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Modelling safety of multistate systems with dependent components and subsystems

Keywords

safety, modelling, dependency, load sharing model

Abstract

This report focuses on the safety analysis in the multistate ageing system after changing the safety state by any of its components, the inside interactions among the remaining components may cause the change of those components' safety states. As we consider multistate systems, in a local, equal and mixed load sharing model of dependencies. For them, we analyze the safety of a multistate parallel-series and "*m* out of *l*"-series system, assuming that the lifetimes of subsystem components in the safety state subset are decreasing according to the local load sharing rule and the lifetimes of subsystems are decreasing according to the equal load sharing rule.

1. Introduction

In the reliability and safety analysis the independence of system components is often assumed, that could taper its applicability in many practical situations. The problem of system components' dependence applies to both two-state systems and multistate systems. In two-state systems reliability analysis we consider dependencies of component failures and we assume that after failure of any system components its load is transmitted to the remaining surviving components causing the decreasing of components lifetimes. In multistate systems reliability analysis instead of components' failures we consider components' departures from the reliability state subsets [Blokus-Roszkowska, Kołowrocki, 2014a,b]. This report focuses on the safety analysis in second of those cases. Namely, in the multistate ageing system after changing the safety state by any of its components, the inside interactions among the remaining components may cause the change of those components' safety states.

The key factor is a model of load sharing that best describes the impact of changes in the safety states of some components on the safety of other components. Commonly used are equal load-sharing rules and local load-sharing. The equal load sharing model, in which the load on the failed two-state component is transferred uniformly among the remaining components, has been studied early by Daniels [Daniels, 1945] and Smith [Smith, 1982, 1983] and later by Pradhan et al. [Pradhan et al., 2010]. Local load sharing rule, in which the load on the failed two-state component is transferred to other components proportionally to their distance from the failed component was introduced by Harlow and Phoenix [Harlow, Phoenix, 1978, 1982] and further analyzed by Phoenix and Smith [Phoenix, Smith, 1983]. In this report, as we consider multistate systems, in a local load sharing model of dependency the mean values of components lifetimes in the safety state subsets are changing dependently to the distance from the component that has got out of the safety state subset. While, in a equal load sharing model of dependency the mean values of components lifetimes in the safety state subsets are changing equally dependently to the number of components that have got out of the safety state subset. A multistate approach to the reliability analysis of systems with dependent components assuming equal load sharing model has been presented by the authors in [Blokus-Roszkowska, Kołowrocki, 2014a,b] and in case of local load sharing model of dependency the results can be found in [Blokus-Roszkowska, Kołowrocki, 2015a,b,c].

There can be found various models of failure dependency of system components as well as different approaches to this problem, both analytical and simulation [Jain, Gupta, 2012], [Singh, Gupta, 2012]. In [Kostandyan, Sørensen, 2014], the authors estimate reliability of a system by structural reliability approach and the failure mechanism is based on a fracture mechanics model. Cheng at all. in [Cheng at all., 2009] use a sequential Monte Carlo simulation to evaluate the reliability of systems time-varying loads and dependent failures.

With more complex structures of systems, analyzing systems composed of several subsystems, we can consider the impact of subsystem component on other components of this subsystem and the changes that have occurred within the subsystem to other subsystems. In such systems, we can meet one model of dependency between components in subsystems and another model of dependency between subsystems of a system. For such mixed load sharing model of components and subsystems dependency, we analyze the safety of a multistate parallel-series and "*m* out of *l*"-series system, assuming that the lifetimes of subsystem components in the safety state subset are decreasing according to the local load sharing rule and the lifetimes of subsystems are decreasing according to the equal load sharing rule.

2. Local load sharing model of components dependency

2.1. Approach description

We suppose as in [Kołowrocki, 2014] and [Kołowrocki, Soszyńska-Budny, 2011] that all components and a system under consideration have the safety state set $\{0,1,...,z\}$, $z \geq 1$, where the state 0 is the worst and the state z is the best. The state of a system and components degrades with time. Further, we consider a multistate series system, defined in [Kołowrocki, 2014] and [Kołowrocki, Soszyńska-Budny, 2011], composed of n ageing and independent components with the safety functions of its components

$$S_i(t,\cdot) = [S_i(t,0),S_i(t,1),...,S_i(t,z)],\ t \geq 0,\ i = 1,2,...,n, \tag{1}$$

