follows (*B, L* transformed from PL-1992):

$$B_P = 49^o\ 30'\ 0.0031027'',\ L_P = 20^o\ 59'\ 59.9936134'',\ h_P = 462.302\text{m},$$
$$B_Q = 49^o\ 36'\ 0.0002671'',\ L_Q = 21^o\ 6'\ 0.0050081'',\ h_Q = 743.855\text{m},$$

Afterwards, in accordance with standard conversion algorithms $(B, L, h) \Leftrightarrow (X, Y, Z)$ geocentric *Cartesian* coordinates are defined:

$$X_P = 3874927.46281\text{m},\ Y_P = 1487445.15369 \text{ m},\ Z_P = 4827208.42911 \text{ m},$$
$$X_Q = 3864599.02185\text{m},\ Y_Q = 1491224.76219 \text{ m},\ Z_Q = 4834639.11090 \text{ m}.$$

For classic method of observational reductions, numerical parameters are calculated, as follows:

$B = 49^{o}\ 33'\ 00''$ = average latitude of geodesic segment;

$R_M = 6372458.3110$ m, $R_N = 6390535.7065$m (main radii of curvature of the ellipsoid);

α = 36.67887 [g] = azimuth of geodesic segment;

$R = 6377813.1051$m= average radius of curvature in the normal section of the ellipsoid in the azimuth α, $a = 6378137.0$ m = semi–major axis; $e^2 = 0.00669438002290$ = first eccentricity squared as GRS80 ellipsoid parameters (Moritz, 2000);

Full reduction is divided into elementary operations: leveled distance $d_s \Rightarrow d_{o,}$ what explains precisely Fig. 2, conversion on the chord and arc of the reference surface $d_o \Rightarrow d_c \Rightarrow d_e$ and mapping $d_e \Rightarrow d_m$.

In the next step, we find the distances: d_c (chord) and d_e (arc) on the reference surface. After substituting into the formula (18) the respective numerical values, $R_s = 6377813.1051$m, $h_{sr} = h_{Ps} + 0.5 \cdot \Delta h_{Ps\text{-}Qs} = 462.3019 + 140.7767 = 603.0786$m, we get: $\delta d_{o\text{-}c} = -1.2547$m and $d_c = 13268.9084$ m. Reducing the chord d_c on the length of arc, according to (15), gives $d_e = 13268.9108$ m.

Empirical reductions enable to calculate scaled difference between measured distances in different configurations, designated on the basis of approximate coordinates. It is important that the coordinates taken in various systems derived from the precise transformation of the same data (coordinates) can be assumed with some errors (for example rounded to 1 m).

From the approximate geocentric coordinates we determine $d_s^{(o)} = 13272.3217$ m while from corresponding geodetic coordinates $d_e^{(0)} = 13268.0827$m. Reduction value obtained by empirical method is $\delta d_{s\text{-}e}^{(0)} = -4.2393$ m.

The last of the conventional distance reductions applies to the cartographic mapping. For this purpose we use formulas specified for the wide area *Gauss-Krüger* mapping (Balcerzak, 1994, 1995). As a result we obtain: $\delta d_{e\text{-}m} = -5.7077$m. It differs of approx. *2.7* mm to the corresponding value of the resulting empirical method (see. Table 2).

In empirical methodology, the conversion can be executed in one stage. For illustrative purposes and controls we show reductions and observational conversion in three variants:

a) the measured slant distance onto an ellipsoid arc,

$$\delta d_{s\text{-}e}^{(0)} = [(13268.0827 - 13272.3217) / 13272.3217] \cdot 13273.1496 = -4.2393\text{m}$$

b) the ellipsoid arc onto a mapping plane,

$$\delta d_{e\text{-}m}^{(0)} = [(13262.3778 - 13268.0827) / 13268.0827] \cdot 13268.9103 = -5.7053\text{m}$$

c) the measured slant distance directly on the mapping plane,

$$\delta d_{s\text{-}m}^{(0)} = [(13262.3778 - 13272.3217) / 13272.3217] \cdot 13273.1496 = -9.9445\text{m}$$

Naturally, the last reduction should be equal to the sum of the first two reductions. An important feature of the last reduction is that it does not require an intermediate passage through the geodesic. The reduction is simply the scaled difference between the measures of length in two spaces, designated by approximate coordinates.

Table 2. Example illustrating classic and empirical reductions (approximated measures are marked in blue)

Data:

	$x^{(0)}$	$y^{(0)}$	H	i/s	ζ	measured slant distance
P	183317	644767	422.334	1.420	38.548	
Q	194627	651695	705.641	0.500	37.714	d_s = 13273.1496m

$x^{(0)}$, $y^{(0)}$ – approximated planar coordinates (rounded to 1 m) of points in PL-1992 system [*Gauss-Krüger* / transverse *Mercator* mapping with application parameters: $L_o = 19^o$ (axis meridian), $m = 0.9993$ (scale in the axis meridian), translation parameters: $x_o = -5300000$m, $y_o = 500000$m], H – normal height, offsets of measurement points i/s (instrument and signal height), ζ – height anomaly of a local quasi-geoid.

