Filus Jerzy K., 0000-0002-5083-5626
Oakton College, Des Plaines, U.S.A., jkfilus6{at}gmail.com
Filus Lidia Z., 0000-0002-4267-5773
Northeastern Illinois University, Chicago, U.S.A., L-Filus{at}neiu.edu

Modeling reliability of systems with repair by stochastic processes with long memory

Keywords

discrete time, unlimited memory stochastic processes, *k*-Markovianity, triangular transformations, method of parameter dependence

Abstract

In modelling reliability of systems with repair by stochastic processes of times between consecutive failures the usual Markovianity assumption was significantly relaxed. Instead of the Markovian stochastic processes, processes with long memory were constructed for the reliability and maintenance applications. The Markovianity restriction on the process's memory could be omitted as two (relatively) new methods of the processes construction were employed. In this work, one of the two available methods, the 'method of triangular transformations', is presented. Other, the 'method of parameter dependence', is shortly described in Section 5. Since using an arbitrarily long memory has serious drawbacks in modelling process we, on the other hand, limited it by introducing the notion of k-Markovianity (k = 1,2,…), where the memory is reduced to the last k previous (discrete) time epochs. The discussion of this kind of problems together with construction of some new classes of stochastic processes with discrete time and their reliability application is provided.

1. Introduction

Consider reliability of a system with repair. After each failure it is repaired and then it works again till next failure. In the framework we consider, times of repair are neglected. Every (random) time between (j – 1)-th and j-th failure (j = 1,2,…) we denote by X_j which is a nonnegative random quantity.

The times X_1, X_2,… form the stochastic process which stands for the model of such repairable system.

In most common literature, the usual assumption about this model is either independence of the random variables X_1, X_2,… or Markovianity of the underlying stochastic process $\{X_j\}$. The foregoing assumption means that the each conditional distribution $F_j(x_j \mid x_1,\ldots,x_{j-1})$ of $(X_j \mid x_1,\ldots,x_{j-1})$ only depends on the condition $X_{j-1} = x_{j-1}$ and is independent on the values $x_1,\ldots,x_{j-2}$.

Thus, at a given time epoch, $X_{j-1} = x_{j-1}$, the future stochastic behavior up to the j-th failure only depends on the previous cycle between (j – 2)-th and (j – 1)-th failure. So that the essential part of history of the system performance is very limited and often this assumption is not realistic.

When the stochastic dependence is present, the Markovianity assumption mostly (with exception of the case of multivariate normal distributions which have very limited application to the considered reliability problems) is forced on researchers by the lack of known properly defined analytical forms of conditional distributions, say, $F_j(x_j \mid x_1,\ldots,x_{j-1})$ that explicitly depend on the values $x_1,\ldots,x_{j-2}$.

Safety and Reliability of Systems and Processes, Summer Safety and Reliability Seminar 2023.

DOI: 10.26408/srsp-2023-06.

In reliability frameworks the need for such conditional distributions is especially vital for the case when they are Weibullian including the exponential case.

The stochastic process, governed by these distributions, we call the *process with good memory*.

It is then vital to find methods for construction of such *analytical forms* of the $F_j(x_j \mid x_1,\ldots,x_{j-1})$ or of the conditional densities $f_j(x_j \mid x_1,\ldots, x_{j-1})$.

Our goal is to present two such methods. One of the two is the *parameter dependence method* (Filus & Filus, 2008), but first we present the (partially equivalent) *triangular transformations method* (Filus et al, 2010) this time in application to the construction of the stochastic processes.

As for the method of triangular transformations, basically, in the case of stochastic processes, the method relies on transforming any renewal process $\{T_j\}$ (the random variables $T_1, T_2,\ldots$ are independent and equally distributed) into the (in general non-Markovian) stochastic process $\{X_j\}$ with different and dependent terms.

There are two basic versions of the transformations: finite dimensional (for random vectors) and infinite dimension (for the processes with discrete times).

2. Random vectors case

For each finite $n = 2,3,\ldots$ define the class of *pseudoaffine* transformations which is the subclass of the most general considered class of triangular transformations of random vectors, say $(T_1,\ldots,T_n)$ into random vectors $(X_1,\ldots,X_n)$. (Here it is to be mentioned that the general triangular transformation is characterized by the only property of having its Jacoby matrix lower triangular.)

The pattern for the finite, say R^n à R^n, *pseudoaffine* transformations is given by the following formula:

$$X_1 = \phi_0 T_1 + \psi_0 \qquad (1)$$

$$X_2 = \phi_1(X_1)T_2 + \psi_1(X_1)$$

$$\ldots$$

$$X_j = \phi_{j-1}(X_1,\ldots,X_{j-1})T_j + \psi_{j-1}(X_1,\ldots,X_{j-1})$$

$$\ldots$$

$$X_n = \phi_{n-1}(X_1,\ldots,X_{n-1})T_n + \psi_{n-1}(X_1,\ldots,X_{n-1}),$$

where, depending on a particular setting that may be encountered, a sense of the foregoing n equalities (1) may vary from case to case. At least, however, it will be assumed they hold with respect to probability distributions of the sides. This or other *weak* meaning (equality with probability one or equality of the random variables' realizations) of the underlying equalities can be adopted.

