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MODIFIED LAPLACE BASED VARIATIONAL ITERATION METHOD FOR THE MECHANICAL VIBRATIONS AND ITS APPLICATIONS

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Abstract: In this paper, we are putting forward the periodic solution of non-linear oscillators by means of variational iterative method (VIM) using Laplace transform. Here, we present a comparative study of the new technique based on Laplace transform and the previous techniques of maximum minimum approach (MMA) and amplitude frequency formulation (AFF) for the analytical results. For the non-linear oscillators, MMA, AFF and VIM by Laplace transform give the same analytical results. Comparison of analytical results of VIM by Laplace transform with numerical results by fourth-order Runge–Kutta (RK) method conforms the soundness of the method for solving non-linear oscillators as well as for the time and boundary conditions of the non-linear oscillators.

Key words: variational iterative method, non-linear oscillator, Laplace transform

1. INTRODUCTION

The variational iterative method was suggested in 1990s to solve an oozing of with fractional deriva-tives and non-linear oscillator [2, 3, 4], and this method is used enormously as a main mathematical instrument for solving non-linear oscillators in various sciences (e.g. see [5–10]). This is a very popu-lar method in the list of methods for non-linear systems, and it includes high citation index articles dealing with the 'variational iterative method' (VIM). This method deals with non-linear oscillators like Fangzhu oscillators [11], fractal Toda oscillator [12], HIV models [13], biological models [14], fractal vibration models [15], microelectromechanical system oscillators [16, 17], fractal/fractional/ two-scale oscillators [18], interconnected spring carts [19, 20], etc. Naveed et al. [21, 22] investigated the ho-motopy perturbation method for the oscillators in nanotechnology. This paper suggests the periodic solution of the governing differential equations (non-linear oscillators) obtained by Laplace transform and VIM. The VIM retains a series of linear equations that can be solved by Laplace transform. This method identifies some obvious benefits, and its Lagrange multiplier is much trouble-free than that of variational theory [23–27]. The general recognition of the Lagrange multiplier by Laplace transform is given by Eq. (5) in [28].

Consider a non-linear oscillator in the equation form as

$$u''(t) + f(u) = 0 \tag{1}$$

with initial conditions $u(0) = A, u'(0) = 0$. Eq. (1) can be written as:

$$u'' + w^2 u + p(u) = 0 \tag{2}$$

where w is unknown frequency, $\mathrm{p(u)} = f(u) - w^2 u$. As claimed by the VIM, the correction functional which is basically a convolution for Eq. (2) is given as [29–33]:

$$u_{n+1}(t) = u_n(t) + \int_0^t \zeta(t,\xi)[u_n''(\xi) + w^2 u_n(\xi) + \tilde{p}(u_n)]d\xi \qquad n = 0,1,2,\ldots \tag{3}$$

where ζ is a general Lagrange multiplier, and it can be choicely determined by immobilised conditions of Eq. (3) with respect to using variational theory [10–13]. The subscript *n* denotes the *n*th approximation of the solution and $\tilde{p}$ is a restricted variation. This convolution gives the value of ζ by making $u_n(t)$ immobilised. This method is applicable to derive the analytical solution for the motion of non-linear unbound vibration of conservative, single degree of freedom systems.

Now we implement this method to justify the motion of two oscillators by making Laplace transform in the well-known VIM to obtain the relationship between amplitude and angular frequency. This method is equally good when compared with the older versions of VIM, Ganji and Azimi [1] maximum minimum approach (MMA), and amplitude frequency formulation (AFF) to non-linear oscillation systems. For the first problem shown in (Fig. 1), consider a block of mass m_1, which is on the horizontal surface, while another block of mass m_2 is just slipped vertical and is also attached with mass m_1. In this system, the length of support is *L*, gravitational acceleration produced due to free motion of blocks is denoted by *g* and *K* is spring constant. If we assume that $u = \frac{x}{L} \ll 1$, then the equation in the range of time and boundary conditions is given as:

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$u'' + \left(\frac{m_2}{m_1}\right)u^2u'' + \left(\frac{m_2}{m_1}\right)uu'^2 + \left(\frac{K}{m_1} + \frac{m_2\boldsymbol{g}}{m_1}\right)u + \frac{m_2\boldsymbol{g}}{2Lm_1}u^3 = 0, \quad u(0) = A, \quad u'(0) = 0$ (4)

where both *u*, the generalised displacement, and *t*, the time variable, are dimensionless.