Symbolic name of the distance type	**The classical methods of reductions** (multi-step reductions)		**The proposed (empirical) methods of reductions**			
			Two-steps conversion from the original slant distance on the ellipsoid arc and from the ellipsoid arc onto the mapping plane		One-step conversion - direct from the original slant distance to the distance on the mapping plane	
	observation and reduced distances [m]	classical reductions [m]	approximated and observed or reduced distance [m]	empirical reductions [m]	approximated and observed or reduced distance [m]	empirical reduction [m]
d_s	13273.1496		13272.3217 13273.1496		13272.3217 13273.1496	
		$\delta d_{s-o} = -2.9865$		$\delta d_{s-e} = -4.2393$		$\delta d_{s-m} = -9.9445$
d_o	13270.1631					
		$\delta d_{o-c} = -1.2547$				
d_c	13268.9084					
		$\delta d_{c-e} = 0.0024$				
d_e	13268.9108		13268.0827 13268.9103			
		$\delta d_{e-m} = -5.7077$		$\delta d_{e-m} = -5.7053$		
d_m	13263.2031		13262.3778 13263.2050		13262.3778 13263.2051	

4. Mapping reductions of geodesic azimuth

The azimuth as well the length of a geodesic segment (considered in the previous Section) may result from the conversion of the *Cartesian* GNSS vector (see e.g. Kadaj, 1997, 1998 – Section 3.6.5). Classic reduction of geodesic azimuth to the direction angle (azimuth topographic) T on the mapping plane is made of two components (Fig. 3): γ – convergence and curvilinear reducing δk (the difference between direction of the chord and the tangent to the mapped geodesic arc on the plane):

$$T_{PQ} = \alpha_{PQ} - \gamma_P + \delta k_{PQ} = \alpha_{PQ} + \delta\alpha_{PQ}; \tag{23}$$

where $\delta\alpha_{PQ} = -\gamma_P + \delta k_{PQ}$ is the total reduction of the geodesic azimuth. Above, only the point name (***P***) is assigned to the convergence value, assuming, a conformal mapping, typical in geodesy applications (then the convergence is constant at a given point).

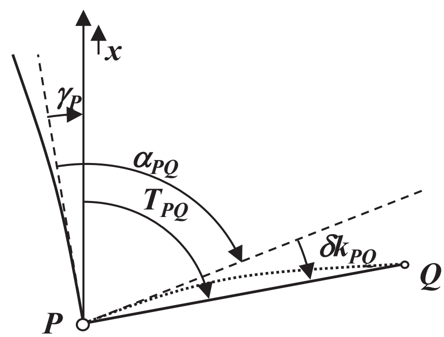


Fig. 3. Geodesic and topographic azimuth

The classic approach of the azimuth reduction by conformal mapping therefore require a separate designation of convergence and reduction of direction. These two components are essential elements of any cartographic mapping, determined by means of appropriate differentials formulas.

Empirical methodology leads to, similarly as distance reductions, to determination of the overall reduction based on the models (measurement approximations) of azimuths (geodetic and topographic) determined on the basis of approximate coordinates:

$$\alpha_{PQ}^{(0)} = G_2\,(B_P^{(0)}, L_P^{(0)}, B_Q^{(0)}, L_Q^{(0)}) \tag{24}$$

(here G_2 is a function determining initial azimuth of the geodesic segment on the ellipsoid)

$$T_{PQ}^{(0)} = Arg\;(\,z_{PQ}\,)\;;\; z_{PQ} = (x_Q^{(0)} - x_P^{(0)}) + i\cdot(y_Q^{(0)} - y_P^{(0)}); \tag{25}$$

(argument of a complex number as the directional angle of the vector ***PQ*** on the mapping plane),

$$\delta\alpha_{PQ}{}^{(0)} = T_{PQ}{}^{(0)} - \alpha_{PQ}{}^{(0)} \tag{26}$$

where the cartographic and geodetic coordinates are in unambiguous relation:

$$(B_P{}^{(0)}, L_P{}^{(0)}) \Leftrightarrow (x_P{}^{(0)}, x_P{}^{(0)}); (B_Q{}^{(0)}, L_Q{}^{(0)}) \Leftrightarrow (y_Q{}^{(0)}, y_Q{}^{(0)}).$$

5. Numerical test for empirical method of the mapping reductions on the example of the geodesic vectors created from GNSS- vectors, by the application of the *Gauss-Krüger* mapping

A set of 10 points located on the ellipsoid GRS80 they way that the distance of consecutive points from the point 1 were increasing from 2 to approx. 523 km was adopted. For further pairs of points 1-2, 1-3, 1-4, ..., 1-10 independent lengths and azimuths of geodesic were determined. Next, we calculated the reduction of vectors mapping to the system PL-1992 (*Gauss – Krüger* mapping with axial meridian $\lambda_o = 19^o$ and scale on the axial meridian m = 0.9993), using two methods, conventional one – according to the formulas PL-1992 (Balcerzak, 1995) and empirical one, described in this paper.

Table 3 summarizes data, and in Table 4 mapping reductions are calculated by two methods. The symbol [1] refers to the classic method based on application of analytical (classic) models for reduction, while the symbol [2] describes in this paper empirical method. For the above mentioned two methods reductions both for distances and azimuths of geodesics were calculated. In columns marked with "numeric error of [1]" there are the differences of given reductions in both methods. As it turned out, these differences are at the same time numerical errors of classic methods, as it can be easily verified what should be measurements of correct pseudo-observations calculated by the *Cartesian* coordinates (with Tab. 4) on the mapping plane. These measures are exactly equal (with an error of rounding the last digit) to values (corrections) calculated using empirical method. Based on these results it can be concluded that the reduction values obtained with use of classic analytical methods for long GNSS vectors (vectors after transformation into a geodesic) can be deflected by significant numerical error.