The functions ϕ_0, $\phi_1(\ )$,…,$\phi_{n-1}(\ )$ and ψ_0, $\psi_1(\ )$,…,$\psi_{n-1}(\ )$, that we call the *parameter functions*, are only assumed to be continuous and ϕ_0, $\phi_1(\ )$,…, $\phi_{n-1}(\ )$ are never zero, with ϕ_0 and ψ_0, being constant.

When all the parameter functions are reduced to constants, formula (1) reduces to the pattern for the class of well known (diagonal) affine mappings. If all ψ_0, $\psi_1(\ )$,…, $\psi_{n-1}(\ )$ reduce to zero we call the transformations *pseudolinear* and if all ϕ_0, $\phi_1(\ )$,…, $\phi_{n-1}(\ )$ reduce to the constant function equal 1 we call them *pseudotranslations*. Notice only at this point that, for any $n = 2, 3,\ldots$ all the triangular transformations form the *algebraic group* with respect to superposition and all the *pseudoaffine*, *pseudolinear* and *pseudotranslations* form subgroups of the *triangular group* with respect to the same group operation with all the groups having as the unity the identity transformation (Filus & Filus, 1999). It is remarkable that all the transformations, given by pattern (1), are so easily (row by row) reversible. The transformation inverse to (1) is given as follows:

$$T_1 = (X_1 - \psi_0) / \phi_0$$

$$T_2 = (X_2 - \psi_1(X_1)) / \phi_1(X_1) \qquad (1^*)$$

$$\ldots$$

$$T_n = (X_n - \psi_{n-1}(X_1,\ldots,X_{n-1})) / \phi_{n-1}(X_1,\ldots,X_{n-1}).$$

Also, the *Jacobi matrix* of any transformation (1) or its inverse (1*) turns out to be *triangular,* so that the jacobian, say J, of the inverse (1*) takes on remarkably simple form as the following arithmetic product:

$$J = \prod_{k=1}^{n} \partial t_k / \partial x_k =$$

$$= [\phi_0]^{-1} \cdot [\phi_1(x_1)]^{-1} \ldots [\phi_{n-1}(x_1,\ldots,x_{n-1})]^{-1}.$$

Therefore, the joint pdf of any output random vector $(X_1,\ldots,X_n)$ can easily be obtained using the ordinary standard procedures for the calculations.
Our interest lies in construction and investigation of the associated stochastic processes.
But, because the finite dimension cases form an integral part of the being developed theory of the stochastic processes, and because these results are not widely enough known within the statistician communities, as well as, since most of the related publications in literature are not always at an immediate easy reach, we have decided to sketch here the most basic findings on the finite dimension stochastic models.

2 A. At least the following four new classes of n-variate pdfs can immediately be obtained by use of transformations (1).

Although this is not necessary in more general settings, for the purpose of reliability application we assume that the input random variables $T_1,\ldots,T_n$, present in (1), are independent, and that all of them belong to one of the four pdf classes i.e., either the Gaussian, or the gamma, or the Weibullian, or, in particular, to the class of exponentials. Considering other classes such as, for example, the Pareto or even some discrete distributions is possible as well. In each of the four cases (whenever the pdfs $f_1(t_1),\ldots,f_n(t_n)$ of the independent random variables $T_1,\ldots,T_n$, are given), the standard procedure yields the joint pdf $g(x_1,\ldots,x_n)$ of the output random vector $(X_1,\ldots,X_n)$ that is given by simple arithmetic product form:

$$g(x_1,\ldots,x_n) =$$

$$= g_1(x_1),\ g_2(x_2 \mid x_1),\ldots,g_n(x_n \mid x_1,\ldots,x_{n-1}), \qquad (2)$$

where $g_1(x_1)$ is the (*initial*) marginal pdf of X_1 which, in most of important practical cases, simply satisfies the condition

$$g_1(x_1) = f_1(x_1),$$

while the remaining $n - 1$ pdfs are the conditional pdfs

$$g_j(x_j \mid x_1,\ldots,x_{j-1}), \text{ of each } X_j,$$

given that

$$X_1 = x_1,\ldots, X_{j-1} = x_{j-1} \text{ as } j = 2, 3,\ldots, n.$$

For all the four classes of the pdfs as specified above, it turns out that whenever pattern (1) is applied, all the n factors, i.e., the n pdfs in the product (2), preserve the original class of the pdfs to which all the ***initial*** pdfs $f_1(t_1),\ldots, f_n(t_n)$ belong.
According to a particular choice of class of the input random vectors $(T_1,\ldots,T_n)$ applied in (1), the so obtained joint pdfs $g(x_1,\ldots,x_n)$ of the corresponding output random vectors $(X_1,\ldots,X_n)$ we classify as: the FF-normal, also called ***pseudonormal***, see (Kotz et al, 2000) or FF-gamma (***pseudogamma***) or FF-Weibullian (*pseudoWeibullian*) or FF-exponential (*pseudoexponential*), respectively.
It is an interesting theoretical fact that each of the above defined four classes of the distributions is ***invariant*** with respect to all the transformations (1).