with the coordinates

$$S_i(t,u) = P(E_i(t) \geq u \mid E_i(0) = z) = P(T_i(u) > t) \text{ for } t \geq 0,\ u = 0,1,...,z,\ i = 1,\dots,n, \tag{2}$$

where $E_i(t)$ is a component E_i state at the moment t, $t \geq 0$, given that it was in the state z at the moment $t = 0$ and $T_i(u)$, $i = 1,2,...,n$, $u = 0,1,...,z$, are independent random variables representing the lifetimes of components E_i in the safety state subset $\{u,u+1,...,z\}$, $u = 0,1,...,z$, while they were in the safety state z at the moment $t = 0$. Similarly, as in [Kołowrocki, Soszyńska-Budny, 2011], we define the safety function of a multistate system as a vector

$$\boldsymbol{S}(t,\cdot) = [\boldsymbol{S}(t,0),\boldsymbol{S}(t,1),\dots,\boldsymbol{S}(t,z)],\ t \geq 0, \tag{3}$$

with its coordinates

$$\boldsymbol{S}(t,u) = P(s(t) \geq u \mid s(0) = z) = P(T(u) > t) \text{ for } t \geq 0,\ u = 0,1,...,z, \tag{4}$$

where $s(t)$ is a system state at the moment t, $t \geq 0$, given that it was in the state z at the moment $t = 0$ and $T(u)$ is a random variable representing the lifetime of a system in the safety state subset $\{u,u+1,...,z\}$, $u = 0,1,...,z$, while it was in the safety state z at the moment $t = 0$. Under this definition $\boldsymbol{S}(t,0) = 1$ for $t \geq 0$ and further we will use 1 instead of $\boldsymbol{S}(t,0)$.

Taking into account the local load sharing model of dependency of series system components, we assume that after changing the safety state subset by one of system components to the worse safety state subset, the lifetimes of remaining system components in the safety state subsets decrease mostly for neighbour components in first line, then less for neighbour components in second line and so on. Further, we call this rule of components dependency a local load sharing (LLS) rule. More exactly, in this rule if the system component E_j, $j = 1,\dots,n$, gets out of the reliability state subset $\{u,u+1,\dots,z\}$, $u = 1,2,\dots,z$, the reliability parameters of remaining system components E_i, $i = 1,\dots,n$, $i \neq j$, are changing dependently of the distance from the component E_j that has got out of the reliability state subset $\{u,u+1,\dots,z\}$, $u = 1,2,\dots,z$. The distance is defined by $d_{ij} = |i-j|$, $i,j = 1,2,\dots,n$ and the meaning of the distance index is illustrated in Figure 1.

We denote by $E[T_i(u)]$ and $E[T_{i/j}(u)]$, $i = 1,2,...,n$, $j = 1,2,...,n$, $u = 1,2,\dots,z$, the mean values of system components' lifetimes $T_i(u)$ and $T_{i/j}(u)$, respectively, before and after departure of one fixed component E_j, $j = 1,\dots,n$, from the safety state subset $\{u,u+1,\dots,z\}$, $u = 1,2,\dots,z$. With this notation, in considered local load sharing rule, the mean values of components lifetimes in the safety state subset $\{\upsilon,\upsilon+1,\dots,z\}$, $\upsilon = u,u-1,\dots,1$, $u = 1,2,\dots,z$, are decreasing according to the following formula:

$$T_{i/j}(\upsilon) = q(\upsilon, d_{ij}) \cdot T_i(\upsilon),\ E[T_{i/j}(\upsilon)] = q(\upsilon, d_{ij}) \cdot E[T_i(\upsilon)],$$
$$i = 1,\dots,n, j = 1,\dots,n,\ \upsilon = u, u-1,\dots,1, \qquad (5)$$

where the coefficients $q(\upsilon, d_{ij})$, $0 < q(\upsilon, d_{ij}) \le 1$, $i = 1,\dots,n$, $j = 1,\dots,n$, and $q(\upsilon,0) = 1$ for $\upsilon = u,u\text{-}1,\dots,1$, $u = 1,2,\dots,z\text{-}1$, are non-increasing functions of components' distance $d_{ij} = |i - j|$ from the component that has got out of the safety state subset $\{u,u+1,\dots,z\}$, $u = 1,2,\dots,z$.
Further, we define the safety function of a component E_i, $i = 1,\dots,n$, after departure of the component E_j, $j = 1,2,\dots,n$, from the safety state subset $\{u,u+1,\dots,z\}$, $u = 1,2,\dots,z$,

