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JAN STOCHEL, A STELLAR MATHEMATICIAN

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Abstract. The occasion for this survey article was the 70th birthday of Jan Stochel, professor at Jagiellonian University, former head of the Chair of Functional Analysis and a prominent member of the Kraków school of operator theory. In the course of his mathematical career, he has dealt, among other things, with various aspects of functional analysis, single and multivariable operator theory, the theory of moments, the theory of orthogonal polynomials, the theory of reproducing kernel Hilbert spaces, and mathematical aspects of quantum mechanics.

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1. INTRODUCTION

Professor Jan Stochel was born on December 22, 1951 in Olkusz, a town in Poland near Kraków. As a child, he and his family moved to Kraków, where he successively completed elementary school in 1965 and then graduated from the Metallurgical and Mechanical Technical School in Kraków Nowa Huta in 1970. He inherited his scientific abilities from his father, with whom he often studied science as a child. In 1970–1973, he studied physics at the Jagiellonian University, and in 1971–1976, he studied mathematics at the same university, where he finished with a master's thesis entitled "Similarity to contractions in Hilbert spaces" under the supervision of Wacław Szymanski. In 1979, he received a PhD degree in mathematics from the Institute of Mathematics of the Polish Academy of Sciences on the basis of a dissertation entitled "Dilations of infinite families of semispectral measures", written under the supervision of Professor Włodzimierz Mlak. In 1992, he received his habilitation degree in mathematics, from the Department of Mathematics and Physics at Jagiellonian University, on the basis of a habilitation dissertation entitled "Positive definite functions on $*$-semigroups and their connections with the theory of seminormal operators". He was awarded the title of professor in 2001 while in 2006 he received the position of full professor at the Institute of Mathematics of the Jagiellonian University. Jan Stochel has mentored 11 PhDs in mathematics, most of whom work at universities. He is the author of more than 120 scientific articles, which are widely cited by many mathematicians. Together with Franciszek Hugon Szafraniec, Jan Stochel (see Figure 1) is the founder of the theory of unbounded subnormal operators. We will discuss his contribution to this topic in more detail in the next section.

Fig. 1. Professor Jan Stochel

At the end of June 2022 an international conference "Operator Theory and Beyond 2022" was held at the Institute of Mathematics of the Jagiellonian University in Kraków to celebrate the occasion of Professor Stochel's 70th birthday (see Figure 2).

Fig. 2. A photo taken during the conference "Operator Theory and Beyond 2022"

2. TRILOGY ON NORMAL EXTENSIONS OF UNBOUNDED OPERATORS

The theory of subnormal operators was initiated by Paul Halmos in 1950 (see [24, 30]). Motivated partly by some operators in Quantum Mechanics, the unbounded counterpart of subnormal operators has been studied extensively by Jan Stochel and F.H. Szafraniec in a series of papers (see, for example, [57–60, 63]).

In this section, we briefly discuss the trilogy on normal extensions of unbounded operators focussing on analytic and functional models of subnormal operators. We begin by recalling the notion of an unbounded subnormal operator.

Definition 2.1. A densely defined linear operator S in $\mathcal{H}$ with domain $\mathcal{D}(S)$ is said to be *subnormal* if there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a densely defined normal operator N in $\mathcal{K}$ with domain $\mathcal{D}(N)$ such that

$$\mathcal{D}(S) \subseteq \mathcal{D}(N) \quad \text{and} \quad Sf = Nf, \ f \in \mathcal{D}(S).$$

Recall that a densely defined linear operator S in $\mathcal{H}$ with domain $\mathcal{D}(S)$ is said to be *cyclic* if there is a vector $f_0 \in \mathcal{D}^\infty(S) = \bigcap_{n=0}^\infty \mathcal{D}(S^n)$ (referred to as a *cyclic vector of* S) such that $\mathcal{D}(S)$ is the linear span $\operatorname{span}\{S^n f_0 : n \geq 0\}$ of the set $\{S^n f_0 : n \geq 0\}$. Suppose S is a cyclic operator in $\mathcal{H}$ with f_0 being a cyclic vector of S. It turns out that $\lambda \in \sigma_p(S^*)^*$, where σ_p denotes the point spectrum, if and only if there is a real number $c_\lambda > 0$ such that $|p(\lambda)| \leq c_\lambda \|p(S)f_0\|_{\mathcal{H}}$ for any complex polynomial p (see [59, Lemma 2]). Further, as established in [58, Proposition 3], S is subnormal if and only if there exists a non-negative measure μ on the complex plane $\mathbb{C}$ (referred to as the *representing measure of* S) such that

$$\int_{\mathbb{C}} |z|^{2n} d\mu(z) < \infty, \quad n \geq 0, \tag{2.1}$$

$$\langle S^m f_0, S^n f_0 \rangle_{\mathcal{H}} = \int_{\mathbb{C}} z^m z^{*n} d\mu(z), \quad m, n \geq 0. \tag{2.2}$$

If S is a cyclic subnormal operator in $\mathcal{H}$ so that (2.1) and (2.2) are satisfied, then S is unitarily equivalent to $M_{z,\mu}$ in $\mathcal{H}_\mu$, where $\mathcal{H}_\mu$ is the $L^2(\mu)$-closure of the vector space $\mathbb{C}[z]$ of complex polynomials in z and where $M_{z,\mu}$ is the operator of multiplication by z with domain $\mathbb{C}[z]$ (refer to [59, Theorem 5]); the triple $(M_{z,\mu}, \mathbb{C}[z], \mathcal{H}_\mu)$ will be referred to as a *functional model* of the cyclic subnormal operator S.