For vectors with a length of approx. 2 km a numerical error of approx. 1 mm is marked in a reduction in length, what may have a meaning in practice, e.g. in precise realization networks. For vectors with a length of approx. 20 km numerical errors of lateral and longitudinal vectors already have significant value of 1-2 cm. In practice, the distances between points of geodetic network can be also greater than 20 km.

In this paper we do not deal with a matter of choice of the method of a network calculation. We do present safe alternative to the method of calculating observational

Table 3. Assumed geodetic coordinates (B, L), corresponding planar coordinates (x,y) in PL-1992 and parameters of geodetics

Point i	geogr. coordinates		coordinates in PL-1992		geodesics on the GRS80 ellipsoid		
	B [o ' "]	*L* [o ' "]	*x* [m]	*y* [m]	Vector i-j	Azimuth $\alpha^{[g]}$	Length *s*[m]
1	50 00 00	19 00 00	236968.4486	500000.0000			
2	50 01 00	19 01 00	238821.1044	501193.6799	1-2	36.4377619	2205.4506
3	50 02 00	19 02 00	240674.0315	502386.5339	1-3	36.4256126	4410.6818
4	50 04 00	19 04 00	244380.6995	504769.7628	1-4	36.4013073	8820.4857
5	50 08 00	19 08 00	251797.2879	509526.2952	1-5	36.3526691	17637.4574
6	50 16 00	19 16 00	266643.4560	518999.5859	1-6	36.2552801	35260.8381
7	50 32 00	19 32 00	296387.5964	537786.4899	1-7	36.0600325	70465.2040
8	51 04 00	20 04 00	356081.7046	574716.9270	1-8	35.6675020	140703.1956
9	52 00 00	21 00 00	461197.2429	637253.1611	1-9	34.9730976	263064.9524
10	54 00 00	23 00 00	689131.3915	762053.6978	1-10	33.4447382	522831.3872

Table 4. Reduction of geodesics in the projection plane. [1] classic (analytical) method, [2] empirical method

Vector i j	~*s* [km]	$\delta\alpha$ [cc]		Numeric error of [1] [cc]	δs [m]		numeric error of [1] [m]
		[1]	[2]		[1]	[2]	
1-2	2	0.01	0.02	+0.01	1.5449	1.5438	0.0011
1-3	4	0.02	0.03	+0.01	3.0895	3.0874	0.0022
1-4	9	0.09	0.10	+0.01	6.1778	6.1735	0.0044
1-5	18	0.37	0.36	0.01	12.3483	12.3397	0.0087
1-6	35	1.47	1.40	0.07	24.6477	24.6305	0.0172 **
1-7	70	5.86	5.29	0.57 *	48.9476	48.9137	0.0340 **
1-8	141	23.22	18.62	4.60 *	95.3401	95.2764	0.0636 **
1-9	263	80.29	49.81	30.48 *	163.9519	163.8638	0.0881 **
1-10	523	309.12	61.95	247.17 *	219.0024	219.2294	-0.2270 **

Estimation of numeric errors
*) transverse error > 0.01 m ($\delta\alpha \cdot s$)
**) distance error > 0.01 m (δs)

reductions (pseudo-observation), especially for the cartographic mapping. We also show that the classic (analytical) methods of determining these reductions for long observation vectors are subject to significant numerical errors. The presented empirical methods are free from such errors.

6. Reducing directions or angles

6.1. Classic and empirical methods – general comparative aspects

Reducing angles, for instance from physical measurement space to the ellipsoid surface or on a mapping plane, may be treated as the difference of corresponding reduction types for two directions of the angle arms. Reduction of direction as n original observation, consists of several components (the forms can be found in textbooks of higher geodesy, e.g. in: Czarnecki, 1994; Szpunar, 1982; Warchałowski, 1952) :

• directional or azimuth reduction due to the vertical deviation from the ellipsoid:

$$\delta k_g = [\ \eta \cdot cos(\alpha) - \xi \cdot sin(\alpha)\ ] \cdot ctg(z) \tag{27}$$

(analogous component is used in reduction of astronomical azimuth on the *Laplace* azimuth, but then the component is added due to deviation of meridian plane $-\eta \cdot tg(B)$, which is eliminated as independent from directions in angular reduction);

• reduction due to ellipsoidal height of the target point:

$$\delta k_h \cong [e'^2 \cdot h /(2 \cdot R_N)] \cdot cos^2(B) \cdot sin(2 \cdot \alpha); \tag{28}$$

• reduction due to angular deviation of the geodesic from the normal section of the ellipsoid:

$$\delta k_e \cong -[e^2 \cdot s^2 / (12 \cdot R_N^2)] \cdot cos^2(B) \cdot sin(2 \cdot \alpha), \tag{29}$$

where:
ξ, η – components of vertical deflections,
e, e' – first and second eccentricities of the ellipsoid: $e^2=(a^2-b^2)/a^2,\ e'^2=(a^2-b^2)/b^2$
a, b – semi-axes of a reference ellipsoid,
R_N – radius of curvature of the prime vertical in the station position (defined in (12)),
B – geodetic latitude of station positions,
h – ellipsoidal height for the marked target point,
s, α – length and azimuth of the geodesic.
The quantities (28) and (29) are expressed in radians and (27) in units of components ξ, η.

