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# GROUND STATES OF COUPLED CRITICAL CHOQUARD EQUATIONS WITH WEIGHTED POTENTIALS

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**Abstract.** In this paper, we are concerned with the following coupled Choquard type system with weighted potentials

$$\begin{cases} -\Delta u + V_1(x)u = \mu_1(I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}])Q(x)|u|^{\frac{\alpha}{N}-1}u + \beta(I_\alpha * [Q(x)|v|^{\frac{N+\alpha}{N}}])Q(x)|u|^{\frac{\alpha}{N}-1}u, \\ -\Delta v + V_2(x)v = \mu_2(I_\alpha * [Q(x)|v|^{\frac{N+\alpha}{N}}])Q(x)|v|^{\frac{\alpha}{N}-1}v + \beta(I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}])Q(x)|v|^{\frac{\alpha}{N}-1}v, \\ u, v \in H^1(\mathbb{R}^N), \end{cases}$$

where $N \geq 3$, $\mu_1, \mu_2, \beta > 0$ and $V_1(x)$, $V_2(x)$ are nonnegative functions. Via the variational approach, one positive ground state solution of this system is obtained under some certain assumptions on $V_1(x)$, $V_2(x)$ and $Q(x)$. Moreover, by using Hardy's inequality and one Pohozǎev identity, a non-existence result of non-trivial solutions is also considered.

**Keywords:** ground states, Choquard equations, Hardy–Littlewood–Sobolev inequality, lower critical exponent.

**Mathematics Subject Classification:** 35B25, 35B33, 35J61.

## 1. INTRODUCTION AND RESULTS

### 1.1. BACKGROUND

In this paper, we consider the following coupled Choquard type elliptic system

$$\begin{cases} -\Delta u + V_1(x)u = \mu_1(I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}])Q(x)|u|^{\frac{\alpha}{N}-1}u \\ \qquad\qquad + \beta(I_\alpha * [Q(x)|v|^{\frac{N+\alpha}{N}}])Q(x)|u|^{\frac{\alpha}{N}-1}u, \\ -\Delta v + V_2(x)v = \mu_2(I_\alpha * [Q(x)|v|^{\frac{N+\alpha}{N}}])Q(x)|v|^{\frac{\alpha}{N}-1}v \\ \qquad\qquad + \beta(I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}])Q(x)|v|^{\frac{\alpha}{N}-1}v, \\ u, v \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 3$, $\mu_1, \mu_2, \beta > 0$, $V_1(x), V_2(x), Q(x) \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ and $\frac{N+\alpha}{N}$ is the Hardy–Littlewood–Sobolev lower critical exponent. Here $I_\alpha$ is the Riesz potential given for each $x \in \mathbb{R}^N \backslash \{0\}$ by $I_\alpha(x) := \frac{A_\alpha}{|x|^{N-\alpha}}$, where

$$A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha},$$

and $\alpha \in (0, N), \Gamma$ is the Euler gamma function.

In the literature, there have been a lot of results on the nonlinear Choquard equations as follows

$$-\Delta u + Vu = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

When $N = 3$, $\alpha = 2$ and $p = 2$, equation (1.2) is called the Choquard–Pekar equation, which goes back to the work by Pekar on quantum theory of a Polaron at rest [1] and to 1976's model of Choquard of electron trapped in its own hole, in an approximation to Hartree–Fock theory of one-component plasma [9]. In 2013, one optimal range of parameters was given by V. Moroz and J. Van Schaftingen [14] to establish the existence of positive ground state solutions to (1.2) with $V = 1$. Thanks to a Pohozǎev identity, they showed that (1.2) with $V = 1$ admits a nontrivial solution in $H^1(\mathbb{R}^N)$ if and only if

$$\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}.$$

The endpoints $2_{\alpha,*} := \frac{N+\alpha}{N}$ and $2^*_\alpha := \frac{N+\alpha}{N-2}$ are sometimes called lower and upper Hardy–Littlewood–Sobolev critical exponents respectively in the sense of the Hardy–Littlewood–Sobolev inequality. Later, in 2015, V. Moroz and J. Van Schaftingen [12] considered the Choquard equation (1.2) with a purely lower critical nonlinearity and established a sufficient condition on the existence of ground state solutions. Subsequently, some open questions raised in [12] are given a partial answer by D. Cassani, J. Van Schafting and J. Zhang in [3], where the authors investigated the existence and nonexistence of ground states to (1.2) for $p = \frac{N+\alpha}{N}$. Very recently, S. Zhou, Z. Liu and J. Zhang [28] studied a class of Choquard type equations with weighted potentials and Hardy–Littlewood–Sobolev lower criti cal exponent as follows

$$-\Delta u + V(x)u = (I_\alpha * Q(x)|u|^{\frac{N+\alpha}{N}})Q(x)|u|^{\frac{\alpha}{N}-1}u, \quad x \in \mathbb{R}^N.$$

By using variational approaches, they investigated the existence of ground state solutions under the asymptotic behaviour of weighted potentials at infinity. Moreover, a non-existence result of nontrivial solutions is also obtained.

Meanwhile, there are also some results concerning the following coupled Choquard systems

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1(I_\alpha * |u|^p)|u|^{p-2}u + \beta(I_\alpha * |v|^p)|u|^{p-2}u, \\ -\Delta v + \lambda_2 v = \mu_2(I_\alpha * |v|^p)|v|^{p-2}v + \beta(I_\alpha * |u|^p)|v|^{p-2}v, \\ u, v \in H^1(\mathbb{R}^N). \end{cases} \tag{1.3}$$

In 2017, J. Wang and J. Shi [17] considered system (1.3) for $N = 3, p = 2$. By using the moving plane method, the authors show the symmetry of positive solutions to problem (1.3) when $\mu_1, \mu_2, \lambda_1, \lambda_2 > 0$ and $\beta \geq 0$. Moreover, the existence and non-existence of positive ground state solutions are also investigated. In 2017, J. Wang *et al.* [19] considered system (1.3) with perturbations and subcritical exponents, and via the Nehari constraint and minimax methods, obtained the multiplicity of non-trivial solutions provided the perturbation is small enough. In 2018, J. Wang and W. Yang [18] considered the existence and nonexistence of the normalized solutions of a system similar to (1.3) with $p = 2, \alpha = N - 2$. Under certain type trapping potentials, they give a precise description on the concentration behavior of minimizer solutions. Furthermore, they also obtained an optimal blowing up rate of the minimizer solutions. In 2019, S. You *et al.* [25, 26] derived the existence of a positive ground state of (1.3) with the upper critical exponents in a bounded smooth domain. Moreover, the limit behavior of positive ground state solutions also are considered as $\beta \to 0$ or $\beta \to -\infty$. In 2020, the following coupled Hartree system was considered in a smooth bounded domain by Y. Zheng *et al.* [27] in the fractional setting