Given a Hilbert space $\mathcal{H}$ and a cyclic operator S in $\mathcal{H}$ with a cyclic vector f_0 of S, one can come up with a sequence $r = \{r_n\}_{n\geq 0}$ of complex polynomials such that $e = \{e_n\}_{n \geq 0}$ with $e_n = r_n(S)f_0$ is an orthonormal basis for $\mathcal{H}$ and such that

$$\operatorname{span}\{e_n : n \geq 0\} = \operatorname{span}\{S^n f_0 : n \geq 0\}$$

(refer to [59]). As observed in the proof of [59, Proposition 6], the polynomials r_n so obtained form a Hamel basis for $\mathbb{C}[z]$. Set

$$\omega_r := \left\{ z \in \mathbb{C} : \sum_{n=0}^\infty |r_n(z)|^2 < \infty \right\}.$$

We define K_r on $\omega_r \times \omega_r$ by

$$K_r(z,w) = \sum_{n=0}^{\infty} r_n(z)^* r_n(w), \quad z, w \in \omega_r.$$

Let $K_{r,z}(w)$ stand for $K_r(z,w)$. Since K_r is a positive definite kernel on ω_r, we can associate with K_r a reproducing kernel Hilbert space $\mathcal{H}_r$ as described in [6]. The reproducing kernels $K_r(z,w)$ are assumed to be jointly continuous in the variables z and w. The following theorem of Stochel and Szafraniec leads to the notion of an analytic model.

Theorem 2.2 ([59, Theorem 6]). *Suppose $\mathcal{H}$, S, r, e, ω_r and $\mathcal{H}_r$ are as described in the preceding paragraph. If the point spectrum $\sigma_p(S^*)$ of S^* is non-empty, then the following are true:*

(a) *$\mathcal{P}_r$, the set of restrictions of members of $\mathbb{C}[z]$ to ω_r, is dense in $\mathcal{H}_r$,*
(b) *the operator M_z of multiplication by z defined on $\mathcal{P}_r$ is cyclic with the cyclic vector 1,*
(c) *there is a unique partial isometry $W : \mathcal{H} \to \mathcal{H}_r$ with its initial space being the closure of*

$$\operatorname{span}\left\{\sum_{n=0}^{\infty} r_n(\lambda)^* e_n : \lambda \in \sigma_p(S^*)^*\right\}$$

and its final space being $\mathcal{H}_r$ and such that $WS = M_z W$, and
(d) *$\omega_r = \sigma_p(S^*)^* = \sigma_p(M_z^*)^*$.*

Suppose for a cyclic operator S in $\mathcal{H}$ having non-empty point spectrum $\sigma_p(S^*)$, W in (c) of Theorem 2.2 turns out to be a unitary of $\mathcal{H}$ onto $\mathcal{H}_r$; in this case the triple $(M_z, \mathcal{P}_r, \mathcal{H}_r)$ will be referred to as an *analytic model* of the cyclic operator S. Suppose one defines M_z^{max} in $\mathcal{H}_r$ by

$$(M_z^{max} f)(z) = z f(z), \quad z \in \omega_r, f \in \mathcal{D}(M_z^{max}),$$

where $\mathcal{D}(M_z^{max}) = \{f \in \mathcal{H}_r : zf \in \mathcal{H}_r\}$. It is easy to check that M_z^{max} is closed. Since M_z^{max} extends M_z, it follows that M_z is closable and that $\overline{M}_z \subset M_z^{max}$, where $\overline{M}_z$ denotes the closure of M_z. If the interior of $\sigma_p(M_z^*)$ is not empty then $M_z^{max} = \overline{M}_z$ (see [59, Proposition 11]). This is perhaps the first known example, where maximal and minimal domain coincide in the sense of [29] (see [29, Section 1.2] for more such examples). In case a cyclic subnormal operator S in a Hilbert space $\mathcal{H}$ admits an analytic model, there is a unitary U from $\mathcal{H}_\mu$ onto $\mathcal{H}_r$ that preserves polynomials; in that case, as was shown in Proposition 9 of [59], one has, for every f in $\mathcal{H}_\mu$,

$$(Uf)(\lambda) = f(\lambda) \text{ for } \mu\text{-almost all } \lambda \in \sigma_p(S^*)^* \cap \operatorname{supp}(\mu).$$

The reader is referred to [59, Corollary 14] for a sufficient condition ensuring an analytic model for a cyclic subnormal operator.