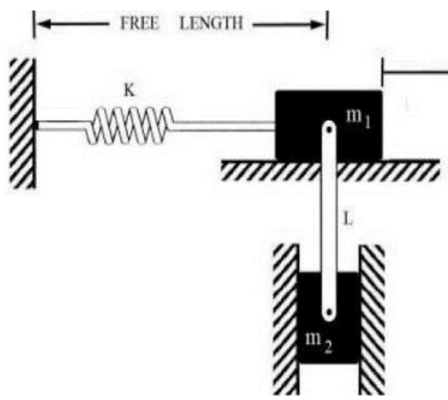

Fig. 1. Geometry of the first Problem

2. APPLICATIONS OF LAPLACE-BASED VIM TO THE FIRST PROBLEM

In order to solve the first problem given in Eq. (4), by making Laplace transform in VIM, we rewrite the problem as:

$u'' + \left(\frac{m_2}{m_1}\right)u^2u'' + \left(\frac{m_2}{m_1}\right)uu'^2 + w_0^2u + \frac{m_2\boldsymbol{g}}{2Lm_1}u^3 = 0,$
$u(0) = A, \quad u'(0) = 0$ (5)

where $w_0^2 = \left(\frac{K}{m_1} + \frac{m_2 g}{Lm_1}\right)$. We can rewrite the equation in the form:

$(1 + \alpha u^2)u'' + \alpha uu'^2 + w_0^2u + \beta u^3 = 0, \quad u(0) = A, \quad u'(0) = 0$ (6)

where $\alpha = \frac{m_2}{m_1}$ and $\beta = \frac{m_2 g}{2Lm_1}$. To solve the above equation, we use VIM by Laplace transform. To approach the correctional functional, we write the above equation in general non-linear oscillator form as:

$u'' + w^2u + p(u) = 0$ (7)

where

$p(u) = \alpha u^2u'' + \alpha uu'^2 + (w_0^2 + w^2)u + \beta u^3$ (8)

The correctional functional is defined as

$u_{n+1}(t) = u_n(t) + \int_0^t \zeta(t,\xi)[u_n''(\xi) + w^2u_n(\xi) + \tilde{p}(u_n)]d\xi$ (9)

where the Lagrange multiplier is given by $\zeta(\xi) = -\frac{1}{w}\sin wt$. Now, by applying Laplace transform to Eq. (9), we have

$L[u_{n+1}(t)] = L[u_n(t)]]$

$-L[\int_0^t \frac{1}{w}\sin w(t-\xi)[u_n''(\xi) + w^2u_n(\xi) + p(u_n)]d\xi$

$L[u_{n+1}(t)] = L[u_n(t)] - \frac{1}{w}L[\sin wt]L[u_n'' + \alpha u_nu_n'^2 + \alpha u_n^2u_n'' + w_0^2u_n + \beta u_n^3]$ (10)

For a first-order approximate solution, put $n = 0$, we get

$L[u_1(t)] = L[u_0(t)] - \frac{1}{w}L[\sin wt]L[u_0'' + \alpha u_0u_0'^2 + \alpha u_0^2u_0'' + w_0^2u_0 + \beta u_0^3]$

Using the trail function $u_0(t) = A\cos wt$, the above equation gets the form

$L[u_1(t)] = L[A\cos wt] - \frac{1}{w}L[\sin wt]L[-Aw^2\cos wt - \alpha A^3w^2\cos^3 wt + \alpha A^3w^2\cos wt\sin^2 wt + w_0^2A\cos wt + \beta A^3\cos^3 wt]$

$L[u_1(t)] = L[A\cos wt] - \frac{1}{w}L[\sin wt]L[-Aw^2\cos wt - \frac{\alpha A^3w^2}{4}(3\cos wt - \cos 3wt) + \frac{\alpha A^3w^2}{4}(\cos wt + \cos 3wt) + w_0^2A\cos wt + \frac{\beta A^3}{4}(3\cos wt - \cos 3wt)]$