$$\begin{cases} (-\Delta)^s u + \lambda_1 u = \alpha_1 \int_\Omega \frac{|u(z)|^{2^*_\mu}}{|x-z|^\mu} \mathrm{d}z |u|^{2^*_\mu - 2} u + \beta \int_\Omega \frac{|v(z)|^{2^*_\mu}}{|x-z|^\mu} \mathrm{d}z |u|^{2^*_\mu - 2} u, & x \in \Omega, \\ (-\Delta)^s v + \lambda_2 v = \alpha_2 \int_\Omega \frac{|v(z)|^{2^*_\mu}}{|x-z|^\mu} \mathrm{d}z |v|^{2^*_\mu - 2} v + \beta \int_\Omega \frac{|u(z)|^{2^*_\mu}}{|x-z|^\mu} \mathrm{d}z |v|^{2^*_\mu - 2} v, & x \in \Omega, \end{cases}$$

where $(-\Delta)^s$ stands for the fractional Laplacian operator of order $0 < s < 1$, $\alpha_1, \alpha_2 > 0$, $\beta \neq 0$, $4s < \mu < N$, $2^*_\mu = \frac{2N-\mu}{N-2s}$ is the fractional upper critical exponent. By applying the Dirichlet-to-Neumann map, the existence of ground state solutions was obtained with some various assumptions on $\alpha_1, \alpha_2, \lambda_1, \lambda_2$ and $\beta$. Very recently, F. Gao *et al.* [7] studied the coupled Hartree system with the upper critical exponent

$$\begin{cases} -\Delta u + V_1(x) u = \alpha_1 (|x|^{-4} * u^2) u + \beta (|x|^{-4} * v^2) u & \text{in } \mathbb{R}^N, \\ -\Delta v + V_2(x) v = \alpha_2 (|x|^{-4} * v^2) u + \beta (|x|^{-4} * u^2) v & \text{in } \mathbb{R}^N, \end{cases}$$

where $N \geq 5$, $\beta > \max\{\alpha_1, \alpha_2\} \geq \min\{\alpha_1, \alpha_2\} > 0$, and $V_1, V_2 \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ nonnegative. By establishing a nonlocal version of the global compactness lemma, the authors proved the existence of a high energy positive solutions when $\|V_1\|_{L^{N/2}}$ and $\|V_2\|_{L^{N/2}}$ are suitably small. Moreover, thanks to moving sphere arguments in the integral form, the classification and uniqueness of positive solutions are also given. For further related results, we would like to refer to [4, 24] for the lower critical case, [5, 6, 8, 16] for the upper critical case, [23] for coupled Choquard systems and other related results [11, 22].

## 1.2. MOTIVATION

In the present paper, we are concerned with the existence of positive ground state solutions of Choquard type systems with lower critical exponents. In [20], H. Wu dealt with the coupled Choquard type system (1.1) with $Q \equiv 1$ and obtained the existence of at least one positive ground state under the following assumptions on potentials:

$(H_1)$ $V_i(x) \geq 0$, $x \in \mathbb{R}^N$, $V_i(x) \in L^\infty(\mathbb{R}^N)$, $i = 1,2$,
$(H_2)$ $\lim\limits_{|x|\to\infty} V_i(x) = 1$, $i = 1,2$,
$(H_3)$ $\liminf\limits_{|x|\to\infty}(1 - V_i(x))|x|^2 \geq \frac{N^2(N-2)}{4(N+1)}$, $i = 1,2$.

Inspired by [28], we aim to investigate the coupled Choquard type system (1.1) with $Q(x) \not\equiv const$. In the following, we perform the variational method to study the existence and nonexistence of ground state solutions to (1.1). The associated functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ is given by

$$\begin{aligned} I(u,v) =& \frac{1}{2}\int\limits_{\mathbb{R}^N} (|\nabla u|^2 + V_1(x)u^2 + |\nabla v|^2 + V_2(x)v^2)\mathrm{d}x \\ &- \frac{N}{2(N+\alpha)} \int\limits_{\mathbb{R}^N} \Big(\mu_1 (I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}])Q(x)|u|^{\frac{N+\alpha}{N}} \\ &\qquad + \mu_2 (I_\alpha * [Q(x)|v|^{\frac{N+\alpha}{N}}])Q(x)|v|^{\frac{N+\alpha}{N}} \\ &\qquad + 2\beta (I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}])Q(x)|v|^{\frac{N+\alpha}{N}}\Big)\,\mathrm{d}x. \end{aligned}$$

Due to the presence of the lower critical exponent $p = \frac{N+\alpha}{N}$, the problem lacks compactness. Similarly to Sobolev critical problems, one Brezis–Nirenberg argument can be adopted to recover compactness. Actually, by imposing some suitable condition on $V_i$'s and $Q$, we can get the existence of a positive ground state solution. For this purpose, we assume that:

$(C_1)$ $\inf_{x\in\mathbb{R}^N} V_i(x) > 0$, $\lim_{|x|\to\infty} V_i(x) = 1$, $i = 1,2$,
$(C_2)$ there exists $\mu \in \mathbb{R}$ such that $\lim_{|x|\to\infty}(1 - V_i(x))|x|^2 = \mu$, $i = 1,2$,
$(Q_1)$ $\inf_{x\in\mathbb{R}^N} Q(x) \geq 0$, $\lim_{|x|\to\infty} Q(x) = 1$,
$(Q_2)$ there exist $\delta \geq 0$ and $\nu_\delta \in \mathbb{R}$ such that $\lim_{|x|\to\infty}(Q(x) - 1)|x|^\delta = \nu_\delta$.

## 1.3. MAIN RESULTS

**Theorem 1.1.** *Assume $(C_1)$, $(C_2)$, $(Q_1)$, $(Q_2)$ hold, then system (1.1) admits a positive ground state solution for all $\beta > 0$, provided one of the following conditions holds:*

(1) $\delta = 0, \mu > \frac{N^2(N-2)}{4(N+1)}, \inf_{x\in\mathbb{R}^N} Q(x) \geq 1$,
(2) $0 < \delta < 2, \nu_\delta > 0$,
(3) $2 < \delta < N, \mu > \frac{N^2(N-2)}{4(N+1)}$.