The most outstanding example of an unbounded subnormal operator is the Creation Operator $A^\dagger$ of Quantum Mechanics. A standard reference for some basic facts pertaining to $A^\dagger$ is [48]. The domain of $A^\dagger$ is the Schwartz space $\mathcal{S}$ ($\subset \mathcal{H} = L^2(\mathbb{R})$) of rapidly decreasing functions on the real line $\mathbb{R}$ and its action on a member f of $\mathcal{S}$ is given by

$$(A^\dagger f)(x) = \frac{1}{\sqrt{2}}\Big(x - \frac{d}{dx}\Big) f(x), \ x \in \mathbb{R}.$$

As follows from the observations in [58], the closure of $A^\dagger$ equals the closure of

$$A_h^\dagger = A^\dagger|_{\mathrm{span}\{H_n : n=0,1,\cdots\}},$$

where H_n is the n-th Hermite function defined by $H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2}$. It may be recalled that $\{\tilde{H}_n = \frac{1}{\sqrt{2^n n!}} H_n\}_{n\geq 0}$ forms a complete orthonormal set in $L^2(\mathbb{R})$ and that $A^\dagger \tilde{H}_n = \sqrt{n+1}\tilde{H}_{n+1}$. If $\{e_n\}_{n\geq 0}$ is an orthonormal basis for $\mathcal{H}$ and S is a linear operator in $\mathcal{H}$ with domain span$\{e_n : n = 0, 1, \ldots\}$ such that $Se_n = \alpha_n e_{n+1}$ for some positive numbers α_n, then S is called a *weighted shift operator*. Thus $A_h^\dagger$ is an example of a weighted shift operator. It can be seen that a weighted shift operator S and its adjoint S^* satisfy

$$\overline{S} f = \sum_{n=0}^{\infty} \langle f, e_n\rangle \alpha_n e_{n+1}, \ f \in \mathcal{D}(\overline{S}), \quad S^* f = \sum_{n=0}^{\infty} \langle f, e_{n+1}\rangle \alpha_n e_n, \ f \in \mathcal{D}(S^*)$$

(refer to [63, Section 1]). It was proved in [58, Theorem 4] that a weighted shift operator S in $\mathcal{H}$ is subnormal if and only if the sequence $\{\|S^n e_0\|_{\mathcal{H}}^2\}_{n\geq 0}$ is a *Stieltjes moment sequence*, that is, there exists a finite non-negative Borel measure μ such that

$$\|S^n e_0\|_{\mathcal{H}}^2 = \int_0^\infty t^n d\mu(t), \ n \geq 0.$$

The weighted shift operator $S = A_h^\dagger$ (and hence $A^\dagger$) is subnormal since $\{\|S^n \tilde{H}_0\|_{L^2(\mathbb{R})}^2 = n!\}_{n\geq 0}$ is a Stieltjes moment sequence with the associated measure μ given by $d\mu(t) = e^{-t}dt$. For an alternate verification of this fact, see [57, Section 9] (see also [8, Example 1]). We record that the point spectrum of S^* is $\mathbb{C}$ (refer to [59, Section 16]).

A functional model of $A_h^\dagger$ is given by the triple $(M_{z,\mu}, \mathbb{C}[z], \mathcal{H}_\mu)$, where $d\mu(z) = \frac{1}{\pi} e^{-|z|^2} dz$, $z \in \mathbb{C}$. The Hilbert space $\mathcal{H}_\mu$ has $\{z^{(n)} = \frac{z^n}{\sqrt{n!}}\}_{n\geq 0}$ as an orthonormal basis and $\mathcal{H}_\mu$ turns out to be the reproducing kernel Hilbert space associated with the positive definite kernel $K(z, w) = e^{z^* w}$ defined on $\mathbb{C} \times \mathbb{C}$. Thus the triple $(M_{z,\mu}, \mathbb{C}[z], \mathcal{H}_\mu)$ may also be looked upon as an analytic model $(M_z, \mathcal{P}_r = \mathbb{C}[z], \mathcal{H}_r)$ of $A_h^\dagger$. The Hilbert space $\mathcal{H}_\mu = \mathcal{H}_r$ is referred to as the *Segal-Bargmann space* (refer to [10, 51]). Let S be a densely defined cyclic subnormal operator in $\mathcal{H}$ with the cyclic vector f_0. It follows from Proposition 6 of [59] that, for any point $\lambda \in \sigma_p(S^*)^*$, there exists a unique vector h_λ in $\mathcal{H}$ such that $p(\lambda) = \langle p(S) f_0, h_\lambda\rangle_{\mathcal{H}}$ for every complex polynomial p. One can then define a function k_S on $\mathbb{C}$ by setting $k_S(\lambda)$ equal to $\|h_\lambda\|^2$ if $\lambda \in \sigma_p(S^*)^*$, and equal to ∞ otherwise. If $\mathrm{int}(\sigma_p(S^*))$ is non-empty and if one defines

$$\gamma(S) = \big\{\lambda \in \mathbb{C} : k_S \text{ is finite and continuous in a neighborhood of } \lambda\big\},$$

then it follows from [59, Theorem 7] that $\gamma(S)$ is an open subset of $\mathrm{int}(\sigma_p(S^*)^*)$, $\mathrm{int}\,(\sigma_p(S^*)^*) \setminus \gamma(S)$ is a nowhere dense subset of $\mathbb{C}$ and $\sigma(\overline{S}) \setminus \sigma_{ap}(\overline{S}) \subset \gamma(S)$ (recall that subnormal operators are closable (see [58]); see also [59, Footnote 9]). If S is a cyclic subnormal operator in $\mathcal{H}$ with an analytic model, then $\gamma(S) = \sigma(\overline{S}) \setminus \sigma_{ap}(\overline{S})$, where $\gamma(S)$ is as in (2.3) (see [59, Theorem 9]). Moreover, if S is a closed subnormal operator in $\mathcal{H}$ with a normal extension N acting in $\mathcal{K}$ and ω is a connected component of $\sigma(N)^c$, then either $\omega \cap \sigma(S) = \emptyset$ or $\omega \subset \sigma(S)$ (see [59, Theorem 2]).