After some simplification, we have the expression as

$L[u_1(t)] = L[A\cos wt] - \frac{1}{w}\left[-Aw^2 - \frac{\alpha A^3w^2}{2} + w_0^2A + \frac{3\beta A^3}{4}\right]L[\sin wt]L[\cos wt] - \frac{1}{w}\left[\frac{\alpha A^3w^2}{4} - \frac{\beta A^3}{4}\right]L[\sin wt]L[\cos 3wt]$

By inverse Laplace transform, the expression for the first-order approximate solution is

$u_1(t) = A\cos wt - \frac{1}{w}\left[-Aw^2 - \frac{\alpha A^3w^2}{2} + w_0^2A + \frac{3\beta A^3}{4}\right](\frac{1}{2}t\sin wt) - \frac{1}{w}\left[\frac{\alpha A^3w^2}{4} - \frac{\beta A^3}{4}\right](\frac{1}{8w}\cos wt - \cos 3wt)$ (11)

In Eq. (11), the second term is a secular term because it grows in amplitude with time, so avoiding the secular term in approximate solution requires that

$-\frac{1}{w}\left[-Aw^2 - \frac{\alpha A^3w^2}{2} + w_0^2A + \frac{3\beta A^3}{4}\right] = 0$

$Aw^2 + \frac{\alpha A^3w^2}{2} = w_0^2A + \frac{3\beta A^3}{4}$

$w^2 = \frac{w_0^2 + \frac{3A^2}{4}\frac{m_2\boldsymbol{g}}{2Lm_1}}{1 + \frac{m_2A^2}{2m_1}}$

This leads to the expression for angular frequency of the system:

$w = \frac{1}{2}\sqrt{\frac{8w_0^2Lm_1 + 3m_2\boldsymbol{g}A^2}{L(m_2A^2 + 2m_1)}}$

This expression for angular frequency of the first problem is exactly the same as obtained by the MMA in Eq. (11) and the AFF method in Eq. (22) by Ganji and Azimi [1] and He in [34]. So, the periodic solution in this case becomes the same as that of MMA and AFF, while the first-order approximate solution is given as

$u_1(t) = A\cos wt - \frac{1}{w}\left[\frac{\alpha A^3w^2}{4} - \frac{\beta A^3}{4}\right](\frac{1}{8w}\cos wt - \cos 3wt).$ (12)

3. APPLICATION OF THE LAPLACE-BASED VIM TO THE SECOND PROBLEM

The second problem deals with the motion of simple pendulum devoted to a spinning rigid frame shown in (Fig. 2), which has the differential equation:

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$$\theta'' + (1 - \Lambda\cos\theta)\sin\theta = 0, \quad \theta'(0) = 0, \; \theta(0) = A \quad (13)$$

where θ is generalised displacement without dimensions, t is the time variable and Λ indicates the relation $\Lambda = \frac{w^2 r}{g}$.

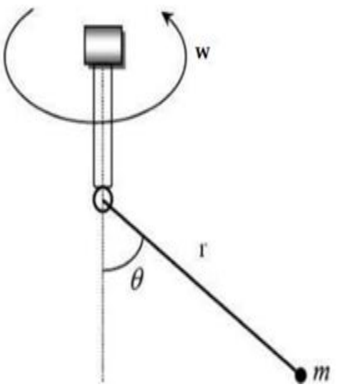

Fig. 2. Geometry of the first Problem

In order to solve this problem by using VIM with Laplace transform, we write Eq. (13) as

$$\theta'' + (1 - \Lambda)\theta + \left(\frac{-1}{6} - \frac{2\Lambda}{3}\right)\theta^3 + \left(\frac{1}{120} - \frac{2\Lambda}{15}\right)\theta^5 = 0, \quad \theta'(0) = 0, \; \theta(0) = A \quad (14)$$

Eq. (14) in the form of general non-linear oscillator has the form

$\theta'' + w^2\theta + p(\theta) = 0$, where $p(\theta) = -w^2\theta + (1 - \Lambda)\theta + \left(\frac{-1}{6} - \frac{2\Lambda}{3}\right)\theta^3 + \left(\frac{1}{120} - \frac{2\Lambda}{15}\right)\theta^5$.