**Theorem 1.2.** *Assume $V_i$, $i = 1,2$, $Q \in \mathcal{C}^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and*

$(C_3)$ $\sup_{x\in\mathbb{R}^N} |x|^2\langle x, \nabla V_i(x)\rangle < \frac{(N-2)^2}{2}, i = 1,2$,

$(Q_3)$ $\inf_{x\in\mathbb{R}^N} Q(x) \geq 0, \quad \inf_{x\in\mathbb{R}^N}\langle x, \nabla Q(x)\rangle \geq 0$,

*then system (1.1) admits only a trivial solution in $H^1(\mathbb{R}^N)$.*

As a special case, for the external Schrödinger potential $V_{\mu,\nu} : \mathbb{R}^N \to \mathbb{R}$,

$$V_{\mu,\nu}(x) = 1 - \frac{\mu}{\nu^2 + |x|^2}, \quad \text{for } \mu \in \mathbb{R}, \nu > 0 \text{ and } x \in \mathbb{R}^N,$$

and the weighted potential $Q_\delta : \mathbb{R}^N \to \mathbb{R}$,

$$Q_\delta(x) = 1 + \frac{\nu_\delta}{1 + |x|^\delta}, \quad \text{for } \nu_\delta \in \mathbb{R}, \delta \geq 0 \text{ and } x \in \mathbb{R}^N,$$

system (1.1) reduces to the following form

$$\begin{cases} -\Delta u + V_{\mu,\nu}(x)u = \mu_1(I_\alpha * [Q_\delta(x)|u|^p])Q_\delta(x)|u|^{p-2}u \\ \qquad\qquad\qquad\qquad + \beta(I_\alpha * [Q_\delta(x)|v|^p])Q_\delta(x)|u|^{p-2}u, \\ -\Delta v + V_{\mu,\nu}(x)v = \mu_2(I_\alpha * [Q_\delta(x)|v|^p])Q_\delta(x)|v|^{p-2}v \\ \qquad\qquad\qquad\qquad + \beta(I_\alpha * [Q_\delta(x)|u|^p])Q_\delta(x)|v|^{p-2}v, \\ u, v \in H^1(\mathbb{R}^N). \end{cases} \tag{1.4}$$

Denote by $\mu^\nu$ the best constant of the embedding

$$H^1(\mathbb{R}^N) \hookrightarrow L^2\left(\mathbb{R}^N, (\nu^2 + |x|^2)^{-1}\mathrm{d}x\right),$$

that is,

$$\mu^\nu := \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2)\mathrm{d}x}{\int_{\mathbb{R}^N} \frac{|u(x)|^2}{\nu^2+|x|^2}\mathrm{d}x}.$$

**Corollary 1.3.** *System* (1.4) *admits a positive ground state solution, provided one of the following conditions holds:*

(1) $\delta = 0$, $\frac{N^2(N-2)}{4(N+1)} < \mu < \mu^\nu$, $\nu_\delta = 0$,
(2) $0 < \delta < 2$, $\mu < \mu^\nu$, $\nu_\delta > 0$,
(3) $2 < \delta < N$, $\frac{N^2(N-2)}{4(N+1)} < \mu < \mu^\nu$, $\nu_\delta \geq -1$,

*and has non-trivial solutions if* $\mu < \frac{(N-2)^2}{4}$ *and* $-1 \leq \nu_\delta \leq 0$.

## 2. PROOFS OF THEOREMS 1.1–1.2

Before proving Theorems 1.1–1.2, we introduce some preliminaries. First, the following Hardy–Littlewood–Sobolev inequality will be frequently used in the sequel.

**Lemma 2.1** (Hardy–Littlewood–Sobolev inequality [10])**.** *Let* $s, r > 1$, $0 < \alpha < N$ *with* $\frac{1}{s} + \frac{1}{r} = 1 + \frac{\alpha}{N}$, $f \in L^s(\mathbb{R}^N)$ *and* $g \in L^r(\mathbb{R}^N)$, *then there exists a positive constant* $\mathcal{C}(s, N, \alpha)$ (*independent of* $f$, $g$) *such that*

$$\left| \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)|x-y|^{\alpha-N}g(y)\mathrm{d}x\mathrm{d}y \right| \leq \mathcal{C}(s, N, \alpha)\|f\|_s\|g\|_r.$$

*In particular, if $s = r = \frac{2N}{N+\alpha}$, the sharp constant is given by*

$$\mathcal{C}_\alpha := \pi^{\frac{N-\alpha}{2}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N+\alpha}{2})} \left[\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right]^{\frac{-\alpha}{N}}.$$

Due to the presence of the lower critical exponent $\frac{N+\alpha}{N}$, the compactness fails in general. To recover the compactness, the following Brezis–Lieb type lemma plays a crucial role in the decomposition of the maximization sequence for $C_*$ given below.

For any $u, v \in L^2(\mathbb{R}^N)$, let

$$G_\infty(u,v) = \int\limits_{\mathbb{R}^N} \Big[\mu_1(I_\alpha * |u|^{\frac{N+\alpha}{N}})|u|^{\frac{N+\alpha}{N}} + \mu_2(I_\alpha * |v|^{\frac{N+\alpha}{N}})|v|^{\frac{N+\alpha}{N}} + 2\beta(I_\alpha * |u|^{\frac{N+\alpha}{N}})|v|^{\frac{N+\alpha}{N}}\Big] \mathrm{d}x$$

and

$$T_\infty(u,v) = \int\limits_{\mathbb{R}^N} \left(|u|^2 + |v|^2\right) \mathrm{d}x$$

and for any $u, v \in H^1(\mathbb{R}^N)$,

$$\begin{aligned} G(u,v) = \int\limits_{\mathbb{R}^N} \Big(&\mu_1(I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}])Q(x)|u|^{\frac{N+\alpha}{N}} \\ &+ \mu_2(I_\alpha * [Q(x)|v|^{\frac{N+\alpha}{N}}])Q(x)|v|^{\frac{N+\alpha}{N}} \\ &+ 2\beta(I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}])Q(x)|v|^{\frac{N+\alpha}{N}}\Big)\mathrm{d}x, \\ T(u,v) = \int\limits_{\mathbb{R}^N} &\left[|\nabla u|^2 + V_1(x)|u|^2 + |\nabla v|^2 + V_2(x)|v|^2\right] \mathrm{d}x. \end{aligned}$$