2 B. Consider first the class (group) of the n-variate FF-normal distributions as wide extension of the n-variate normals. Although the normal distributions can hardly be applied in reliability as models for life-times they can be applied with a pretty good accuracy as models for (random) system strength. That is why we consider them in below.

When constructing the FF-normal by use of transformation (1), see *pseudonormal* (an *old version*) in (Kotz et al, 2000, pages 217–218**)** we assume that the independent input random variables $T_1,\ldots,T_n$ have each the normal $N(\mu_j,\sigma_j)$ distribution ($j = 1,2,\ldots,n$).
In this case the FF-normal pdf $g(x_1,\ldots,x_n)$ is given by the product (2) with each j-th factor being a normal pdf, with respect to the argument x_j.
The factor $g_1(x_1)$ is an ordinary normal $N(\mu_1,\sigma_1)$ density. In most of potential applications it is assumed to be identical to $f_1(t_1)$ of T_1 (where $t_1 = x_1$).
For each $j = 2,\ldots,n$ the corresponding j-th conditional normal (!) pdf in (2) is given by the formula:

$$g_j(x_j \mid x_1,\ldots,x_{j-1})$$

$$= (\sigma_j \mid \varphi_{j-1}(x_1,\ldots,x_{j-1}) \mid \sqrt{2\pi})^{-1}$$

$$\exp[-(x_j - \mu_j - \psi_{j-1}(x_1,\ldots,x_{j-1}))^2 /$$

$$2\sigma_j^2(\varphi_{j-1}(x_1,\ldots,x_{j-1}))^2]. \qquad (3)$$

The foregoing pdfs are exactly the normals $N(\mu_j + \psi_{j-1}(x_1,\ldots,x_{j-1});\ \sigma_j\,|\varphi_{j-1}(x_1,\ldots,x_{j-1})|)$ with respect to the x_j variable, while $\varphi_{j-1}(x_1,\ldots,x_{j-1})$, and $\psi_{j-1}(x_1,\ldots,x_{j-1})$ are the parameter functions

that the pdfs $g_j(x_j \mid x_1,...,x_{j-1})$ inherit from the given defining *pseudoaffine* transformation (1).
The constants μ_j, σ_j are inherited from the (assumed) normal $N(\mu_j;\sigma_j)$ pdf $f_j(t_j)$ of the T_j, as $j = 1, 2,\ldots,n$.
Notice, that, unlike in the case of the *n*-variate normal pdfs, the corresponding regression functions:

$$E[X_j \mid X_1 = x_1,\ldots,X_{j-1} = x_{j-1}] =$$

$$= \mu_j + \psi_{j-1}(x_1,...,x_{j-1}),$$

in general are not linear in $x_1,...,x_{j-1}$.
The above regressions may be chosen to be any continuous functions in their arguments. In particular, adding a quadratic form in variables $x_1,...,x_{j-1}$, to the usual linear form as present in the normal case, seems to be the right choice for initial investigations. In addition to that, the conditional standard deviations

$$[\mathrm{Var}(X_j \mid x_1,...,x_{j-1})]^{½} = \sigma_j \mid \varphi_{j-1}(x_1,...,x_{j-1}) \mid$$

may also be an arbitrary continuous function in the values $x_1,...,x_{j-1}$, contrary to the (classical) normal cases where these must remain a constant.
These facts suggest that, possibly, a number of various (unknown) stochastic dependencies, that may be described in the FF-normal models scope, cannot be expressed in the classical normal models framework.
An expected (by the authors) benefit of this fact in stochastic modelling, (that follows the possibility of replacing the usual linear regression functions by a much more general continuous one) may be a significant gain in precision of the stochastic description of real life phenomena including those outside the scope of reliability problems. The improvement in modelling precision is the main practical motive for the FF-normal pdfs construction.
As for the three other classes of the constructed multivariate pdfs, as directly applied to reliability theory, notice that all constructed pdfs are given in the form of product (2), where all the *n* factors belong to exactly one of the four, mentioned above, pdf classes.
Now, in order to provide an illustration of a similar (to the FF-normal) bivariate FF-exponential pdf structure, consider the following scheme of *pseudoaffine* (more particularly, *pseudolinear*) transformations of the random vectors (T_1,T_2) à (X_1,X_2):