The correctional functional to approximate solution is defined as

$$\theta_{n+1}(t) = \theta_n(t) + \int_0^t \zeta(t,\xi)[\theta_n''(\xi) + w^2\theta_n(\xi) + \tilde{p}(\theta_n)]d\xi$$

By using Lagrange multiplier $\zeta(\xi) = -\frac{1}{w}\sin wt$, we have the iterative formula

$$\theta_{n+1}(t) = \theta_n(t) - \frac{1}{w}\int_0^t \sin(t - \xi)[\theta'' + (1 - \Lambda)\theta + \left(\frac{-1}{6} - \frac{2\Lambda}{3}\right)\theta^3 + \left(\frac{1}{120} - \frac{2\Lambda}{15}\right)\theta^5]d\xi$$

Now, by applying Laplace transform on the above iterative formula, we get

$$L[\theta_{n+1}(t)] = L[\theta_n(t)] - \frac{1}{w}L[\sin wt]L[\theta'' + (1 - \Lambda)\theta + \left(\frac{-1}{6} - \frac{2\Lambda}{3}\right)\theta^3 + \left(\frac{1}{120} - \frac{2\Lambda}{15}\right)\theta^5]$$

For the first-order approximate solution, use $n = 0$ and trail function $u_0(t) = A\cos wt$.

$$L[\theta_1(t)] = L[A\cos wt] - \frac{1}{w}L[\sin wt]L[-Aw^2\cos wt + (1 - \Lambda)A\cos wt + \left(\frac{-1}{6} - \frac{2\Lambda}{3}\right)A^3\cos^3 wt + \left(\frac{1}{120} - \frac{2\Lambda}{15}\right)A^5\cos^5 wt]$$

$$L[\theta_1(t)] = L[A\cos wt] - \frac{1}{w}L[\sin wt]L[-Aw^2\cos wt + (1 - \Lambda)A\cos wt + \left(\frac{-1}{6} - \frac{2\Lambda}{3}\right)\frac{A^3}{4}(\cos 3wt + 3\cos wt) + \left(\frac{1}{120} - \frac{2\Lambda}{15}\right)\frac{A^5}{16}(\cos 5wt + 5\cos 3wt + 10\cos wt)]$$

After simplification, we get

$$L[\theta_1(t)] = L[A\cos wt] - \frac{1}{w}\left(-Aw^2 + (1 - \Lambda)A - \frac{A^3}{8} + \frac{A^3\Lambda}{2} + \frac{A^5}{192} - \frac{A^5\Lambda}{12}\right)L\left(\frac{t}{2}\sin wt\right) - \frac{1}{w}\left(-\frac{A^3}{24} + \frac{A^3\Lambda}{6} + \frac{A^5}{384} - \frac{A^5\Lambda}{24}\right)L\left(\frac{1}{8w}(\cos wt - \cos 3wt)\right)$$

By applying inverse Laplace, the expression reduces to

$$\theta_1(t) = A\cos wt - \frac{1}{w}\left(-Aw^2 + (1 - \Lambda)A - \frac{A^3}{8} + \frac{A^3\Lambda}{2} + \frac{A^5}{192} - \frac{A^5\Lambda}{12}\right)\left(\frac{t}{2}\sin wt\right) - \frac{1}{w}\left(-\frac{A^3}{24} + \frac{A^3\Lambda}{6} + \frac{A^5}{384} - \frac{A^5\Lambda}{24}\right)\left(\frac{1}{8w}(\cos wt - \cos 3wt)\right)$$

Here, in this equation, the second term is a secular term because it grows in amplitude with time, so avoiding the secular term in approximate solution required that

$$-\frac{1}{w}\left(-Aw^2 + (1 - \Lambda)A - \frac{A^3}{8} + \frac{A^3\Lambda}{2} + \frac{A^5}{192} - \frac{A^5\Lambda}{12}\right) = 0$$

$$w^2 = \left((1 - \Lambda) - \frac{A^2}{8} + \frac{A^2\Lambda}{2} + \frac{A^4}{192} - \frac{A^4\Lambda}{12}\right)$$

The expression for the angular frequency is given as:

$$w = \sqrt{1 - \Lambda - \frac{A^2}{8} + \frac{A^2\Lambda}{2} + \frac{A^4}{192} - \frac{A^4\Lambda}{12}}$$