**Lemma 2.2** (Brezis–Lieb type Lemma)**.** *Assume that $(C_1)$ and $(Q_1)$ hold and let $\{u_n\}, \{v_n\}$ be bounded in $H^1(\mathbb{R}^N)$ and for some $u, v \in L^2(\mathbb{R}^N)$, such that $(u_n, v_n) \to (u, v)$ strongly in $L^2_{loc}(\mathbb{R}^N) \times L^2_{loc}(\mathbb{R}^N)$ as $n \to \infty$, then, up to a subsequence, there holds that*

$$\lim_{n\to\infty} [G(u_n, v_n) - G(u,v) - G_\infty(u_n - u, v_n - v)] = 0.$$

*Proof.* Without loss of generality, we assume that $(u_n, v_n) \to (u, v)$ almost everywhere in $\mathbb{R}^N$ as $n \to \infty$. Similarly as in [14, 28], we have

$$\begin{aligned}&\int_{\mathbb{R}^N} (I_\alpha * [Q(x)|u_n|^{\frac{N+\alpha}{N}}])Q(x)|u_n|^{\frac{N+\alpha}{N}}\,\mathrm{d}x\\&= \int_{\mathbb{R}^N} (I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}])Q(x)|u|^{\frac{N+\alpha}{N}}\,\mathrm{d}x\\&\quad + \int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^{\frac{N+\alpha}{N}})|u_n - u|^{\frac{N+\alpha}{N}}\,\mathrm{d}x + o_n(1),\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N} (I_\alpha * [Q(x)|v_n|^{\frac{N+\alpha}{N}}])Q(x)|v_n|^{\frac{N+\alpha}{N}}\,\mathrm{d}x\\&= \int_{\mathbb{R}^N} (I_\alpha * [Q(x)|v|^{\frac{N+\alpha}{N}}])Q(x)|v|^{\frac{N+\alpha}{N}}\,\mathrm{d}x\\&\quad + \int_{\mathbb{R}^N} (I_\alpha * |v_n - v|^{\frac{N+\alpha}{N}})|v_n - v|^{\frac{N+\alpha}{N}}\,\mathrm{d}x + o_n(1).\end{aligned}$$

So, to prove that

$$\lim_{n\to\infty} [G(u_n, v_n) - G(u, v) - G_\infty(u_n - u, v_n - v)] = 0,$$

it suffices to show

$$\begin{aligned}&\int_{\mathbb{R}^N} (I_\alpha * [Q(x)|u_n|^{\frac{N+\alpha}{N}}])Q(x)|v_n|^{\frac{N+\alpha}{N}}\,\mathrm{d}x\\&= \int_{\mathbb{R}^N} (I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}])Q(x)|v|^{\frac{N+\alpha}{N}}\,\mathrm{d}x\\&\quad + \int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^{\frac{N+\alpha}{N}})|v_n - v|^{\frac{N+\alpha}{N}}\,\mathrm{d}x + o_n(1).\end{aligned}$$

From the Brezis–Lieb lemma [21], we know that as $n \to \infty$,

$$\begin{cases}|u_n|^{\frac{N+\alpha}{N}} - |u_n - u|^{\frac{N+\alpha}{N}} \to |u|^{\frac{N+\alpha}{N}} & \text{in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N),\\ |v_n|^{\frac{N+\alpha}{N}} - |v_n - v|^{\frac{N+\alpha}{N}} \to |v|^{\frac{N+\alpha}{N}} & \text{in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N).\end{cases} \tag{3.1}$$

Then according to Lemma 2.1,

$$\begin{aligned}&\left\| I_\alpha * \left(|u_n|^{\frac{N+\alpha}{N}} - |u_n - u|^{\frac{N+\alpha}{N}} - |u|^{\frac{N+\alpha}{N}}\right)\right\|_{L^{\frac{2N}{N-\alpha}}}\\&\le C\left\| |u_n|^{\frac{N+\alpha}{N}} - |u_n - u|^{\frac{N+\alpha}{N}} - |u|^{\frac{N+\alpha}{N}}\right\|_{L^{\frac{2N}{N+\alpha}}} = o_n(1).\end{aligned}$$

Noting that $\lim_{|x|\to\infty} Q(x) = 1$, we get that

$$\Big\| I_\alpha * \Big[ Q(x)(|u_n|^{\frac{N+\alpha}{N}} - |u_n - u|^{\frac{N+\alpha}{N}} - |u|^{\frac{N+\alpha}{N}}) \Big] \Big\|_{L^{\frac{2N}{N-\alpha}}} = o_n(1).$$

Therefore, as $n \to \infty$,

$$\begin{cases} I_\alpha * \Big[ Q(x)(|u_n|^{\frac{N+\alpha}{N}} - |u_n - u|^{\frac{N+\alpha}{N}}) \Big] \to I_\alpha * \Big[ Q(x)|u|^{\frac{N+\alpha}{N}} \Big] & \text{in } L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N), \\ I_\alpha * \Big[ Q(x)(|v_n|^{\frac{N+\alpha}{N}} - |v_n - v|^{\frac{N+\alpha}{N}}) \Big] \to I_\alpha * \Big[ Q(x)|v|^{\frac{N+\alpha}{N}} \Big] & \text{in } L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N). \end{cases} \tag{3.2}$$

Let

$$\begin{aligned} &\int_{\mathbb{R}^N} (I_\alpha * [Q(x)|u_n|^{\frac{N+\alpha}{N}}]) Q(x)|v_n|^{\frac{N+\alpha}{N}} \mathrm{d}x \\ &= \int_{\mathbb{R}^N} (I_\alpha * [Q(x)|u_n - u|^{\frac{N+\alpha}{N}}]) Q(x)|v_n - v|^{\frac{N+\alpha}{N}} \mathrm{d}x + J_1 + J_2 + J_3, \end{aligned}$$

where

$$\begin{cases} J_1 = \int_{\mathbb{R}^N} \Big( I_\alpha * [Q(x)(|u_n|^{\frac{N+\alpha}{N}} - |u_n - u|^{\frac{N+\alpha}{N}})] \Big) Q(x) \Big( |v_n|^{\frac{N+\alpha}{N}} - |v_n - v|^{\frac{N+\alpha}{N}} \Big) \mathrm{d}x, \\ J_2 = \int_{\mathbb{R}^N} \Big( I_\alpha * [Q(x)(|v_n|^{\frac{N+\alpha}{N}} - |v_n - v|^{\frac{N+\alpha}{N}})] \Big) Q(x)|u_n - u|^{\frac{N+\alpha}{N}} \mathrm{d}x, \\ J_3 = \int_{\mathbb{R}^N} \Big( I_\alpha * [Q(x)(|u_n|^{\frac{N+\alpha}{N}} - |u_n - u|^{\frac{N+\alpha}{N}})] \Big) Q(x)|v_n - v|^{\frac{N+\alpha}{N}} \mathrm{d}x. \end{cases}$$

Firstly, it follows from (3.1) that

$$J_1 = \int_{\mathbb{R}^N} \Big( I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}] \Big) Q(x)|v|^{\frac{N+\alpha}{N}} \mathrm{d}x + o_n(1).$$

Since $v_n$ is bounded in $H^1(\mathbb{R}^N)$, without loss of generality, we assume that

$$v_n \rightharpoonup v \text{ in } H^1(\mathbb{R}^N), \quad v_n \to v \ a.e \text{ in } \mathbb{R}^N.$$