$$X_1 = a\, T_1$$

$$X_2 = \phi(X_1)\, T_2,$$

where T_1, T_2 are assumed to be independent random variables, and for $k = 1, 2$, T_k has a following one parameter exponential pdf

$$f_k(t_k) = (1/\theta_k) \exp[-\, t_k/\theta_k].$$

Moreover, the symbol *a* in the latter transformation denotes a positive real number, and $\phi(x_1)$ is only assumed to be a positive continuous real function.
Also, (only for the reliability purposes) we should assume that $\phi(0) = 1$.
Simple calculations yield the following general formula for the scheme for bivariate FF-exponential pdfs of the random vectors (X_1, X_2):

$$g(x_1, x_2) = g_1(x_1)\, g_2(x_2 \mid x_1) =$$

$$(a\theta_1)^{-1}\exp[-\, x_1/(a\theta_1)](\theta_2\phi(x_1))^{-1}\exp[-\, x_2/(\theta_2\phi(x_1))].$$

Specifically, we consider a subclass determined by class of the parameter functions

$$\phi(x_1) = 1 + Ax_1^r,$$

(with *A*, and *r*, being positive real constants) as a particularly interesting case.
Other interesting class is determined by the choice: $\phi(x_1) = \cosh(cx_1)$, where *c* is a nonzero real number. However for the considered in this paper model of reliability of system with repair more proper parameter function would be a function $\phi(x_1)$ such that $\mid \phi(x_1) \mid \leq 1$ and with $\phi(x_1)$ being a decreasing function in x_1 (for example one may choose $\phi(x_1) = \exp[-\, ax_1]$ or $\phi(x_1) = 1/(1 + ax^{\alpha})$ with parameters $a > 0$, $\alpha > 0$ to be statistically estimated.
As it turns out, many analytical calculations as applied in order to obtain the pdf's parameters, such as (all) moments or correlations, are easy (Filus & Filus 2001).

3. Extension of random vectors to stochastic processes

Letting n à ∞ in the *pseudoaffine* formula (1) one obtains the following extension of the pattern of the *pseudoaffine* transformations that will be called an infinite *pseudoaffines*:

$$X_1 = \phi_0 T_1 + \psi_0 \qquad (1^{**})$$

…

$$X_j = \phi_{j-1}(X_1,\dots,X_{j-1})T_j + \psi_{j-1}(X_1,\dots,X_{j-1})$$

…,

where $j = 2, 3,\dots$

The time j, present in the above formula, is assumed to take on all positive integer values, while all the transformations satisfying (1**) are thought of as R^∞ **à** R^∞ transformations.

R^∞ here is understood as the class of all sequencies of real numbers, possibly endowed with the Frechet metrics.

The basic pattern of construction given ahead, mainly relies on transforming through (1**) some stochastic processes $\{T_1,T_2,\dots\}$, that usually are chosen to belong to certain important, well known, classes of the processes (for example, normal).

As a result, in each such case, a relatively *new class* of the corresponding stochastic processes $\{X_1,X_2,\dots\}$ with the discrete time j is determined. Similar constructions of stochastic processes with a continuous time can be found in (Filus & Filus, 2008).

(For the validity of that construction pattern realize that every so obtained stochastic processes $\{X_1,X_2,\dots\}$ can be equivalent to the (unique) sequence $\{h^{(n)}(x_1,\dots,x_n)\}_{n=2,3,\dots}$ of the well defined joint pdfs of the random vectors $(X_1,\dots,X_n)$, each being the output of the first n rows in (1). In such a case, the consistency of the underlying pdfs is obvious).

The various analytical properties, of the processes, often turn out to be very interesting, and seem to promise to be useful in applications including reliability applications but not only.

In some cases, in particular if the distributions of all the input random variables $T_1,T_2,\dots$ are Weibullian, the classes of the obtained processes sometimes can be fruitfully extended even more when the infinite *pseudoaffine* scheme (1**) is replaced by the following, more general, infinite *pseudopower* (R^∞ **à** R^∞) transformations scheme:

$$X_1 = \phi_0 T_1^{\alpha(0)} + \psi_0 \qquad (4)$$

…

$$X_j = \phi_{j-1}(X_1,\dots,X_{j-1})T_j^{\,\alpha(j-1)\,(X_1,\dots,X_{j-1})} +$$

$$+\ \psi_{j-1}(X_1,\dots,X_{j-1})$$

where, for $j = 2,3,\dots$ and $n = \infty$, the symbols $\alpha(j-1)\,(X_1,\dots,X_{j-1})$, being the exponents at the arguments T_j, will be called ***exponent parameter functions*** with the random variables $X_1,\dots,X_{j-1}$ as their arguments, while the exponential parameter function $\alpha(0)$ is considered to be a nonzero real constant.