Additionally, when applying cartographic mapping, corresponding directional reduction δk_m (reducing the geodesic to the chord on the plane) is taken into account. This value defines the formula of a mapping – in the examples we use the *Gauss-Krüger* conformal mapping.

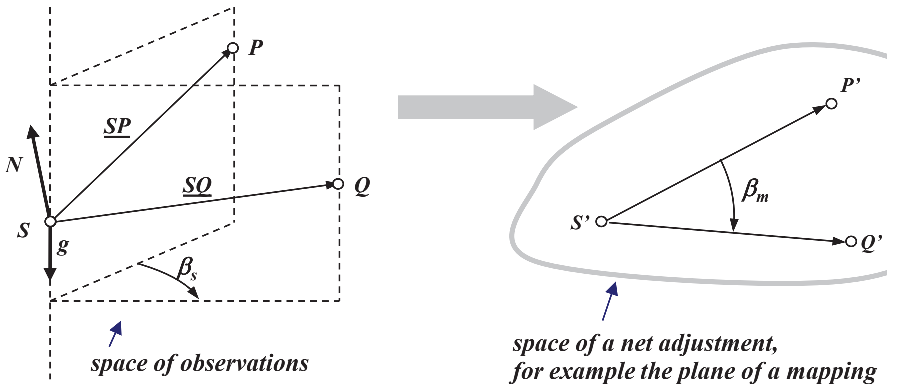


Fig. 4. Conversion of the angle to the mapping plane

Introducing the point names ***P, S, Q*** in the angle designation (target left, the station, the target right) as an element of a geodetic network (Fig. 4), we create the angle reduction as the difference of appropriate directional reductions

$$\delta\beta_{(.)}(\boldsymbol{P},\boldsymbol{S},\boldsymbol{Q}) = \delta k_{(.)}(\boldsymbol{S},\ \boldsymbol{Q}) - \delta k_{(.)}(\boldsymbol{S},\boldsymbol{P}), \tag{30}$$

where the symbol (.) replaces conventional designation of the reduction.

Afterwards, we are considering empirical methodology to reduce the observation angle in particular. As we know from general considerations included in p. 2, the observational reduction of any kind consists on the direct transformation from measurement space to a mathematical space, where the network adjustment, based on the approximation of the observation measures in two spaces, is made with the use of the approximation coordinates. As a result, empirical reduction is determined by the difference of these measures.

Let us consider the situation of the measured angle on the surface of Earth (Fig. 4), and its image on the mapping plane. The same as with distance reductions, we arrange major variants of angle reductions (index points were omitted for simplicity):

A) $\beta_s \Rightarrow \beta_m$: a variant which includes all elementary reductions from the measured angle to the angle reduced to the mapping plane.

B) $\beta_s \Rightarrow \beta_e$: Reduction of measured angle only on the reference ellipsoid (e.g. to network adjustment on the ellipsoid).

C) $\beta_e \Rightarrow \beta_m$: Variant represents only reduction of the angle mapping, i.e. the conversion from the ellipsoid on a mapping plane.

6.2. Formulas of empirical methodology

Directional observations can be considered as readings on the horizontal circle (protractor) of theodolite at a random zero position. In this case we can also assume that zero coincides with any direction (we often tell that the directions are reduced to the initial). Then the readings in other directions will be single angles measured in relation to the adopted target. Therefore, the reduction of directions can be brought to reductions of angles.

Another problem are functional and stochastic models in a process adjusting networks with directional observations. In the traditional models and adjustment methods for horizontals geodetic networks, each subset of directional observations introduces one additional unknown to the observational system – an orientation parameter (as the azimuth of zero-reading on protractor). The orientation parameters (so-called nuisance parameters) can be eliminated, creating the angles as differences of directions. There is a combination of those pseudo-observations (angles set) that leads to the identical adjustment results as in the original system by the application of least squares solution. It is so-called *Schreiber's* set of angles (see: Kadaj, 2008). In this case the problems of reduction of directional observations lead equivalently to the reduction of angles.

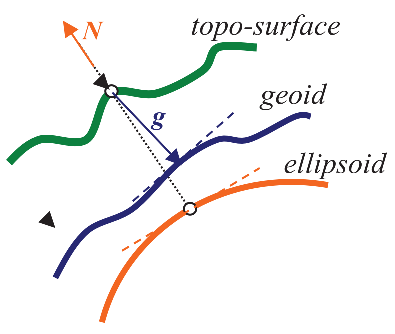


Fig. 5. The normal vector of the ellipsoid and the vector of gravity

Just like with any use of empirical methods we assume that the known approximate coordinates of the points defining the geometric element, as well as parameters for measuring the physical space to bind with a specific mathematical space. Let us, for example, assume approximate geodetic coordinates ($B_S^{(0)}, L_S^{(0)}, h_S^{(0)}$), ($B_P^{(0)}, L_P^{(0)}, h_P^{(0)}$), ($B_Q^{(0)}, L_Q^{(0)}, h_Q^{(0)}$) of three points defining the geometric element of the measured angle (Fig. 4) and vertical deviation component (ξ, η) at the point position as a local physical feature of a measuring space (Fig. 5).