Due to the fact that $|u_n - u|^{\frac{N+\alpha}{N}}$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, we have $|u_n - u|^{\frac{N+\alpha}{N}} \rightharpoonup 0$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ as $n \to \infty$. Thanks to (3.2),

$$J_2 = \int_{\mathbb{R}^N} \Big( I_\alpha * \Big[ Q(x)|v|^{\frac{N+\alpha}{N}} \Big] \Big) Q(x)|u_n - u|^{\frac{N+\alpha}{N}} \mathrm{d}x + o_n(1) = o_n(1).$$

Similarly, $J_3 = o_n(1)$, as $n \to \infty$. Thus, we get

$$\begin{aligned} &\int_{\mathbb{R}^N} \Big( I_\alpha * [Q(x)|u_n|^{\frac{N+\alpha}{N}}] \Big) Q(x)|v_n|^{\frac{N+\alpha}{N}} \mathrm{d}x \\ &= \int_{\mathbb{R}^N} \Big( I_\alpha * [Q(x)|u_n - u|^{\frac{N+\alpha}{N}}] \Big) Q(x)|v_n - v|^{\frac{N+\alpha}{N}} \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} \Big( I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}] \Big) Q(x)|v|^{\frac{N+\alpha}{N}} \mathrm{d}x + o_n(1). \end{aligned}$$

Similarly as in [28], we get that

$$\begin{aligned}&\int_{\mathbb{R}^N} \left(I_\alpha * [Q(x)|u_n-u|^{\frac{N+\alpha}{N}}]\right) Q(x)|v_n-v|^{\frac{N+\alpha}{N}} \mathrm{d}x\\ &=\int_{\mathbb{R}^N} \left(I_\alpha * |u_n-u|^{\frac{N+\alpha}{N}}\right) |v_n-v|^{\frac{N+\alpha}{N}} \mathrm{d}x + o_n(1).\end{aligned}$$

The proof is complete. □

Set

$$C_\infty := \sup\left\{G_\infty(u,v) : T_\infty(u,v)=1, u\in L^2(\mathbb{R}^N)\right\}$$

and

$$C_* := \sup\left\{G(u,v) :\ T(u,v)=1, u\in H^1(\mathbb{R}^N)\right\}.$$

Then by Lemma 2.1,

$$0<C_*, C_\infty<\infty.$$

**Lemma 2.3** (Compactness)**.** *If $C_*>C_\infty$, then $C_*$ can be achieved.*

*Proof.* For any maximization sequence $\{(u_n,v_n)\}_{n=1}^\infty \subset H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ of $C_*$, i.e., as $n\to\infty$, $G(u_n,v_n)\to C_*$ with $T(u_n,v_n)=1$. Without loss of generality, we assume that $u_n, v_n$ are non-negative for all $n$ and for some $u_0, v_0\in H^1(\mathbb{R}^N)$, $(u_n,v_n)\to(u_0,v_0)\geq 0$ weakly in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, strongly in $L^2_{loc}(\mathbb{R}^N)\times L^2_{loc}(\mathbb{R}^N)$ and a. e. in $\mathbb{R}^N$ as $n\to\infty$. Moreover, set

$$\omega_n = u_n-u_0, \quad z_n = v_n - v_0,$$

then thanks to Lemma 2.2,

$$C_* = G(u_0,v_0)+G_\infty(w_n,z_n)+o_n(1), \tag{3.3}$$

where $o_n(1)\to 0$ as $n\to\infty$. we have

$$1 = T(u_0,v_0)+T(w_n,z_n)+o_n(1). \tag{3.4}$$

On the other hand, by the definition of $C_*$ and $C_\infty$, it easy to know that

$$G(u,v)\leq C_*\left(T(u,v)\right)^{\frac{N+\alpha}{N}}, \quad \text{for any } u,v\in H^1(\mathbb{R}^N)$$

and

$$G_\infty(u,v)\leq C_\infty\left(T_\infty(u,v)\right)^{\frac{N+\alpha}{N}}, \quad \text{for any } u,v\in L^2(\mathbb{R}^N).$$

Obviously, $T(u_0,v_0)\in[0,1]$. Then by (3.3) and (3.4),

$$\begin{aligned}C_* &\leq C_*\left(T(u_0,v_0)\right)^{\frac{N+\alpha}{N}} + C_\infty\left(T_\infty(w_n,z_n)\right)^{\frac{N+\alpha}{N}}+o_n(1)\\ &\leq C_*\left(T(u_0,v_0)\right)^{\frac{N+\alpha}{N}} + C_\infty\left(T(w_n,z_n)\right)^{\frac{N+\alpha}{N}}+o_n(1)\\ &= C_*\left(T(u_0,v_0)\right)^{\frac{N+\alpha}{N}} + C_\infty\left(1-T(u_0,v_0)\right)^{\frac{N+\alpha}{N}}+o_n(1)\\ &\leq C_* T(u_0,v_0)+C_\infty(1-T(u_0,v_0))+o_n(1),\end{aligned}$$

where we used the fact that $\frac{N+\alpha}{N} > 1$. It follows that

$$C_* \le C_* T(u_0, v_0) + C_\infty (1 - T(u_0, v_0))$$

and then $C_* \le C_\infty$ if $T(u_0, v_0) < 1$. So $T(u_0, v_0) = 1$ and $G(u_0, v_0) = C_*$. The proof is complete. □

In the following, we give a lower bound estimate for $C_*$. For any $u \in L^2(\mathbb{R}^N)$, set

$$\overline{G}(u) := \int_{\mathbb{R}^N} \left( I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}] \right) Q(x)|u|^{\frac{N+\alpha}{N}} \mathrm{d}x,$$

$$\overline{G}_\infty(u) := \int_{\mathbb{R}^N} \left( I_\alpha * |u|^{\frac{N+\alpha}{N}} \right) |u|^{\frac{N+\alpha}{N}} \mathrm{d}x,$$

and

$$c_\infty = \sup \left\{ \overline{G}_\infty(u) : \int_{\mathbb{R}^N} u^2 \mathrm{d}x = 1 \right\}.$$

Moreover, for any $u \in H^1(\mathbb{R}^N)$, let

$$\overline{T}_i(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + V_i(x)u^2)\mathrm{d}x, \quad i = 1, 2.$$