If a linear operator S in $\mathcal{H}$ is subnormal then its normal extension N in $\mathcal{K}$ is said to be *minimal of spectral type* if the only closed subspace of $\mathcal{K}$ reducing N and containing $\mathcal{H}$ is $\mathcal{K}$ itself. Given a normal extension N of S, one can always guarantee a minimal normal extension of spectral type (refer to [60, Section 2]): Let S in $\mathcal{H}$ be subnormal with a normal extension N in $\mathcal{K}$. Let

$$\mathcal{H}_s[N] := \bigvee \{E(\sigma)f : f \in \mathcal{H}, \sigma \text{ is a Borel subset of } \mathbb{C}\},$$

where $E(\cdot)$ denotes th spectral measure of N. Since $\mathcal{H}_s[N]$ reduces N, one can define a linear operator N_s in $\mathcal{H}_s[N]$ by $N_s f = Nf$ for every $f \in \mathcal{D}(N_s) = \mathcal{D}(N) \cap \mathcal{H}_s[N]$. It then follows from [60, Proposition 1] that N_s is a minimal normal extension of S of spectral type. An essential feature of this kind of minimal normal extension is the following variant of the spectral inclusion theorem.

Theorem 2.3 ([60, Corollary 2])**.** *Let N be a minimal normal extension of spectral type of S. Then*

$$\partial\sigma(S) \subseteq \sigma_{ap}(S) \subseteq \sigma_{ap}(N) = \sigma(N) \subseteq \sigma(S).$$

This spectral inclusion theorem plays an important role in obtaining polynomial approximation results in the functional model spaces (see [18, Corollary 2.5 and Example 2.6]).

3. COMPLEX MOMENT PROBLEM

Inverse problems naturally occur in many branches of science and mathematics. An inverse problem entails finding the values of one or more parameters using the values obtained from observed data. Moment problems are a special class of inverse problems, and they arise naturally in statistics, spectral analysis, geophysics, image recognition, and economics. While the classical theory of moments dates back to the beginning of the 20th century, the systematic study of truncated moment problems began only in the waning years of the 20th century. In his 1987 seminal paper [41], Landau wrote, “The moment problem is a classical question in analysis, remarkable not only for its own elegance, but also for the extraordinary range of subjects, theoretical and applied, which it has illuminated.”

Jan Stochel has made numerous and lasting contributions to the theory of complex moment problems, including some outstanding research with F.H. Szafraniec in a series of papers. Some of these results include a novel approach to the complex moment problem. Here a function (for example, of two variables) is deduced from its integrals calculated along all possible lines. This problem is intimately connected with image reconstruction for X-ray computerized tomography. Some of these results include linking positive linear functionals L acting on polynomials p in z and $\bar{z}$ with d-tuples $N \equiv (N_1, \ldots, N_d)$ of multiplication operators on the Hilbert space $L^2(\mu)$, where μ is a Radon measure on $\mathbb{C}^d$. This is done using the functional calculus for normal d-tuples of operators, via the cyclic vector $1 \in L^2(\mu)$, as follows:

$$L(p) := \langle p\,(N, N^*)\,1, 1\rangle = \int\limits_{\mathbb{C}^d} p(z, \bar{z}) d\mu(z).$$

This approach leads to a fruitful interplay between multivariable operator theory, the theory of positive linear functionals on the space of polynomials, and the theory of complex moment problems. Together with [61], it represents a predecessor of the unprecedented connections, beginning in the early 1990's, among real algebraic geometry, optimization theory, the theory of quadratures in numerical analysis, the theory of moments (full and truncated), the mathematics of finance, and the theory of realizability of point processes.

As a simple example of the results obtained by Stochel and his collaborators, we recall that the solubility of the moment problem in two variables cannot be fully characterized in terms of the positivity of the associated moment sequence. This is a consequence of the existence of nonnegative polynomials in two variables that do not admit a representation as a sum of squares of polynomials. Stochel and Szafraniec describe a series of additional conditions which allow a positive definite sequence to become a moment sequence, with a representing measure. These conditions have to do with the support of the representing measure, which must belong to a suitable class of algebraic curves. Along the way, the authors prove a boundedness criterion for formally normal operators in Hilbert spaces. In this way, results about moment sequences can be derived from criteria for essential normality for unbounded Hilbert space operators. The work makes contact with the well-known 1991 pioneering result of K. Schmüdgen, which created a new bridge between operator theory and real algebraic geometry.