Approximate geodetic coordinates can be converted to the corresponding geocentric *Cartesian* coordinates ($X_S^{(0)}, Y_S^{(0)}, Z_S^{(0)}$), ($X_P^{(0)}, Y_P^{(0)}, Z_P^{(0)}$), ($X_Q^{(0)}, Y_Q^{(0)}, Z_Q^{(0)}$). In *Cartesian* geocentric coordinates – we can also specify the vector parallel to the vertical one at the point position – we denote it as follows:

$\boldsymbol{g} = [g_x, g_y, g_z]^T$. This vector will be calculated using the known relation between the topocentric system specified on the surface of the ellipsoid at the point of coordinates (*B, L*) (we substitute the appropriate coordinates of the station), and the geocentric system:

$$\boldsymbol{g} = \boldsymbol{U} \cdot \boldsymbol{z} \tag{31}$$

where

$$\boldsymbol{U} = \begin{bmatrix} -sin(B) \cdot cos(L) & -sin(L) & cos(B) \cdot cos(L) \\ -sin(B) \cdot sin(L) & cos(L) & cos(B) \cdot sin(L) \\ cos(B) & 0 & sin(B) \end{bmatrix} \tag{32}$$

is orthonormal rotation matrix (see e.g. Thomson (1976, Section 2.1.3) and Kadaj (2001, Section 4.3)):

$$\boldsymbol{U}^T \cdot \boldsymbol{U} = \boldsymbol{I} \text{ (unit matrix)} \Rightarrow \boldsymbol{U}^{-1} = \boldsymbol{U}^T \Rightarrow \boldsymbol{z} = \boldsymbol{U}^T \cdot \boldsymbol{g}, \tag{33}$$

however, $\boldsymbol{z}$ is the unit vector with components created by vertical deviation components (Fig. 6):

$$\boldsymbol{z} = [u, v, w]^{\mathbf{T}};$$
$$u = -c \cdot tg(\xi), \; v = -c \cdot tg(\eta), \; w = -c, \; c = 1/sqrt\,[tg^2(\xi) + tg^2(\eta) + 1]. \tag{34}$$

From the properties (36) of the transformation (34) it can be concluded that the resultant vector $\boldsymbol{g}$ of the vertical direction as well as the vector $\boldsymbol{z}$ are both unit vectors, i.e. $g_x^2 + g_y^2 + g_z^2 = 1$.

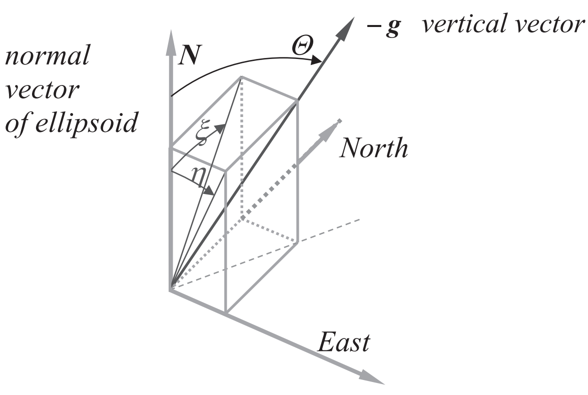


Fig. 6. Components of vertical deviations in a topocentric system

The obtained components of the vector $\boldsymbol{g}$ can be easily verified, by calculating the angle Θ between the vector $\boldsymbol{g}$, and the ellipsoid normal vector $\boldsymbol{N}$ at the station position (Fig. 5). This angle should be near to the value (only for small angles) $sqrt(\xi^2 + \eta^2)$. Previously, we determine the normal vector $\boldsymbol{N}$ normalizing it to the unit length ($\boldsymbol{n}$ vector);

$$\boldsymbol{N} = (N_x, N_y, N_z) = (1/2) \cdot a \cdot (2 \cdot X_e / a^2, 2 \cdot Y_e / a^2, 2 \cdot Z_e / b^2) = (X_e / a, Y_e / a, Z_e \cdot a / b^2); \quad (35)$$

$$\boldsymbol{n} = \boldsymbol{N} / |\boldsymbol{N}|; |\boldsymbol{N}| = (N_x{}^2 + N_y{}^2 + N_z{}^2)^{1/2} \quad (36)$$

where X_e, Y_e, Z_e – orthogonal projected coordinates of the station on the ellipsoid, resulting from the conversion of approximate geodetic positions $(B^{(0)}, L^{(0)}, 0) \Rightarrow (X_e, Y_e, Z_e)$. The angle between the indicated vectors (these are the unit vectors) will be calculated, e.g. by the scalar product:

$$\Theta = \arccos [-(\boldsymbol{n} \bullet \boldsymbol{g})] \quad (37)$$

(• – symbol of the scalar product).