It can be seen in [12] that $c_\infty$ is achieved by $u_\varepsilon = \varepsilon^{\frac{N}{2}} U(\varepsilon x)$ for any $\varepsilon > 0$, with

$$\overline{G}_\infty(u_\varepsilon) = c_\infty, \quad \int_{\mathbb{R}^N} u_\varepsilon^2 \mathrm{d}x = 1,$$

where

$$U(x) = C\lambda^{\frac{N}{2}} (\lambda^2 + |x|^2)^{-\frac{N}{2}},$$

for some fixed constant $C > 0$ and $\lambda \in \mathbb{R}^+$ as parameters. Let

$$v_\varepsilon = \frac{u_\varepsilon}{\sqrt{1 + s_m^2}},$$

then it can be seen in [20] that $(s_m v_\varepsilon, v_\varepsilon)$ is a maximizer of $C_\infty$, that is,

$$G_\infty(s_m v_\varepsilon, v_\varepsilon) = C_\infty, \quad T_\infty(s_m v_\varepsilon, v_\varepsilon) = 1,$$

where $s_m$ is a minimum point of function $g(s) : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$g(s) = \frac{1 + s^2}{\left( \mu_2 + \mu_1 s^{\frac{2(N+\alpha)}{N}} + 2\beta s^{\frac{N+\alpha}{N}} \right)^{\frac{N}{N+\alpha}}}.$$

It is easy to know

$$C_\infty = \frac{c_\infty}{[g(s_m)]^{\frac{N+\alpha}{N}}}.$$

**Lemma 2.4** (Energy estimate)**.** *Assume* $(C_1)$–$(C_2)$*,* $(Q_1)$–$(Q_2)$ *hold, then we have* $C_* > C_\infty$ *provided one of the following conditions holds:*

(1) $\delta = 0$, $\mu > \frac{N^2(N-2)}{4(N+1)}$, $\inf_{x\in\mathbb{R}^N} Q(x) \geq 1$,
(2) $0 < \delta < 2$, $\nu_\delta > 0$,
(3) $2 < \delta < N$, $\mu > \frac{N^2(N-2)}{4(N+1)}$.

*Proof.* Observe that for any $\varepsilon > 0$,

$$(1+s_m^2)\int_{\mathbb{R}^N} |v_\varepsilon|^2 \mathrm{d}x = 1, \quad \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 \mathrm{d}x = \frac{\varepsilon^2}{1+s_m^2}\int_{\mathbb{R}^N} |\nabla U|^2 \mathrm{d}x < +\infty.$$

Let

$$m_\varepsilon^1 = \overline{T}_1(u_\varepsilon), \quad m_\varepsilon^2 = \overline{T}_2(u_\varepsilon), \quad m_\varepsilon := T(s_m v_\varepsilon, v_\varepsilon),$$

then

$$m_\varepsilon = \frac{s_m^2 m_\varepsilon^1 + m_\varepsilon^2}{1+s_m^2}.$$

Observe that

$$m_\varepsilon^i = 1 + \varepsilon^2 \mathcal{I}_\mu^i(\varepsilon), \quad i = 1, 2,$$

where

$$\mathcal{I}_\mu^i(\varepsilon) = \varepsilon^{-2}\int_{\mathbb{R}^N} \left[|\nabla u_\varepsilon|^2 + (V_i(x)-1)|u_\varepsilon|^2)\right] \mathrm{d}x.$$

Similarly as that in [28], we have

$$\mathcal{I}_\mu^i(\varepsilon) = a_\mu + o_\varepsilon(1), \quad i = 1, 2,$$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$ and

$$a_\mu := \left[\frac{N^2(N-2)}{4(N+1)} - \mu\right]\int_{\mathbb{R}^N} \frac{|U(x)|^2}{|x|^2}.$$

Obviously,

$$a_\mu = \begin{cases} > 0, & \text{if}\, \mu < \frac{N^2(N-2)}{4(N+1)}, \\ = 0, & \text{if } \mu = \frac{N^2(N-2)}{4(N+1)}, \\ < 0, & \text{if } \mu > \frac{N^2(N-2)}{4(N+1)}. \end{cases}$$

Then, as $\varepsilon \to 0$, $m_\varepsilon^i = 1 + a_\mu \varepsilon^2 + o(\varepsilon^2)$, $i = 1, 2$. It follows that

$$m_\varepsilon = 1 + a_\mu \varepsilon^2 + o(\varepsilon^2).$$

Let $w_\varepsilon := \frac{v_\varepsilon}{\sqrt{m_\varepsilon}}$, then $T(s_m w_\varepsilon, w_\varepsilon) = 1$ and $G(s_m w_\varepsilon, w_\varepsilon) \leq C_*$. In the following, we show that $G(s_m w_\varepsilon, w_\varepsilon) > C_\infty$ for $\varepsilon > 0$ small. In fact,

$$G(s_m w_\varepsilon, w_\varepsilon) = \left(\mu_1 s_m^{\frac{2(N+\alpha)}{N}} + 2\beta s_m^{\frac{N+\alpha}{N}} + \mu_2\right)\overline{G}(w_\varepsilon).$$

Let $\tilde{u}_\varepsilon = \frac{u_\varepsilon}{\sqrt{m_\varepsilon^1}}$, then $\overline{T}_1(\tilde{u}_\varepsilon) = 1$ and it can be seen in [28] that, as $\varepsilon \to 0$,

$$\overline{G}(\tilde{u}_\varepsilon) \begin{cases} \geq c_\infty - \frac{N+\alpha}{N} c_\infty a_\mu \varepsilon^2 + o(\varepsilon^2), & \text{if } \delta = 0 \text{ or } \delta > 2, \\ = c_\infty + b_{\delta,\alpha} \nu_\delta \varepsilon^\delta + o(\varepsilon^\delta), & \text{if } 0 < \delta < 2, \end{cases}$$

where

$$b_{\delta,\alpha} = 2 \int\limits_{\mathbb{R}^N} (I_\alpha * |U|^{\frac{N+\alpha}{N}})[|x|^{-\delta}|U|^{\frac{N+\alpha}{N}}]dx > 0.$$

Obviously,

$$\overline{G}(w_\varepsilon) = \frac{\overline{G}(\tilde{u}_\varepsilon)}{(1+s_m^2)^{\frac{N+\alpha}{N}}} \left(\frac{m_\varepsilon^1}{m_\varepsilon}\right)^{\frac{N+\alpha}{N}} = \frac{\overline{G}(\tilde{u}_\varepsilon)}{(1+s_m^2)^{\frac{N+\alpha}{N}}}(1 + o(\varepsilon^2)),$$

and then

$$G(s_m w_\varepsilon, w_\varepsilon) = \frac{(1+o(\varepsilon^2))\overline{G}(\tilde{u}_\varepsilon)}{[g(s_m)]^{\frac{N+\alpha}{N}}}.$$

Since

$$(1+o(\varepsilon^2))\overline{G}(\tilde{u}_\varepsilon) \begin{cases} \geq c_\infty - \frac{N+\alpha}{N} c_\infty a_\mu \varepsilon^2 + o(\varepsilon^2), & \text{if } \delta = 0 \text{ or } \delta > 2, \\ = c_\infty + b_{\delta,\alpha} \nu_\delta \varepsilon^\delta + o(\varepsilon^\delta), & \text{if } 0 < \delta < 2, \end{cases}$$

one of the assumptions (1)–(3) implies that, for $\varepsilon$ small,

$$G(s_m w_\varepsilon, w_\varepsilon) > \frac{c_\infty}{[g(s_m)]^{\frac{N+\alpha}{N}}} = C_\infty.$$