In a truly trailblazing research accomplishment, Stochel and Szafraniec (see [62]) discovered a polar decomposition approach to the moment problem. Consider a double-indexed sequence $\gamma \equiv \{\gamma_{m,n}\}$ of complex numbers, where the indices m and n run over the integer lattice points of the nonnegative quarter plane; that is, whenever $m, n \geq 0$. Solving the moment problem entails, in this case, finding a positive Borel measure μ on $\mathbb{C}$ such that $\gamma_{m,n} = \int z^m z^n d\mu(z)$ (all $m, n \geq 0$). It is well known that the existence of a representing measure for γ implies that γ is positive definite, that is, the associated moment matrix must be positive semidefinite. It is also known that this condition is not sufficient for the solubility of the complex moment problem. Now suppose that we ask γ to admit a positive definite extension Γ to the integer lattice points of the northeast half-plane determined by the diagonal $m+n=0$; that is, Γ must be positive definite for all pairs (m, n) such that $m+n \geq 0$. In [62], the authors proved that γ has a representing measure if and only if the above mentioned extension Γ exists.

This superb result was highlighted in a Featured Mathematical Review [25], alongside another superb result obtained by Putinar and F.-H. Vasilescu (see [47]). The two articles represented outstanding additions to our existing knowledge, in terms of providing new criteria for existence and uniqueness of representing measures, and for localization of the support of such measures. They introduced original ideas, methods and techniques that had a lasting impact on subsequent developments of the theory. Both articles appealed to the notion of extendability, in different but compatible directions, and consonant with the main approach to truncated moment problems that was being developed at the time. The key ingredient needed was the idea of building a new moment problem, essentially equivalent to (and extending) the original one, but in a higher-dimensional setting, where positivity alone provides the necessary and sufficient condition, just as in the single-variable case.

Throughout his mathematical career, Stochel has excelled in many dimensions, and this has allowed him to access a myriad of different mathematical areas with great success. His research has amazing breadth and depth, and his ideas have nurtured and enhanced the Kraków school in analysis. Stochel belongs to that special breed of mathematicians that take a leading mathematical subject to a higher level. His contributions are numerous, deep, and cover many diverse aspects of functional analysis, single and multivariable operator theory, the theory of orthogonal polynomials, the theory of reproducing kernel Hilbert spaces, the theory of moments, and mathematical aspects of quantum mechanics. Stochel's ideas are often brilliant, and address fundamental problems; the solutions found indicate a profound understanding of the intrinsic structure of the mathematical entities under consideration, and of their interconnections with other areas of research.

4. LAMBERT'S PHENOMENON

Let H be a separable, complex Hilbert space. A linear bounded operator $S : H \longrightarrow H$ is called *subnormal* if there exists a larger Hilbert space $H \subset K$ and a normal, bounded operator N acting on K which leaves the closed subspace H invariant and coincides with S on it: $S = N|_H$. While the typical examples (functional models) of normal operators are multipliers $N = M_z$ with the complex variable on a Lebesgue space $L^2(\mu)$ of a positive, compactly supported measure on the complex plane, a quintessential subnormal operator is the same multiplier $S = M_z$, this time restricted to the closure $R^2(\mu)$ in $L^2(\mu)$ of complex, rational functions with poles off the support of μ. Deep questions of complex approximation theory, such as completeness of specific systems of functions or existence of bounded point evaluations have a non-trivial translation, and often are solved, in the Hilbert space framework we have mentioned.

While the theory of *bounded* subnormal operators is considered at a post maturity stage, see for instance Conway's monograph [24], the pitfalls revealed by *unbounded* (sub)normal operators are numerous and quite treacherous. Jan Stochel has obstinately attacked these challenges for several decades, sometimes seconded by F.H. Szafraniec and more recently by his younger colleagues. We refer below only to a couple of recent, outstanding contributions in this direction.

Several complementary characterizations of operator subnormality exist, as charted at a glorious, almost naive initial stage, by Halmos [31]. It is clear that subnormality has to do with the existence of a positive measure attached to the normal extension, hence a 2D moment problem is lurking around in many impersonations. More surprising is a characterization in terms of a 1D moment problem, as stated in a celebrated result of Lambert [38]: *A linear bounded operator S with a cyclic vector ξ is subnormal, if and only if there exists a positive measure σ on the semi-axis, admitting all moments, such that*

$$\|S^n\xi\|^2 = \int_0^\infty t^n \sigma(dt), \quad n \geq 0.$$

Now, this is a classical moment problem, well charted since the foundational work of Stieltjes [55,56]. Hidden above, in the innocent looking power moment representation, is a powerful interplay between function theory of a complex variable, continued fractions and the geometry of Hilbert space.

It is unquestionable the merit of Jan Stochel to have pursued an analog of this dictionary in the case of a closed, unbounded operator S. The major discovery of Jabłoński–Jung–Stochel [33] is the existence of a far from subnormal unbounded composition operator which generates a Stieltjes moment sequence, in the spirit of Lambert's Theorem. See also [39]. This is done with a superb control of the intricate indeterminate Stieltjes moment data. As a byproduct, the authors correct a result of Simon [54] which parameterizes von Neumann extensions of a closed real symmetric operator with deficiency indices $(1,1)$. The pathological examples of indeterminate Stieltjes measures, among them the ubiquitous log-normal distribution, turn this article into an invaluable source of hard analysis observations emanating from a classical theme.