Now we can easily obtain an approximation of the angle measurement in the measurement space as the dihedral angle between two vertical planes, intersecting along the edge parallel to the vector $\boldsymbol{g}$. The angle between the planes replace the angle between the normal vectors of planes. Thus, we determine the normal vectors of respective planes first – in any case, as the vector product of a vector $\boldsymbol{g}$ and a vector formed from the respective coordinate differences (vector lying in the vertical plane):

$$\begin{aligned}\boldsymbol{w}_{SP} = \underline{\boldsymbol{SP}} \times \boldsymbol{g} &= ((X_P{}^{(0)} - X_S{}^{(0)}), (Y_P{}^{(0)} - Y_S{}^{(0)}), (Z_P{}^{(0)} - Z_S{}^{(0)})) \times (g_x, g_y, g_z) = \\ &= (((Y_P{}^{(0)} - Y_S{}^{(0)}) \cdot g_z - (Z_P{}^{(0)} - Z_S{}^{(0)}) \cdot g_y), \\ &-((X_P{}^{(0)} - X_S{}^{(0)}) \cdot g_z + (Z_P{}^{(0)} - Z_S{}^{(0)}) \cdot g_x), \\ &((X_P{}^{(0)} - X_S{}^{(0)}) \cdot g_y - (Y_P{}^{(0)} - Y_S{}^{(0)}) \cdot g_x)) \end{aligned} \quad (38)$$

$$\begin{aligned}\boldsymbol{w}_{SQ} = \underline{\boldsymbol{SQ}} \times \boldsymbol{g} &= ((X_Q{}^{(0)} - X_S{}^{(0)}), (Y_Q{}^{(0)} - Y_S{}^{(0)}), (Z_Q{}^{(0)} - Z_S{}^{(0)})) \times (g_x, g_y, g_z) = \\ &= (((Y_Q{}^{(0)} - Y_S{}^{(0)}) \cdot g_z - (Z_Q{}^{(0)} - Z_S{}^{(0)}) \cdot g_y), \\ &-((X_Q{}^{(0)} - X_S{}^{(0)}) \cdot g_z + (Z_Q{}^{(0)} - Z_S{}^{(0)}) \cdot g_x), \\ &((X_Q{}^{(0)} - X_S{}^{(0)}) \cdot g_y - (Y_Q{}^{(0)} - Y_S{}^{(0)}) \cdot g_x)) \end{aligned} \quad (39)$$

(× – symbol of the vectorial product) $\boldsymbol{w}_{SP}$, $\boldsymbol{w}_{SQ}$ denote normal vectors of vertical planes, $\underline{\boldsymbol{SP}}$, $\underline{\boldsymbol{SQ}}$ are the vectors created respectively of the coordinates differences. As a result, the angle between the normal vectors can be determined from the scalar product:

$$(\beta_s)_{PSQ}{}^{(0)} = \arccos (\boldsymbol{w}_{SP} \bullet \boldsymbol{w}_{SQ} / (|\boldsymbol{w}_{SP}| \cdot |\boldsymbol{w}_{SQ}|)) \quad (40)$$

Depending on the destination system of the angle transformation (ellipsoid or mapping plane), according to the options presented in Section 5.1., we determine a corresponding approximate measure for the transformed angle. In the case of an ellipsoid, we use the algorithms, based on tasks of higher geodesy, calculating azimuths for two geodesic (some numerical methods can be found e.g. in Czarnecki, 1994; Szpunar, 1982; Zakatow, 1976). Then, the measure of the angle is expressed as

$$(\beta_e)_{PSQ}^{(0)} = \alpha_{SQ}^{(0)} - \alpha_{SP}^{(0)} = G_2\,(B_S^{(0)}, L_S^{(0)}, B_Q^{(0)}, L_Q^{(0)}) - G_2\,(B_S^{(0)}, L_S^{(0)}, B_Q^{(0)}, L_Q^{(0)}) \tag{41}$$

$$(\beta_m)_{PSQ}^{(0)} = T_{SQ}^{(0)} - T_{SP}^{(0)} = Arg[(x_Q - x_S) + i\cdot(y_Q - y_S)] - Arg[(x_P - x_S) + i\cdot(y_P - y_S)] \tag{42}$$

where $T_{jk}^{(0)}$, $T_{ji}^{(0)}$ – directional angle of vectors <u>***SQ***</u>, <u>***SP***</u> on the mapping plane as the arguments of the corresponding complex coordinates – we count them with the coordinates of the points obtained from the transformations resulting from the adopted mapping system:

$$(B_{(.)}^{(0)}, L_{(.)}^{(0)}) \Rightarrow (x_{(.)}^{(0)}, y_{(.)}^{(0)}), \tag{43}$$

where the symbol (.) replaces points names: ***P, S, Q***.

For individual variants A), B), C), included in Section 6.1 empirical values of angle reduction will be:

$$(\delta\beta_{s\text{-}m})_{PSQ}^{(0)} = (\beta_m)_{PSQ}^{(0)} - (\beta_s)_{PSQ}^{(0)} \text{ and } (\beta_m)_{PSQ}^{(obs)} = (\beta_s)_{PSQ}^{(obs)} + (\delta\beta_{s\text{-}m})_{PSQ}^{(0)} \qquad \text{(A)}$$
$$(\delta\beta_{s\text{-}e})_{PSQ}^{(0)} = (\beta_e)_{PSQ}^{(0)} - (\beta_s)_{PSQ}^{(0)} \text{ and } (\beta_e)_{PSQ}^{(obs)} = (\beta_s)_{PSQ}^{(obs)} + (\delta\beta_{s\text{-}e})_{PSQ}^{(0)} \qquad \text{(B)}$$
$$(\delta\beta_{e\text{-}m})_{PSQ}^{(0)} = (\beta_m)_{PSQ}^{(0)} - (\beta_e)_{PSQ}^{(0)} \text{ and } (\beta_m)_{PSQ}^{(obs)} = (\beta_e)_{PSQ}^{(obs)} + (\delta\beta_{e\text{-}m})_{PSQ}^{(0)} \qquad \text{(C)}$$

(symbol „(obs)“ means that the appropriate measure is the observation angle or its reduced value, while the symbol „$^{(0)}$“ – as it has already been explained, means the empirical measure, determined using the approximate coordinates).