The expression for angular frequency of the second problem is exactly the same as obtained by the MMA in Eq. (30) and the AFF method in Eq. (35) by Ganji and Azimi [1]. So, the periodic solution in this case becomes the same as that of MMA and AFF, while the approximate solution is

$$\theta_1(t) = A\cos wt - \frac{1}{w}\left(-\frac{A^3}{24} + \frac{A^3\Lambda}{6} + \frac{A^5}{384} - \frac{A^5\Lambda}{24}\right) * \left(\frac{1}{8w}(\cos wt - \cos 3wt)\right) \quad (15)$$

4. RESULTS AND DISCUSSION FOR THE FIRST PROBLEM

In this section, we have compared the numerical solution of non-linear oscillator (4) obtained by fourth-order Runge–Kutta (RK) method with analytical solutions obtained by Laplace based VIM. The analytical solution by VIM using Laplace transform coincides analytically with MMA and AFF techniques. In (Fig. 3), the comparison between analytical solution by VIM, VIM with Laplace and numerical solution by fourth-order RK method shows the validity of Laplace-based VIM.

In this session, we have characterised the error analysis of the analytical solution by VIM with Laplace transform and numerical solution by fourth order RK method. In Table 1, the error terms, e_1 and e_2 are by VIM and VIM with Laplace transform, respectively. Error e_2 conforms the validity of the solution by VIM with Laplace transform.

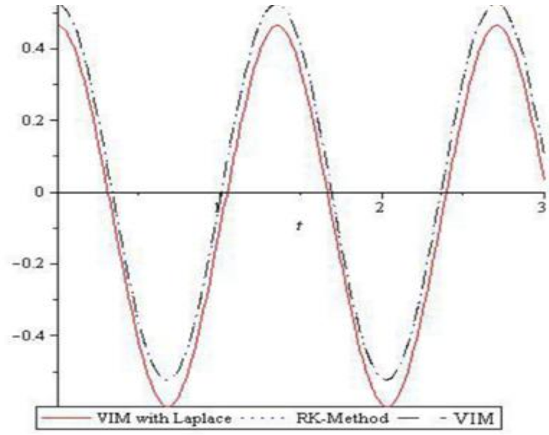

Fig. 3. Comparison among VIM, VIM with Laplace and fourth-order RK method in the first problem for $= \frac{\pi}{6}$; g = 9.81 $\mathrm{m\ s^{-2}}$, K = 100 $\mathrm{N\ m^{-2}}$, $\mathrm{m_1} = 5\ \mathrm{kg}$, $\mathrm{m_2} = 1\ \mathrm{kg}$, L= $1\ \mathrm{m}$, $\alpha = \frac{1}{5}$, $\beta = 0.981\ \mathrm{s^{-2}}$

Tab. 1. Error analysis for the first problem

Time steps	Previous results	Our results	RK method	e_1	e_2
1	0.468130637	0.402613907	0.469219893	0.001089256	0.066606
2	0.313478406	0.239576597	0.315842007	0.002363601	0.0762654
3	0.0924086651	0.021854038	0.093445360	0.001036695	0.0715913
4	−0.148239947	−0.209104415	−0.149924001	0.001684053	0.0591804
5	−0.357480565	−0.418137275	−0.359780544	0.002299979	0.0583567
6	−0.490980817	−0.561137359	−0.491696084	0.000715266	0.0694413
7	−0.520455648	−0.594098506	−0.520518333	0.042237427	0.0735802
8	−0.439660145	−0.505017274	−0.441122041	0.001461896	0.0638952
9	−0.265712668	−0.324704000	−0.267960752	0.002248084	0.0567432
10	−0.035467936	−0.100595290	−0.035762460	0.000294524	0.0648328

5. RESULTS AND DISCUSSION FOR THE SECOND PROBLEM

In this sectio, we have compared the numerical solution of non-linear oscillator (13) obtained by fourth-order RK method and the analytical solution. The analytical solution by VIM using Laplace transform coincides analytically with MMA and AFF techniques. In (Fig. 4), the comparison among analytical solution by VIM, VIM with Laplace and numerical solution by fourth-order RK method shows the validity of VIM with Laplace.

In this session, we have characterised the error analysis of the analytical solution by VIM with Laplace transform and numerical solution by fourth-order RK method. In Table 2, the error terms $\mathrm{e_1}$ and $\mathrm{e_2}$ are by VIM and VIM with Laplace transform, respectively. Error $\mathrm{e_2}$ conforms the validity of the solution by VIM with Laplace transform.