The analysis of this class of exotic unbounded operators, acting as weighted shifts on specific graphs, is continued by Budzynski, Jablonski, Jung and Stochel [16]. The amount of ingenuity and new observations pertaining to the analysis of indeterminate Stieltjes moment problems put the results of the four authors on the first rank of achievements in linear analysis recorded during the last decade.

The two articles by Stochel and collaborators are a compulsory reading for any researcher touching 1D moment problems of all sorts.

5. WEIGHTED SHIFTS ON DIRECTED TREES

The theory of a weighted shift on directed tree has been first introduced by Stochel and his collaborators in the paper [32] in 2012. This considerably generalizes the notion of a weighted shift which is a classical object of operator theory. The main goal of this study is to implement some methods of graph theory into operator theory. As opposed to the standard graph theory which concerns mostly finite graphs, the models for this operator are mainly dealt with infinite directed trees because operator properties of weighted shifts on finite directed trees are trivial. In the pioneer paper [32], the authors established almost all fundamental properties necessary to deal with weighted shifts on directed trees. After that, this new class has developed extensively in several areas of operator theory for ten years since 2012, and provided many good results and exotic examples. The number of results is far too large for us to summarize them here, and thus below we will focus on a few important topics.

Let $\mathcal{T} = (V, E)$ be a directed tree, where V and E stand for the sets of vertices and edges of $\mathcal{T}$, respectively. If $\mathcal{T}$ has a root, which is denoted by **root**, then we write $V^\circ := V \setminus \{\mathbf{root}\}$. And we write $\mathbf{par}(v)$ for the parent of a vertex $v \in V$. Let $l_2(V)$ be the Hilbert space of all square summable complex functions on V equipped with the standard inner product. Given $\lambda = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$, we define the operator S_λ in $l^2(V)$ by

$$\mathcal{D}(S_\lambda) = \left\{ f \in l^2(V) : \Lambda_{\mathcal{T}} f \in l^2(V) \right\},$$
$$S_\lambda f = \Lambda_{\mathcal{T}} f, \quad f \in \mathcal{D}(S_\lambda),$$

where $\Lambda_{\mathcal{T}}$ is the map defined on functions $f : V \to \mathbb{C}$ via

$$(\Lambda_{\mathcal{T}} f)(v) = \begin{cases} \lambda_v \cdot f(\mathbf{par}(v)) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \mathbf{root}. \end{cases}$$

We first look at Stochel's results in [32]. The structure of this operator is closely related to both properties of the weights and the underlying trees. For example, a directed tree which admits an injective weighted shift must be leafless, where a leaf is a vertex with no children. A densely defined weighted shift on a directed tree is shown to be circular, and the adjoint of such an operator is described. This allows for a description of the polar decomposition and the characterization of Fredholm and semi-Fredholm weighted shifts. This can be done in terms of the underlying tree; such a tree is then called a Fredholm tree. A complete characterization of the hyponormality of weighted shifts on directed trees is given. Also the characterization for a bounded subnormal weighted shift S_λ on a directed tree $\mathcal{T} = (V, E)$ is given, which is that S_λ is subnormal if and only if each vertex $u \in V$ induces a Stieltjes moment sequence. As a useful model to examine the set of all vertexes which induce Stieltjes moment sequences, the authors considered the directed tree $\mathcal{T}_{\eta,\kappa}$ consisting of one branching vertex, η branches and a trunk of length κ. The subnormality and complete hyperexpansivity of bounded weighted shift S_λ on $\mathcal{T}_{\eta,\kappa}$ are characterized. In particular, the techniques elaborated in [32] have been used by other authors in different areas of mathematics including von Neumann algebras. In 2018, Stochel applied the subnormality of weighted shift S_λ on $\mathcal{T}_{\eta,\kappa}$ to the study of the subnormal completion problem. Stochel and his collaborators have obtained several characterizations solving the subnormal completion problem for weighted shifts S_λ on $\mathcal{T}_{\eta,\kappa}$ in [26]. In a subsequent paper [27], the authors obtained equivalent conditions written in terms of two-parameter sequences for the existence of a subnormal completion in which the resulting measures are 2-atomic. For $\eta = 2$, they obtain a solution written explicitly in terms of initial data.

Stochel has made remarkable achievements in the unbounded weighted shifts on directed trees. A criterion for subnormality of unbounded operators was established by the paper [23] in 2011. By using it, Stochel and his collaborators obtained a criterion for subnormality of unbounded weighted shifts on directed trees written in terms of consistent systems of measures in the next year ([12]). In a subsequent paper [13], they invented formulas for subnormality of weighted shift S_λ on directed tree $\mathcal{T}_{\eta,\kappa}$. Furthermore, Stochel and his collaborators have studied normality and quasinormality of densely defined weighted shifts S_λ on directed trees in 2013. Namely, in [34] they proved that formally normal weighted shifts on directed trees are always bounded and normal. Also, they developed a new class of operators, called weakly quasinormal, which is characterized by means of the strong commutant of their moduli, and characterized weak quasinormality of unbounded operators in [35] and [37]. This characterization was applied to the densely defined quasinormal weighted shifts S_λ to characterize the weak quasinormality of such operator S_λ in those papers.