6. 3. Numerical example of reductions of angles

Just like in Section 3.3, we assume the approximate coordinates (*Cartesian* – geocentric) for three points defining a model of angle in observational space. In order to implement the observational reduction (target: to the ellipsoid or onto the plane of a mapping), we accept additional elements representing the gravity field: the vertical deviation component at the point position and height anomalies (quasi-geoid height) on all three points. The input data and calculation results are shown in Table 5.

As we can see from the description of the table, the classic reduction was determined as multi step reductions and the empirical method is shown in three different cases. Indirect empirical reductions (consisting of 4 or 2 components) are presented for testing and comparison with the classic methodology. In any case, empirical method allows direct conversion of an observation from the measurement space to a specific target system. If this transformation is supposed to take place on the mapping plane, going through the procedure of geodesic calculating is omitted. This is the essential difference to the classic principles that assume that reductions of observations to the geodesic precede the reductions on the projection plane.

Some intermediate results of sample calculations, not given in Table 5, are as follows:

The components of the unit vector $\boldsymbol{z}$:

u = 6.02086231529398e-05, v = 4.13590672216337e-05, w = 0.99999999733217

Transformation matrix $\boldsymbol{S}$:

$$\begin{bmatrix} 7.09900125675893E-01 & -3.58367949545300E-01 & 6.06311985947868E-01 \\ -2.72505126714097E-01 & 9.33580426497202E-01 & 2.32741365416285E-01 \\ 6.49448048330184E-01 & 0.00000000000000E+00 & 7.60405965600031E-01 \end{bmatrix}$$

The unit vector of vertical direction $\boldsymbol{g}$:

$g_x = 0.606254420457072,\ g_y = 0.232763569652505,\ g_z = 0.760445065944196$

The normalized normal vector $\boldsymbol{n}$ of ellipsoid:

$n_x = 0.606311985947878,\ n_y = 0.232741365416051,\ n_z = 0.760405965600092$

The angle between the vectors $\Theta = 46.49$[cc]

The value of the resultant vertical deviation $(38.33^2+26.33^2)^{1/2} = 46.50$ [cc]

Vectors of coordinate differences:

$\underline{\boldsymbol{SP}} = (-10328.0,\ 3779.0,\ 7431.0)$

$\underline{\boldsymbol{SQ}} = (4993.0,\ 10988.0,\ -7048.0)$

The unit normal vectors of vertical planes:

$\boldsymbol{w}_{SP} = \underline{\boldsymbol{SP}} \times \boldsymbol{g} = (8.62129966380252E-02, 9.31337769725911E-01, -3.53804010566101E-01)$

$\boldsymbol{w}_{SQ} = \underline{\boldsymbol{SQ}} \times \boldsymbol{g} = (7.15317597776904E-01, -5.77460157308181E-01, -3.93523189954911E-01)$

The angle between these vectors $\beta_o^{(0)} = \mathbf{121.876409}$ [g].

Similarly, the normal vectors of planes of the ellipsoid normal section:

$\boldsymbol{u}_{SP} = \underline{\boldsymbol{SP}} \times \boldsymbol{n} = (8.62141869785415E-02, 9.31338204103903E-01, -3.53802577068265E-01)$

$\boldsymbol{u}_{SQ} = \underline{\boldsymbol{SQ}} \times \boldsymbol{n} = (7.15275921188159E-01, -5.77475431277768E-01, -3.93576527296771E-01)$

The angle between these vectors: $\beta_g^{(0)} = \mathbf{121.876336}$ [g] (the angle reduced due to the vertical deviation).

Vector of coordinate differences assuming that the target points are projected orthogonally on the ellipsoid:

$\underline{\boldsymbol{SP}_e}$ = (-10777.61707, 3605.50729, 6864.73572)
$\underline{\boldsymbol{SQ}_e}$ = (4566.65138, 10823.34330, -7581.23706)

The normal vectors of normal sections planes, passing through the projections $\boldsymbol{P}_{e,}$ $\boldsymbol{Q}_e$ (orthogonal projections of target points ***P, Q*** on the ellipsoid surface):
$\boldsymbol{u}_{SPe} = \underline{\boldsymbol{SP}_e} \times \boldsymbol{n}$ =(8.62143062851846E-02, 9.31338161835913E-01, -3.53802659260592E-01)
$\boldsymbol{u}_{SQe} = \underline{\boldsymbol{SQ}_e} \times \boldsymbol{n}$ =(7.15275973733384E-01, -5.77475312956758E-01, -3.93576605408977E-01)

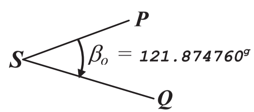
The angle between these vectors, being also the angle reduced due to the ellipsoidal target points:
$\beta_h^{(0)}$ =121.876317 [g]

Another reduction comes from the difference between normal sections of the ellipsoid and tangents of geodesic – arms of the angle. The reduced angle as the difference between geodesic azimuths obtained on the basis of approximate coordinates, will be: $\beta_e^{(0)} = \alpha_{SQ}^{(0)} - \alpha_{SP}^{(0)}$ = 121.876317 [g] which basically means (with an accuracy of rounding errors 0.01cc) that it is identical to the angle obtained from geodesic azimuths.