The proof is complete. □

**Lemma 2.5.** *Assume that $(C_1)$,$(C_2)$,$(Q_1)$,$(Q_2)$ hold. If $\beta > 0$, then*

$$C_* > \max\{C_{*1}, C_{*2}\},$$

*where*

$$C_{*i} = \mu_i \sup\{\overline{G}(u) : \overline{T}_i(u) = 1\}, \quad i = 1, 2.$$

*Proof.* It has been shown in [28] that $C_{*i}, i = 1, 2$ can be achieved, i.e., there exist $u_*, v_* \in H^1(\mathbb{R}^N)$ such that

$$C_{*1} = \mu_1 \overline{G}(u_*), \quad \overline{T}_1(u_*) = 1, \quad C_{*2} = \mu_2 \overline{G}(v_*), \quad \overline{T}_2(v_*) = 1.$$

For any fixed $t > 0$ large, we have

$$G(tu_*, u_*) = \left(\mu_1 t^{\frac{2(N+\alpha)}{N}} + 2\beta t^{\frac{N+\alpha}{N}} + \mu_2\right) \overline{G}(u_*)$$

and

$$T(tu_*, u_*) = t^2 \overline{T}_1(u_*) + \overline{T}_2(u_*) = t^2 + \overline{T}_2(u_*).$$

Let $\bar{u}_* = \frac{u_*}{\sqrt{t^2+\overline{T}_2(u_*)}}$, we have $T(t\bar{u}_*, \bar{u}_*) = 1$ and

$$C_* \geq G(t\bar{u}_*, \bar{u}_*) = \frac{\mu_1 t^{\frac{2(N+\alpha)}{N}} + 2\beta t^{\frac{N+\alpha}{N}} + \mu_2}{\left(t^2 + \overline{T}_2(u_*)\right)^{\frac{N+\alpha}{N}}} \overline{G}(u_*)$$
$$= \frac{\mu_1 t^{\frac{2(N+\alpha)}{N}} + 2\beta t^{\frac{N+\alpha}{N}} + \mu_2}{\mu_1 \left(t^2 + \overline{T}_2(u_*)\right)^{\frac{N+\alpha}{N}}} C_{*1}.$$

Obviously, as $t \to \infty$,

$$\mu_1 t^{\frac{2(N+\alpha)}{N}} + 2\beta t^{\frac{N+\alpha}{N}} + \mu_2 = t^{\frac{2(N+\alpha)}{N}} \left[\mu_1 + 2\beta t^{-\frac{N+\alpha}{N}} + o\left(t^{-\frac{N+\alpha}{N}}\right)\right]$$

and

$$\mu_1 \left(t^2 + \overline{T}_2(u_*)\right)^{\frac{N+\alpha}{N}} = \mu_1 t^{\frac{2(N+\alpha)}{N}} \left[1 + \frac{N+\alpha}{N} \overline{T}_2(u_*) t^{-2} + o(t^{-2})\right].$$

Thanks to $\frac{N+\alpha}{N} < 2$, we know, for $t > 0$ large enough,

$$\frac{\mu_1 t^{\frac{2(N+\alpha)}{N}} + 2\beta t^{\frac{N+\alpha}{N}} + \mu_2}{\mu_1 \left(t^2 + \overline{T}_2(u_*)\right)^{\frac{N+\alpha}{N}}} > 1,$$

which implies that $C_* > C_{*1}$. Similarly, for any fixed $t > 0$ large, we consider the quantity $G(v_*, tv_*)$ and get that

$$C_* \geq \frac{\mu_2 t^{\frac{2(N+\alpha)}{N}} + 2\beta t^{\frac{N+\alpha}{N}} + \mu_1}{\left(t^2 + \overline{T}_1(v_*)\right)^{\frac{N+\alpha}{N}}} \overline{G}(v_*)$$
$$= \frac{\mu_2 t^{\frac{2(N+\alpha)}{N}} + 2\beta t^{\frac{N+\alpha}{N}} + \mu_1}{\mu_2 \left(t^2 + \overline{T}_1(v_*)\right)^{\frac{N+\alpha}{N}}} C_{*2} > C_{*2}.$$

The proof is complete. □

*Proof of Theorem 1.1.* As an immediate consequence of Lemma 2.3 and 2.4, there exist $u_*, v_* \in H^1(\mathbb{R}^N)$ such that $G(u_*, v_*) = C_*$ and $T(u_*, v_*) = 1$. Since $G(|u_*|, |v_*|) = C_*$ and $T(|u_*|, |v_*|) = 1$, without loss of generality, we assume that $u_*$ and $v_*$ are non-negative. By the Lagrange multiplier theorem, there holds that for some $\kappa \in \mathbb{R}$ such that

$$G'(u_*, v_*) = \kappa T'(u_*, v_*) \quad \text{in } H^{-1}(\mathbb{R}^N) \times H^{-1}(\mathbb{R}^N),$$

where

$$G'(u_*, v_*) = (\nabla_u G(u_*, v_*), \nabla_v G(u_*, v_*))$$

and

$$T'(u_*, v_*) = (\nabla_u T(u_*, v_*), \nabla_v T(u_*, v_*)).$$

That is, in the weak sense, $(u_*, v_*)$ satisfies

$$\begin{cases} \frac{N\kappa}{N+\alpha}(-\Delta u + V_1(x)u) = \mu_1(I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}])Q(x)|u|^{\frac{\alpha}{N}-1}u \\ \qquad\qquad + \beta(I_\alpha * [Q(x)|v|^{\frac{N+\alpha}{N}}])Q(x)|u|^{\frac{\alpha}{N}-1}u, \\ \frac{N\kappa}{N+\alpha}(-\Delta v + V_2(x)v) = \mu_2(I_\alpha * [Q(x)|v|^{\frac{N+\alpha}{N}}])Q(x)|v|^{\frac{\alpha}{N}-1}v \\ \qquad\qquad + \beta(I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}])Q(x)|v|^{\frac{\alpha}{N}-1}v, \\ u, v \in H^1(\mathbb{R}^N). \end{cases}$$