We now introduced two celebrated Stochel's results which provide some surprising examples. The first is a result in the paper [33]. In 1974, Lambert proved that a bounded linear operator is subnormal if and only if it generates Stieltjes moment sequences. But this assertion is not true in the case of unbounded subnormal operators. To the best of our knowledge, the only known examples of non-subnormal operators generating Stieltjes moment sequences are those appearing from formally normal ones. Unfortunately, the operators so constructed, though closable, are not closed. Stochel and his collaborators invented the first example of a non-hyponormal operator which generates Stieltjes moment sequences in the paper [33] in 2012. This example is constructed on the basis of the theory of weighted shifts on directed trees consisting of only one branching vertex and an infinite number of branches as above. More details on this are provided in the other part of this article. The second is a result in [15]. In 1940 Naimark gave a remarkable example of a closed symmetric operator whose square has trivial domain (see [43, 44]). After that, Chernoff published a short example of a semibounded closed symmetric operator whose square has trivial domain in the paper [22] in 1983. In the same year Schmüdgen found out another pathological behavior of domains of powers of closed symmetric operators related to density with respect to graph norms (see [50]). In [13], Stochel and his collaborators obtained a result that for any positive integer n there exists a subnormal weighted shift on a directed tree whose nth power is closed and densely defined while its $(n+1)$-st power has trivial domain. This answers completely the open problem in [36].

Stochel's achievement in this theory has influenced various topics in operator theory for the decade since 2012, and many papers related to this topic have been published so far by himself and other operator theorists (see, for example, [20, 28, 42]).

6. WEIGHTED COMPOSITION OPERATORS

Weighted composition operators are an important object of investigation for mathematicians. A significant motivation for studying such operators is the Banach–Stone theorem, which states that if A is a surjective linear isometry between two Banach spaces $C(X)$ and $C(Y)$ of real-valued continuous functions on a compact Hausdorff topological spaces X and Y equipped with the supremum norm, then there exist a continuous real-valued function w on Y and a homeomorphism $\phi\colon Y \to X$ such that $|w| \equiv 1$ and $Af = w \cdot (f \circ \phi)$ for all $f \in C(X)$.

Preliminary research on weighted composition operators in L^2-spaces was conducted in Parrott's Ph.D. thesis [46]. We will now give a definition of these operators. Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space. Given measurable transformations $\phi\colon X \to X$ and $w\colon X \to \mathbb{C}$, we define the *weighted composition operator* $C_{\phi,w}$ in $L^2(\mu)$ with a symbol ϕ and a weight w by

$$\mathcal{D}(C_{\phi,w}) = \big\{f \in L^2(\mu)\colon w \cdot (f \circ \phi) \in L^2(\mu)\big\},$$
$$C_{\phi,w}(f) = w \cdot (f \circ \phi),\ f \in \mathcal{D}(C_{\phi,w}).$$

If $w \equiv 1$, then we call $C_{\phi,w}$ a composition operator and abbreviate it to C_ϕ.

Research related to composition operators has been conducted by many mathematicians over the past 70 years. Nordgren, in his survey article [45] has given a solid foundation for the study of composition operators, opening a new area of investigation in operator theory. One of Stochel's more important results related to composition operators is published in the paper [14] in which Stochel and his collaborators invented the first-ever criteria for the subnormality of unbounded densely defined composition operators in L^2-spaces, with no restrictions on their domains, which becomes a new characterization of subnormality in the bounded case. This criterion states that if an injective densely defined composition operator has a measurable family of probability measures that satisfies the so-called consistency condition, then it is subnormal.

In the monograph [17] Stochel and his collaborators have given a solid foundation for the general theory of bounded and unbounded weighted composition operators. Although unbound weighted composition operators $C_{\phi,w}$ have been studied much earlier, the authors of these papers usually made some additional restrictive assumptions that, among other things, guarantee that the corresponding operator C_ϕ is densely defined. It is worth mentioning here that it may happen that a weighted composition operator $C_{\phi,w}$ is an isometry while the corresponding composition operator C_ϕ is even not well defined. This means that the approach proposed in previous papers excludes a variety of weighted composition operators. To study weighted composition operators in full generality, two fundamental concepts were introduced in monograph [17]. The first of these is the Radon-Nikodym derivative $\frac{\mathrm{d}\mu_w \circ \phi^{-1}}{\mathrm{d}\mu}$, where $\mathrm{d}\mu_w = |w|^2 \mathrm{d}\mu$ and the second is the conditional expectation $\mathsf{E}(\,\cdot\,; \phi^{-1}(\mathscr{A}), \mu_w)$ (both of these notions are discussed in detail in [17]). The monograph begins by examining the basic issues related to weighted composition operators, namely their well-definiteness, closedness, dense definiteness and boundedness. Another of the more important results of the monograph is a characterization of unbounded quasinormal weighted composition operators, which looks formally the same as in the case of bounded composition operators. Since one of the main topics of Stochel's research is related to abstract and concrete classes of subnormal operators, weighted composition operators are a great research field in this case. As was known earlier, the basic characterizations of subnormal operators of the Halmos–Bram type are not true for unbounded subnormal operators and the well-known general characterizations of unbounded subnormal operators do not seem to be useful here. Thus, characterizing unbounded weighted composition operators was a major challenge. In [17] it was shown that fundamental concepts introduced in [17] allowed the authors to adapt the ideas introduced in the paper [14] to give useful criteria for subnormality of weighted composition operators. The idea itself is based on an essential generalization of Lambert's construction of a quasinormal extension of a bounded subnormal composition operator given in [40]. It was known that in the case of unbounded operators, the subnormality of an operator is not equivalent to the fact that the operator generates Stieltjes moment sequences. Thus, a natural question was to characterize weighted composition operators that generate Stieltjes moment sequences. The answer to this question given in [17] also allowed the authors to obtain counterparts of Lambert's characterizations of subnormality for bounded weighted composition operators. Another achievement of [17] is the characterization of the class of seminormal weighted composition operators and some of its subclasses (seminormal weighted composition operators were previously investigated by Campbell and Hornor, but under some restrictive assumptions. In [17], this characterization is in full generality). It turns out that it is often convenient to consider weighted composition operators over discrete measure spaces; in this case, more handy characterizations of the fundamental objects under consideration can be given. This approach allows, among other things, to give a surprising example of an isometric weighted composition operator $C_{\phi,w}$ for which the corresponding composition operator C_ϕ is not even well defined. Monograph [17] also contains subtle considerations on the relationship of the weighted composition operator $C_{\phi,w}$ and the product $M_w C_\phi$ of the multiplication operator M_w and the corresponding composition operator C_ϕ.