The greatest values achieved by the angle reduction due to cartographic mapping. Even if it is the conformal mapping, the reduction defines the angle change due to non-rectilinear mapping of geodesics on the plane. We assume, for example, the conformal *Gauss – Krüger* (transverse *Mercator*) projection in the special Polish implementation: PL-1992. Determination of the angle reduction using the empirical method will lead to the comparison of the relevant angles measures computed in the two systems based on the corresponding approximate coordinates. The initial approximate geocentric or geodetic coordinates will be transformed to the PL-1992 and then (on a plane) the angle between two chords will be calculated. We receive in this way the measure of the mapped angle: $\beta_m^{(0)}$ =121.878876 [g]. The difference $\delta\beta_{e\text{-}m}^{(0)} = \beta_m^{(0)} - \beta_e^{(0)}$ = 25.59 [cc] is the empirical value of the angle reduction.

As shown in Table 5, the empirical determination of reduction leads to approximate measure angle in various systems based on the approximate coordinates, and then the formation of the corresponding differences of those measurements. In particular, direct reduction of observations from the measurement space on the mapping plane is thus possible. In this case, the calculation of azimuth of geodesic is not necessary. In comparison with classical multi stage reducing, the empirical methods can be used in one step. It implies so the essential decrease the quantity and the cost of calculations. Moreover, the high numerical exactitude of empirical methods is independent from mutual distance of network points.

Table 5. The example comparison of classic and proposed (empirical) methods of angle reductions



Classical multi-step reduction: $\beta_o \Rightarrow \beta_g \Rightarrow \beta_h \Rightarrow \beta_e \Rightarrow \beta_m$
Empirical one-step reduction between two systems, for example: $\beta_o \Rightarrow \beta_m$
The differences between two systems are computed by using of approximated coordinates

Data:

Point name	Approximated coordinates of sub-network points (rounded to meters)			Approximated ellipsoidal height	The components of the vertical deviations and approximated geodetic coordinates of the station (position)			
	$X^{(0)}$ [m]	$Y^{(0)}$ [m]	$Z^{(0)}$ [m]	$h^{(0)}$ [m]	ξ ["]	η ["]	B [º]	L [º]
P	3864599	1491224	4834639					
S	3874927	1487445	4827208	744 702	12.42	8.53	49.5	21.0
Q	3879920	1498433	4820160					

Example of an angle observation vor – and after reductions [g]	**The classical method -** 4 components of reductions [cc]	**The proposed (empirical) method** Approximated angle in the different status [g]	Reductions in 4 components [cc]	Reductions in 2 components [cc]	Reduction in 1 component [cc]
β_o =121.874760		$\beta_o^{(0)}$ =121.876409			
	$\delta\beta_{o\text{-}g}$ = –0.74		$\delta\beta_{o\text{-}g}$ = –0.73		
β_g=121.874686		$\beta_g^{(0)}$ =121.876336		$\delta\beta_{o\text{-}e}$ = –0.92	
	$\delta\beta_{g\text{-}h}$ = –0.19		$\delta\beta_{g\text{-}h}$ = –0.19		
β_h=121.874667		$\beta_h^{(0)}$ =121.876317			$\delta\beta_{o\text{-}m}$ = +24.67
	$\delta\beta_{h\text{-}e}$ = 0.00		$\delta\beta_{h\text{-}e}$ = 0.00		
β_e =121.874667		$\beta_e^{(0)}$ =121.876317		$\delta\beta_{e\text{-}m}$ = +25.59	
	$\delta\beta_{e\text{-}m}$ = 25.60		$\delta\beta_{e\text{-}m}$ = 25.59		
β_m=121.877227		$\beta_m^{(0)}$ =121.878876			

7. Conclusion

Empirical method of observational reducing (on the ellipsoid or on the mapping plane) is characterized by high accuracy, superior to classic methods, in particular regarding distance mapping. As shown in the test in Section 5, the published classic forms and algorithms are not suitable for a use of a very long GNSS vector, leading to unacceptable numerical error.

In application of empirical methods, approximate coordinates of all the points that define the geometry of the network elements are necessary. Regardless reduction problems, the approximate coordinates of the network points are a input subset of data to solve the problem of a non-linear network adjustment. During the iterative solving (the *Gauss-Newton* procedure is typically used), the approximate coordinates are successively (iteratively) improved. At the same time, the empirically determined reductions converge to certain optimum values, despite being a relatively large change of approximate coordinates, but sufficient for the convergence of *Gauss-Newton* process, should not have any significant influence on the values of the same reductions.

Empirical methods can be applied to a sequence of elementary reductions, not necessarily only at the mapping stage. Starting from the physical observational space we need to use, in addition to the approximate coordinates, additional information arising from physical fields (eg. characteristic curve of refraction, the components of vertical deviations, height anomalies), also included in the classic methods of reduction.

This paper did not attempt to cover all possible cases of reductions of geodetic observations. The aim was mainly to show the capabilities and properties (especially in terms of precision) of empirical methodology for the observational reductions. These examples, that also serve as models for all other analogies, could be applied.

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