Realize that if for all $j = 1,2,\dots$ the conditions

$$\alpha(0) = \alpha(1)\,(X_1) = \dots = \alpha(j-1)\,(X_1,\dots,X_{j-1}) = 1$$

are satisfied then scheme (4) reduces to the *pseudoaffine* scheme (1). This fact indicates significant gain of generality once we replace the set of transformations (1) by those defined by (4). Realize that even if we assume that the coefficients $\alpha(j-1)\,(X_1,\dots,X_{j-1})$ are only real non-zero constants the gain of generality still remains significant enough.

(As it readily can be verified, *pseudopower* transformations (4) are triangular and form the subgroup of all the triangulars. Moreover, the *pseudoaffine* transformations' group is the subgroup of the group of *pseudopowers*. However, either FF-normals or FF-gamma (including FF-exponentials) are *not anymore invariant* under the group of (4) while the invariance under all the (4) still takes place for the FF-Weibullians i.e., the image process $\{X_j\}$ of the FF-Weibullian input process $\{T_j\}$ under any transformation (4) remains FF-Weibullian).

The most striking fact, associated with the construction of this kind of models, is simplicity.

This, allows to preserve an analytical tractability for the most of important cases.

First realize that in general the stochastic processes obtained by the transformation method as well as by the *method of parameter dependence* have *good memory* in the sense that for every j the conditional probability distribution (or density) of the term X_j, given all the past: $X_1 = x_1,\dots,X_{j-1} = x_{j-1}$ is explicitly given in an analytical form. To illustrate this let us give the following simple example.

Example

Set in the transformation (1**) all the translations

$\psi_{j-1}(X_1,\dots,X_{j-1})$ to zero obtaining, in such a way, ***pseudolinear*** version of (1**). As an input stochastic process $\{T_j\}$ take sequence $T_1, T_2,\dots$ of independent random variables each having the same exponential density:

$$f_k(t_k) = (1/\theta)\exp[-t_k/\theta],\ k = 1, 2,\dots$$

Using standard calculations for each $j = 2, 3,\dots$ one obtains the conditional density of X_j, given all the past $X_1 = x_1,\dots, X_{j-1} = x_{j-1}$ in the following simple analytical form

$$g_j(x_j \mid x_1,\dots,x_{j-1}) =$$

$$= (1/\theta \mid \phi_{j-1}(x_1,\dots,x_{j-1})|)\exp[-x_j/\theta \mid \phi_{j-1}(x_1,\dots,x_{j-1})|] \qquad (5)$$

theoretically, no matter how big is the time j!
The complexity of (5) only depends on a complexity of a chosen *parameter function*

$$\phi_{j-1}(x_1,\dots,x_{j-1}).$$

For an example of such a choice may serve the following simple function:

$$\phi_{j-1}(x_1,\dots,x_{j-1}) = 1 + A_1x_1^r + A_2x_2^r + \dots +$$

$$+ A_{j-1}x_{j-1}^r \qquad (5^*)$$

where A_k, $k = 1,\dots,j-1$ are nonnegative real parameters and r is a nonzero real (in particular $r = 1$ or $r = 2$).
The latter choice for $\phi_{j-1}(x_1,\dots,x_{j-1})$ provides relatively easy analytical computations.
However, this model is not realistic when modelling reliability of systems with repair.
The reason for this is that if to apply the parameter function (5*) in model (5) then as time j grows the conditional expectation $E[Xj \mid x_1,\dots,x_{j-1}]$ considered as the function of j grows while, in general, the reliability of the system after each repair decrease (in the sense of that conditional expectation). To avoid this discrepancy one should, for example, replace option (5*) in model (5) for more suitable following option:

$$\phi_{j-1}(x_1,\dots,x_{j-1}) =$$

$$= \exp[-(A_1x_1^r + A_2x_2^r + \dots + A_{j-1}x_{j-1}^r)] \qquad (5^{**})$$

or we may use the model:

$$\phi_{j-1}(x_1,\dots,x_{j-1}) =$$

$$= (1 + A_1x_1^r + A_2x_2^r + \dots + A_{j-1}x_{j-1}^r)^{-1}. \qquad (5^{***})$$