Finally, it is worth mentioning that the class of weighted composition operators is a broad class of operators and includes, among others, multiplication operators, partial (weighted and unweighted) composition operators, and weighted shifts on countable directed trees.

7. THE CAUCHY DUAL SUBNORMALITY PROBLEM FOR 2-ISOMETRIES

We need a few definitions to explain the Cauchy dual subnormality problem for 2-isometries. Following [1–3], we say that a bounded linear operator T on a complex Hilbert space $\mathcal{H}$ is a 2-*isometry* if

$$I - 2T^*T + T^{*2}T^2 = 0.$$

It turns out that any 2-isometry T is left-invertible, that is, T^*T is an invertible operator on $\mathcal{H}$. In the context of the wandering subspace problem, Shimorin [52] coined the term *Cauchy dual operator*, defined as $T' := T(T^*T)^{-1}$, where T is a left-invertible bounded linear operator on $\mathcal{H}$.

The Cauchy dual subnormality problem (for short, CDSP) asks *whether the Cauchy dual operator of a* 2-*isometry is subnormal.* This problem manifests the rich interplay between positive definite and negative definite functions on abelian semigroups. Indeed, CDSP can be considered as the non-commutative variant of the fact from harmonic analysis on semigroups that the reciprocal of a Bernstein function $f\colon [0,\infty) \to (0,\infty)$ is completely monotone (see [49, Theorem 3.6]).

CDSP has an affirmative solution for operator-valued unilateral weighted shifts. This is a consequence of [4, Theorems 2.5 and 3.3], and it can be obtained from [7, Proposition 6] in the scalar case.

Theorem 7.1. *The Cauchy dual operator of a* 2*-isometric operator-valued weighted shift is subnormal.*

The proof of this theorem exploits the so-called *kernel condition*:

$$T^*T(\ker T^*) \subseteq \ker T^*.$$

Another early result towards the solution of CDSP asserts that the Cauchy dual of any concave operator is a hyponormal contraction (see [53, Equation (26)]). Later this fact was generalized in [19, Theorem 3.1] by deducing the power hyponormality of the Cauchy dual of any concave operator. Around the same time, CDSP was settled affirmatively for Δ_T-regular 2-isometries in [9, Theorem 3.4] (see also [21, Corollary 6.2] for the solution for yet another subclass of 2-isometries).

Related to the counterexamples to CDSP from [4], we share here a short story. In the summer of 2016, the second author of [4] was visiting Prof. Jan Stochel for a week. During the first meeting with Prof. Jan Stochel, he explained the CDSP (indeed as a conjecture), some partial solutions and an example arising from weighted shifts on directed trees. Prof. Jan Stochel, after a while, pointed out a difficulty in deducing the subnormality of the shift operator at the root! He suggested looking at a criterion for a Stieltjes moment sequence to have a backward extension (see [32, Lemma 6.1.2]), and we all (the last three authors of [4]) spent almost an entire evening along this direction. By the end of the day, we could not decide whether the Cauchy dual in that example is subnormal. The meeting was over, and we all went in different directions. The second author was in a grocery shop, and suddenly he got a phone call from Prof. Jan Stochel, and while in a tram and with a lot of excitement, he said, "The subnormality of the Cauchy dual of the shift in question forces the kernel condition!" As weighted shifts on directed trees need not satisfy the kernel condition, this fact together with the notion of "perturbed kernel condition" (see [4, Section 6]) yields the desired counterexamples. In particular, it was shown in [4, Examples 6.6 and 7.10] that there exist 2-isometric weighted shifts on directed trees (that include adjacency operators) whose Cauchy dual is not necessarily subnormal. Recently, a class of cyclic 2-isometric composition operators without subnormal Cauchy dual has been constructed in [5, Theorem 4.4]. In all known counterexamples of the Cauchy dual subnormality problem for 2-isometries, the defect operator $T^*T - I$ is of infinite rank.

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