Obviously, $\kappa = \frac{N+\alpha}{N}C_* > 0$. By virtue of the maximum principle, $u_*$, $v_*$ are positive. To remove the multiplier, let

$$u_\theta = \theta u_*(x), \quad v_\theta = \theta v_*(x), \quad \theta = C_*^{-\frac{N}{2\alpha}},$$

then $(u_\theta, v_\theta)$ is a weak solution of problem (1.1) and

$$\begin{aligned} I(u_\theta, v_\theta) &= \frac{1}{2}T(u_\theta, v_\theta) - \frac{N}{2(N+\alpha)}G(u_\theta, v_\theta) \\ &= \frac{1}{2}\theta^2 T(u_*, v_*) - \frac{N}{2(N+\alpha)}\theta^{\frac{2(N+\alpha)}{N}}G(u_*, v_*) \\ &= \frac{1}{2}\theta^2 - \frac{N}{2(N+\alpha)}\theta^{\frac{2(N+\alpha)}{N}}C_* \\ &= \frac{\alpha}{2(N+\alpha)}C_*^{-\frac{N}{\alpha}} > 0. \end{aligned}$$

In the following, we show that $(u_\theta, v_\theta)$ is a ground state solution of problem (1.1). In fact, for any nontrivial solution $(u, v)$ of problem (1.1), $T(u, v) = G(u, v) > 0$ and

$$I(u, v) = \frac{\alpha}{2(N+\alpha)}G(u, v).$$

Let

$$u_\tau(x) = \frac{u(x)}{\sqrt{T(u, v)}}, \quad v_\tau(x) = \frac{v(x)}{\sqrt{T(u, v)}},$$

then $T(u_\tau, v_\tau) = 1$ and

$$C_* \geq G(u_\tau, v_\tau) = G(u, v)[T(u, v)]^{-\frac{N+\alpha}{N}}.$$

It follows that

$$G(u, v) \leq C_*[T(u, v)]^{\frac{N+\alpha}{N}} = C_*[G(u, v)]^{\frac{N+\alpha}{N}},$$

and then

$$I(u, v) \geq \frac{\alpha}{2(N+\alpha)}C_*^{-\frac{N}{\alpha}}.$$

Due to Lemma 2.5, we know $u_*, v_* \not\equiv 0$. The proof is complete. □

In the following, we give a Pohožaev identity, which helps us to show a non-existence result of nontrivial solutions. We also refer to [2] for the applications of Pohožaev type identities on the attainability of some embedding inequalities.

**Proposition 2.6** (Pohožaev Identity)**.** *If* $(u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ *is a solution of system* (1.1)*, then the following Pohožaev identity holds:*

$$\begin{aligned}
&\frac{N-2}{2}\int\limits_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)\mathrm{d}x+\frac{1}{2}\int\limits_{\mathbb{R}^N}[NV_1(x)+\langle x,\nabla V_1(x)\rangle]u^2\mathrm{d}x\\
&+\frac{1}{2}\int\limits_{\mathbb{R}^N}[NV_2(x)+\langle x,\nabla V_2(x)\rangle]v^2\mathrm{d}x\\
&=\frac{N}{2}\int\limits_{\mathbb{R}^N}\Big[\mu_1(I_\alpha*[Q(x)|u|^{\frac{N+\alpha}{N}}])Q(x)|u|^{\frac{N+\alpha}{N}}+\mu_2(I_\alpha*[Q(x)|v|^{\frac{N+\alpha}{N}}])Q(x)|v|^{\frac{N+\alpha}{N}}\\
&\qquad+2\beta(I_\alpha*[Q(x)|u|^{\frac{N+\alpha}{N}}])Q(x)|v|^{\frac{N+\alpha}{N}}\Big]\mathrm{d}x\\
&\quad+\frac{N}{N+\alpha}\int\limits_{\mathbb{R}^N}\Big[\mu_1(I_\alpha*[Q(x)|u|^{\frac{N+\alpha}{N}}])\langle x,\nabla Q(x)\rangle|u|^{\frac{N+\alpha}{N}}\\
&\qquad+\mu_2(I_\alpha*[Q(x)|v|^{\frac{N+\alpha}{N}}])\langle x,\nabla Q(x)\rangle|v|^{\frac{N+\alpha}{N}}\\
&\qquad+\beta(I_\alpha*[Q(x)|u|^{\frac{N+\alpha}{N}}])\langle x,\nabla Q(x)\rangle|v|^{\frac{N+\alpha}{N}})\\
&\qquad+\beta(I_\alpha*[Q(x)|v|^{\frac{N+\alpha}{N}}])\langle x,\nabla Q(x)\rangle|u|^{\frac{N+\alpha}{N}}\Big]\mathrm{d}x.
\end{aligned}$$

*Proof.* The proof is similar to [13] and [12]. We omit the details here. □

*Completion of the proof of Theorem 1.2.* For any solution $(u,v) \in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ of problem (1.1), using $(u,v)$ as a test function, we have

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N}(|\nabla u|^2+V_1(x)|u|^2+|\nabla v|^2+V_2(x)|v|^2)\mathrm{d}x\\
&=\int\limits_{\mathbb{R}^N}\Big[\mu_1(I_\alpha*[Q(x)|u|^{\frac{N+\alpha}{N}}])Q(x)|u|^{\frac{N+\alpha}{N}}+\mu_2(I_\alpha*[Q(x)|v|^{\frac{N+\alpha}{N}}])Q(x)|v|^{\frac{N+\alpha}{N}}\\
&\qquad+2\beta(I_\alpha*[Q(x)|u|^{\frac{N+\alpha}{N}}])Q(x)|v|^{\frac{N+\alpha}{N}}\Big]\mathrm{d}x.
\end{aligned}$$

Thanks to Proposition 2.6,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(-|\nabla u|^2 - |\nabla v|^2 + \frac{1}{2}\left[\langle x, \nabla V_1(x)\rangle u^2 + \langle x, \nabla V_2(x) v^2\right]\right) \mathrm{d}x \\
&= \frac{N}{N+\alpha} \int_{\mathbb{R}^N} \Big[\mu_1 (I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}])\langle x, \nabla Q(x)\rangle |u|^{\frac{N+\alpha}{N}} \\
&\qquad + \mu_2 (I_\alpha * [Q(x)|v|^{\frac{N+\alpha}{N}}])\langle x, \nabla Q(x)\rangle |v|^{\frac{N+\alpha}{N}} \\
&\qquad + \beta (I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}])\langle x, \nabla Q(x)\rangle |v|^{\frac{N+\alpha}{N}}) \\
&\qquad + \beta (I_\alpha * [Q(x)|v|^{\frac{N+\alpha}{N}}])\langle x, \nabla Q(x)\rangle |u|^{\frac{N+\alpha}{N}}\Big] \mathrm{d}x.
\end{aligned}$$

Then by $(C_3)$, $(Q_3)$ and Hardy's inequality, if $u$, $v$ are nontrivial, then

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) \mathrm{d}x < \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \left(\frac{|u(x)|^2}{|x|^2} + \frac{|v(x)|^2}{|x|^2}\right) \mathrm{d}x \le \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) \mathrm{d}x,$$

which is a contradiction. The proof is complete. □

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