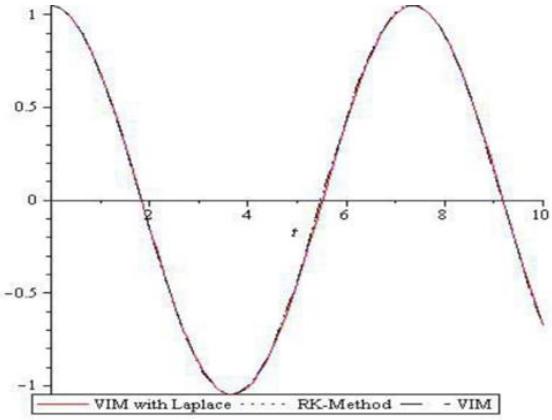

Fig. 4. Comparison among VIM, VIM with Laplace and fourth-order RK method in the second problem for $\mathrm{A} = \frac{\pi}{3}$ and $\Lambda = 0.25$

Tab. 2. Error analysis for the second problem

Time steps	Previous results	Our results	RK method	e_1	e_2
1	1.043371279	1.043420173	1.043429864	0.039474666	0.024021587
2	1.031920422	1.032112440	1.032149462	0.000229041	0.000037022
3	1.012928661	1.013347597	1.013424679	0.000496018	0.000077082
4	0.986534778	0.987247802	0.987370428	0.000835650	0.000122626
5	0.952931653	0.953984220	0.954149663	0.001218009	0.000165443
6	0.912364844	0.913777051	0.913974673	0.001609828	0.000197622
7	0.865130800	0.866895416	0.867108671	0.001977871	0.000213255
8	0.811574689	0.813657009	0.813866137	0.002291448	0.000209128
9	0.752087881	0.754427405	0.754613127	0.002525246	0.000185722
10	0.687105084	0.689618923	0.689765626	0.002660543	0.000146703

6. CONCLUSIONS

In this paper, VIM by Laplace transform is applied to non-linear oscillators to compute the analytical results. Earlier, two techniques, MMA and AFF, were used for analytical results. Our technique, VIM with Laplace, coincides analytically with MMA and AFF, but is graphically slightly different than that of the numerical solution by fourth-order RK method, MMA and AFF.

REFERENCES

1. Ganji DD, Azimi M. Application of max min approach and amplitude frequency formulation to nonlinear oscillation systems. UPB Scientific Bulletin. 2012 Jan 1;74(3):131-40.
2. Suleman M, Lu D, Yue C, Ul Rahman J, Anjum N. He–Laplace method for general nonlinear periodic solitary solution of vibration equations. Journal of Low Frequency Noise, Vibration and Active Control. 2019 Dec;38(3-4):1297-304.
3. He JH. A short remark on fractional variational iteration method. Physics Letters A. 2011 Sep 5;375(38):3362-4.