In the foregoing case the stochastic dependence of time X_j from the past, say $X_1,\dots,X_{j-1}$ remains strong (as $j \to \infty$) and in general for the conditional expected values we have

$$E[X_j \mid x_1,\dots,x_{j-1}] \to 0 \ \text{as}\ j \to \infty.$$

The last relation is very realistic if one models times of functioning of the systems after, say ($j - 1$)-th repair ($j = 2,3,\dots$) when system reliability after each repair decreases.
In this case one may consider end of the system exploitation after first r repairs ($r = 1, 2,\dots$) such that $E[X_r \mid x_1,\dots,x_{r-1}] < t_0$, where $t_0 > 0$ is considered a *minimal reasonable time of the system work*. After that r-th cycle the system is assumed to be replaced by a new one. Notice that, in this framework, r is discrete random variable and its expectation is a real, say $E[r] = r^*$. The value r^* may be applied as a measure of efficiency of the system maintenance process as well as (when the maintenance base offers service to a large number, say N, of the considered systems) can be used to determine the necessary resources needed for running a proper maintenance for the systems (cars, by example). Between others, the quantity r^* can be used for estimation of expected amounts of spare parts so important for the efficiency of all the repairs.
Many other models like (5*), (5**) and (5***) may be employed in the considered setting. This subject is to be more developed in future.

4. The *k*-Markovianity

As one can see, the memory of the stochastic process defined by (1) or (4) theoretically can be arbitrarily long i.e., may contain the whole process's history. However, as j grows, this becomes impractical because the model, in its complexity, grows to infinite expression and, consequently, the number of parameters to be estimated grows without limitation. Such a model can only be applied to the system with repair when the time between the repairs quickly decreases so that number of the repairs till the (last) r-th repair is relatively small.

In general, however, one needs to limit the memory of these stochastic processes to reduce their complexity and number of parameters. For this goal we may, between others, reduce the *good* (unlimited) memory by applying the notion of k-Markovianity ($k = 0,1,2,\ldots$) with the regular Markovianity when $k = 1$ and independence when $k = 0$.

For that purpose we first introduce the notion of *k-Markovian triangular* (here only *pseudoaffine* or *pseudopower) transformations* that will correspond to the notion of the k-Markovian stochastic processes.

Definition

Let $n = 3,4,\ldots$. Consider any *pseudoaffine* transformation, subject to the scheme (1), all whose parameter functions satisfy:

$$\phi_{j-1}(x_1,\ldots,x_{j-1}) = \phi_{j-1}(x_{j-k}, x_{j-k+1},\ldots, x_{j-2}, x_{j-1}),$$

$$\psi_{j-1}(x_1, \ldots,x_{j-1}) =$$

$$= \psi_{j-1}(x_{j-k}, x_{j-k+1},\ldots,x_{j-2}, x_{j-1}), \quad (6)$$

while, additionally, for the *pseudopower* transformations (4) we also have:

$$\alpha(j-1)\,(X_1,\ldots,X_{j-1}) =$$

$$\alpha(j-1)\,(x_{j-k}, x_{j-k+1},\ldots, x_{j-2}, x_{j-1}) \quad (6^*)$$

(i.e., for each $j = 2,\ldots,n$, and a fixed $k = 0,1,2,\ldots, j-1$, the above functions do not depend on the values $x_1,\ldots,x_{j-k-1}$). A class of such transformations will be called (finite or infinite, according to the cases $n < \infty$, and $n = \infty$, respectively) class of *k-th degree Markovian pseudoaffine* transformations or, shorter *k-Markovian transformations*. Consequently, a *pseudoaffine* transformation is *Markovian* if it is k-Markovian, with $k =1$.

As for $k = 0$ the parameter functions (6) reduce to constants and the resulting *zero degree pseudoaffine transformations* are the regular affine.

The stochastic process $\{X_j\}$ is a k-Markovian ($k = 1, 2,\ldots$) if for any integer $j \geq k + 1$, the conditional pdf $g_j\,(x_j \mid \ldots)$ of the random variable X_j, given a past $X_1,\ldots,X_{j-1}$, ($j = 2,3,\ldots$), satisfies the condition:

$$g_j\,(x_j \mid x_1,\ldots,x_{j-1}) =$$

$$= g_j\,(x_j \mid x_{j-k}, x_{j-k+1},\ldots,x_{j-2}, x_{j-1}) \quad (7)$$

If all the random variables $X_1, X_2,\ldots$ are mutually independent then we will consider the process to be *zero-Markovian* or *of Markovianity zero* i.e., $k = 0$.

Proposition

1. For each $k = 0,1,2,\ldots$, any k-Markovian stochastic process can be obtained using a 0-Markovian process $\{T_j\}$ ($T_1, T_2,\ldots$ are independent) by applying to it some k-Markovian *pseudoaffine* or *pseudopower* transformation.
2. If the k-Markovian transformation is applied to an r-Markovian stochastic process then the resulting stochastic process will be $(k + r)$-Markovian.

The proof requires only simple standard calculations.

One obtains the following Corollary.

Corollary

For any fixed value of k, the property of the k-Markovianity of a stochastic processes is invariant under any (ordinary) affine (so, in particular linear) transformation.

For the above statement to hold it is enough to realize that the regular affine transformations are *pseudoaffine* of the Markovianity zero.

Remark: Introducing the k-Markovian processes as the stochastic models has the following advantages.