DOI 10.2478/ama-2022-0012

4. He JH. Variational iteration method–a kind of non-linear analytical technique: some examples. International journal of non-linear mechanics. 1999 Jul 1;34(4):699-708.
5. He JH. Variational principles for some nonlinear partial differential equations with variable coefficients. Chaos, Solitons & Fractals. 2004 Mar 1;19(4):847-51.
6. He JH. Variational approach to (2+ 1)-dimensional dispersive long water equations. Physics Letters A. 2005 Feb 7;335(2-3):182-4.
7. ul Rahman J, Mohyuddin MR, Anjum N, Zahoor S. Mathematical Modelling and Simulation of Mixing of Salt in 3-Interconnected Tanks. Journal of Advances in Civil Engineering. 2015;1(1):1-6.
8. Anjum N, Ain QT. Application of He's fractional derivative and fractional complex transform for time fractional Camassa-Holm equation. Thermal Science. 2020;24(5 Part A):3023-30.
9. Anjum N, He JH. Analysis of nonlinear vibration of nano/ microelectromechanical system switch induced by electromagnetic force under zero initial conditions. Alexandria Engineering Journal. 2020 Dec 1;59(6):4343-52.
10. Ain QT, Anjum N, He CH. An analysis of time-fractional heat transfer problem using two-scale approach. GEM-International Journal on Geomathematics. 2021 Dec;12(1):1-0
11. He JH, El-Dib YO. Homotopy perturbation method for Fangzhu oscillator. Journal of Mathematical Chemistry. 2020 Nov; 58(10): 2245-53.
12. He JH, El-Dib YO, Mady AA. Homotopy perturbation method for the fractal toda oscillator. Fractal and Fractional. 2021 Sep;5(3):93.
13. Suleman M, Lu D, He JH, Farooq U, Hui YS, Rahman JU. Numerical investigation of fractional HIV model using Elzaki projected differential transform method. Fractals. 2018 Oct 5;26(05):1850062.
14. Ul Rahman J, Lu D, Suleman M, He JH, Ramzan M. He–Elzaki method for spatial diffusion of biological population. Fractals. 2019 Aug 13;27(05):1950069.
15. He CH, Liu C, He JH, Gepreel KA. Low frequency property of a fractal vibration model for a concrete beam. Fractals. 2021;29(5):2150117-33.
16. Anjum N, He JH. Higher-order homotopy perturbation method for conservative nonlinear oscillators generally and microelectro-mechanical systems' oscillators particularly. International Journal of Modern Physics B. 2020 Dec 30;34(32):2050313.
17. Tian D, Ain QT, Anjum N, He CH, Cheng B. Fractal N/MEMS: from pull-in instability to pull-in stability. Fractals. 2021 Mar 10;29(02):2150030.
18. Ain QT, Anjum N, He CH. An analysis of time-fractional heat transfer problem using two-scale approach. GEM-International Journal on Geomathematics. 2021 Dec;12(1):1-0.
19. Ain QT, He JH, Anjum N, Ali M. The fractional complex transform: A novel approach to the time-fractional Schrödinger equation. Fractals. 2020 Nov 2;28(07):2050141.
20. ul Rahman J, Mohyuddin MR, Anjum N, Butt R. Modelling of Two Interconnected Spring Carts and Minimization of Energy. DJ Journal of Engineering and Applied mathematics. 2016;2(1):7-11.
21. Ali M, Anjum N, Ain QT, He JH. Homotopy perturbation method for the attachment oscillator arising in nanotechnology. Fibers and Polymers. 2021 Jun;22(6):1601-6.
22. Rahman JU, Suleman M, Anjum N. Solution of unbounded boundary layer equation using modified homotopy perturbation method. Int. J. Macro Nano Phys. 2018;3(1):11-5.
23. He JH. Some asymptotic methods for strongly nonlinear equations. International journal of Modern physics B. 2006 Apr 20;20(10):1 141-99.
24. Noor MA, Mohyud-Din ST. Variational iteration method for solving higher-order nonlinear boundary value problems using He's polynomials. International Journal of Nonlinear Sciences and Numerical Simulation. 2008 Jun 1;9(2):141-56.
25. He JH. Generalized equilibrium equations for shell derived from a generalized variational principle. Applied Mathematics Letters. 2017 Feb 1;64:94-100.
26. He JH. An alternative approach to establishment of a variational principle for the torsional problem of piezoelastic beams. Applied Mathematics Letters. 2016 Feb 1;52:1-3.
27. Wu Y, He JH. A remark on Samuelson's variational principle in economics. Applied Mathematics Letters. 2018 Oct 1;84:143-7.
28. Anjum N, He JH. Laplace transform: making the variational iteration method easier. Applied Mathematics Letters. 2019 Jun 1;92:134-8.
29. He JH. Variational iteration method—some recent results and new interpretations. Journal of computational and applied mathematics. 2007 Oct 1;207(1):3-17.
30. He JH, Wu XH. Variational iteration method: new development and applications. Computers & Mathematics with Applications. 2007 Oct 1;54(7-8):881-94.
31. He JH. Variational iteration method for autonomous ordinary differential systems. Applied mathematics and computation. 2000 Sep 11;114(2-3):115-23.
32. He JH. Variational theory for linear magneto-electro-elasticity. International Journal of Nonlinear Sciences and Numerical Simulation. 2001 Dec 1;2(4):309-16.
33. He J. Variational iteration method for delay differential equations. Communications in Nonlinear Science and Numerical Simulation. 1997 Dec 1;2(4):235-6.
34. He JH. An improved amplitude-frequency formulation for nonlinear oscillators. International Journal of Nonlinear Sciences and Numerical Simulation. 2008 Jun 1;9(2):211-2.

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