1. By incorporating more (than in a regular Markovian model) information on the past, that significantly influence the *present* probability distribution of, say, quantity X_j, one obtains a more accurate description of modeled realities in the sense of better fit to data. Also, future predictions on probability distributions of random variables, say, $X_{j+1}, X_{j+2},\ldots$ made *at a time instant j*, are expected to be more precise than those based on ordinary Markovian models.
2. The usually met increment in computational complexity, associated with attempts such as the above, in this case is not that dramatic. As it turns out, most of the important calculations

associated with the k-Markovian (for $k \geq 2$) stochastic processes, in general can be handled in an analytical way.

3. Possibilities of direct application of the k-Markovian stochastic processes as models, for reliability of systems with repair seems to be evident. One may consider the stochastically *dependent* times $X_1, X_2,\ldots$ between failures (each followed by a repair) as a k-Markovian stochastic process, for some $k = 1, 2,\ldots$ to be chosen.

5. Parameter dependence method

All the conditional and multivariate probability distributions, so far considered, we can obtain by the method of some triangular transformations. However, they may also be obtained by simpler method of the *parameter dependence* (Filus & Filus, 2013). Actually, it is required only to obtain the sequence of the conditional distributions (here the densities) $g_j(x_j \mid x_1,\ldots, x_{j-1})$ of the random variables X_j, ($j = 2, 3,\ldots$) given realizations $x_1,\ldots,x_{j-1}$ of ***earlier*** random variables $X_1,\ldots,X_{j-1}$.

Using the method of parameter dependence, it is obtained from the densities $f_j(t_j, \theta_j)$ of the random variable T_j which are independent from the past $T_1,\ldots,T_{j-1}$, and θ_j is a scalar or vector parameter of the density $f_j(t_j, \theta_j)$ of the T_j.

Realize that when using transformation method for the examples, given above, in one of the cases the normal density $N(\mu_j, \sigma_j)$ of T_j were turned into the conditional density

$$g_j(x_j \mid x_1,\ldots,x_{j-1}) =$$

$$= N(\mu_j + \psi_{j-1}(x_1,\ldots,x_{j-1}); \sigma_j \mid \varphi_{-1}(x_1,\ldots,x_{j-1})|) \text{ of } X_j.$$

But this can be done differently. Let $\theta_j = (\mu_j, \sigma_j)$ be the vector parameter of the normal density of T_j. All what has happened was transformation of the original parameters:

$$\mu_j \text{ à } m_j = \mu_j + \psi_{j-1}(x_1,\ldots,x_{j-1})$$

and

$$\sigma_j \text{ à } s_j = \sigma_j\, |\varphi_{j-1}(x_1,\ldots,x_{j-1})|,$$

so that the values of ***new parameters*** m_j and s_j became continuous (only) functions of the past values $x_1,\ldots,x_{j-1}$.

This also happens in other cases. For example, when using the ***pseudolinear*** transformation and when T_j has the exponential density $f_j(t_j, \theta_j)$ then to obtain the conditional density $g_j(x_j \mid x_1,\ldots,x_{j-1})$ from $f_j(t_j, \theta_j)$ it is enough to change the parameter θ_j to its new value $\theta_j \mid \varphi_{j-1}(x_1,\ldots,x_{j-1})|$.

In the general case the following assignments for random variables:

$$T_j^w \text{ à } X_j$$

we call the *weak transformation*.

This transformation is equivalent to the following (***strong*** or ***deterministic***) transformation of the densities:

$$f_j(t_j, \theta_j) \text{ à } g_j(x_j \mid x_1,\ldots,x_{j-1}),$$

which is defined through the parameter's assignment:

$$\theta_j \text{ à } \theta_j(x_1,\ldots,x_{j-1}),$$

where the parameter θ_j on the left-hand side is the original constant parameter of the density $f_j(t_j, \theta_j)$ of T_j.

The final formula which defines the conditional density is:

$$g_j(x_j \mid x_1,\ldots,x_{j-1}) = f_j(t_j, \theta_j(x_1,\ldots,x_{j-1})),$$

where the continuous function $\theta_j(x_1,\ldots,x_{j-1})$, belongs to a family of parameter functions and is estimated and then verified by statistical methods. This is to be recommended to choose $\theta_j(x_1,\ldots,x_{j-1})$ from some parametrized family of continuous function that has its own parameters so that $\theta_j(x_1,\ldots,x_{j-1}) = \theta_j(x_1,\ldots,x_{j-1}; A, B, C,\ldots)$, where $A, B, C,\ldots$ are some numerical parameters to be estimated and then verified. Thus, the eventual statistical methods to be employed are mostly parametric. The choice among the parametric families (for example, polynomial functions, exponential, logarithmic etc.), as governed by best fit to a given data, also should be made by use of statistical methods. It looks like a lot of the statistical work can be involved. This is, however, left for future or, eventually, for other researchers.

6. Conclusions

1. Extension (together with the possibility of new constructions) of Markov stochastic process

paradigm to the case of long memory processes not only increase the flexibility of the models but, independently of that, the extension allows to use additional information incorporated to the new model for estimating *tendencies* of (times between repairs in our case) the behaviour of the process realizations. It may be important not only to know the recent process's behaviour but also how that behaviour evaluates in time. For example, having model (5) together with its specification (5***) we better may predict next time-length of the operation (or next few cycles) if we know the tendency according to which the consecutive times of system operating decrease.

2. The complexity of the underlying expressions is the price for the new advantages. However, the increment of that complexity is not so dramatic.
3. More concerns may follow the fact that number of parameters to be estimated by statistical methods may increase significantly. Reducing number of the parameters by incorporating the idea of *k*-Markovianity, especially if *k* is small, may help the problem but then information on the tendency of process behaviour may be limited. This, in turn, makes eventual future process's predictions less accurate. Finding, in such a case, an optimal decision on the size of the number *k* may create optimality problems which, if applied, enrich the underlying theory. That kind of eventual solutions will strongly depend on sizes of statistical samples available.
4. The purpose of extending Markovian processes toward the processes with long memory is to enhance an accuracy of the models. From that point of view, however, a serious obstacle for the intended accuracy improvement is an increment of several parameters, each of them resulting with an additional error associated with its statistical estimation. Also, the complexity of an underlying analytical expressions grows. The problem met at this point requires solutions as to choose the proper model in balancing between increment of the new model accuracy and arising difficulties above mentioned. The decision on the model choice (such as a choice of a proper parameter functions) must be done separately in each particular case or a class of cases. Again, many depends on amount of statistical data available that can be applied (to predictions, for example).
5. Two different methods of the stochastic processes construction were presented. The method of parameter dependence is, however, simpler and the class of the stochastic processes as obtained by this method is wider. Also, every process obtained by applying a transformation can be obtained by the parameter dependence. However, the big advantage of triangular transformations is the possibility of easy sampling from the underlying random vectors $(X_1,\ldots,X_n)$ ($n = 2,3,\ldots$) which for large n approximate the processes. For that this is enough to transform any sample from random vectors $(T_1,\ldots,T_n)$ of independent variables having some typical distribution. Such samples are, in general, at hand. This may open ways for simulation procedures and modelling.
6. As mentioned in the chapter, triangular transformations form algebraic groups with unity, with respect to their composition. This is worth to notice a theoretical fact that, for each $n = 2,3,\ldots$ the group operation (here the composition) induces a binary operation in sets of probability densities, say $g(x_1,\ldots,x_n)$ of the output random vectors $(X_1,\ldots,X_n)$. Recall, sequences of these densities determine the constructed processes. Nice fact about it is that the groups of the densities $g(x_1,\ldots,x_n)$ are *isomorphic* with corresponding groups of the transformations. More on the algebraic structure of the, here defined, n-dimensional FF-normal densities (for all $n \geq 2$) as well as more detailed theory of the finite dimensional pseudonormals) can be found in (Filus & Filus, 1999, 2001). It seems to be obvious that the above mentioned algebraic structure of the finite dimensional triangular transformations and the corresponding FF-normal densities, described in (Filus & Filus 1999), can (easily) be extended to classes of infinitely dimensional R^{∞} à R^{∞} transformations and the corresponding classes of stochastic processes with good memory. The propositions formulated in (Filus & Filus 1999) can be extended to the infinite dimensions as well. This problem, interesting on its own grounds, is of rather theoretical nature with no direct application to the here considered reliability and maintenance settings.

References

Filus, J. & Filus, L. 1999. A class of generalized multivariate normal densities (revisited), *Report,* No. 99–1109, Northeastern Illinois University, Chicago.

Filus, J. & Filus, L. 2001. On some bivariate pseudonormal densities. *Pakistan Journal of Statistics* 17(1), 1–19.

Filus, J. & Filus, L. 2008. Construction of new continuous stochastic processes. *Pakistan Journal of Statistics* 24(3), 227–251.

Filus, J., Filus, L. & Arnold, B. 2010. Families of multivariate distributions involving triangular transformations. *Communications in Statistics – Theory and Methods* 39(1), 107–116.

Filus, J. & Filus, L. 2013. A method for multivariate probability distributions construction via parameter dependence. *Communications in Statistics: Theory and Methods* 42(4), 716–721.

Kotz, S., Balakrishnan, N. & Johnson, N. 2000. *Continuous Multivariate Distributions*, *Volume 1: Models and Applications*, A Wiley-Interscience Publication, John Wiley & Sons, INC, Second Edition, New York – Chichester – Weinheim – Brisbane – Singapore – Toronto.