$$S_{i/j}(t,\cdot) = [1, S_{i/j}(t,1),\dots, S_{i/j}(t,z)],\ t \ge 0,$$
$$i = 1,\dots,n, j = 1,\dots,n, \qquad (6)$$

with the coordinates given by

$$S_{i/j}(t,\upsilon) = P(T_{i/j}(\upsilon) > t),\ t \ge 0,\ \upsilon = u, u-1,\dots,1,$$
$$u = 1,2,\dots,z-1,$$

$$S_{i/j}(t,\upsilon) = P(T_{i/j}(\upsilon) > t) = P(T_i(\upsilon) > t) = S_i(t,\upsilon),$$
$$\upsilon = u+1,\dots,z,\ u = 1,2,\dots,z-1. \qquad (7)$$

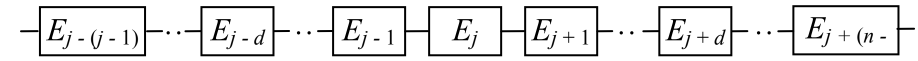

Figure 1. The meaning of the distance *d*.

2.2. Safety of multistate series system with dependent components

In this section we formulate the theorem concerned with safety of a series system with dependent components.

Proposition 1. If in a multistate series system components are dependent according to the local load sharing rule and have safety functions (1)-(2), then its safety function is given by the vector

$$\boldsymbol{S}_{LLS}(t,\cdot) = [1, \boldsymbol{S}_{LLS}(t,1),\dots, \boldsymbol{S}_{LLS}(t,z)],\ t \ge 0, \qquad (8)$$

with the coordinates

$$\boldsymbol{S}_{LLS}(t,u) = \prod_{i=1}^{n} S_i(t,u+1)$$
$$+ \int_0^t \sum_{j=1}^{n} [\tilde{f}_j(a,u+1) \cdot \prod_{\substack{i=1 \\ i \ne j}}^{n} S_i(a,u+1) \cdot S_j(a,u)$$
$$\cdot \prod_{i=1}^{n} S_{i/j}(t-a,u) da,\ u = 1,2,\dots,z-1, \qquad (9)$$

$$\boldsymbol{S}_{LLS}(t,z) = \prod_{i=1}^{n} S_i(t,z), \qquad (10)$$

where:
$S_i(t,u+1)$ – the safety function coordinate of a component E_i, $i = 1,\dots,n$, i.e. the probability that its lifetime in the safety state subset $\{u+1,\dots,z\}$, $u = 1,2,\dots,z\text{-}1$, is greater than t,

$\tilde{f}_j(t,u+1)$ – the density function coordinate of a component E_j, $j = 1,\dots,n$, corresponding to the distribution function $\tilde{F}_j(t,u+1)$, defined as the probability of a component E_j, $j = 1,\dots,n$, departure from the safety state subset $\{u+1,\dots,z\}$, $u = 1,2,\dots,z\text{-}1$, before the time t, under condition that its lifetime in the safety state subset $\{u,u+1,\dots,z\}$, $u = 1,2,\dots,z$, is greater than t, given by

$$\tilde{F}_j(t,u+1) = 1 - \frac{S_j(t,u+1)}{S_j(t,u)},$$
$$u = 1,2,\dots,z-1,\ t \ge 0, \qquad (11)$$

$S_j(t,u)$ – the safety function coordinate of a component E_j, $j = 1,\dots,n$, i.e. the probability that its lifetime in the safety state subset $\{u,u+1,\dots,z\}$, $u = 1,2,\dots,z$, is greater than t,
$S_{i/j}(t,u)$ – the safety function coordinate of a component E_i, $i = 1,\dots,n$, i.e. the probability that its lifetime in the safety conditional state subset $\{u,u+1,\dots,z\}$, $u = 1,2,\dots,z\text{-}1$, after departure from the safety state subset $\{u+1,\dots,z\}$, $u = 1,2,\dots,z\text{-}1$, by the component E_j, $j = 1,\dots,n$, is greater than t, such that

$$S_{i/j}(t-a,u) = \frac{S_{i/j}(t,u)}{S_i(a,u)},\ u = 1,2,\dots,z-1,$$
$$0 < a < t,\quad t \ge 0. \qquad (12)$$

Further, we consider a homogeneous multistate series system composed of components dependent according to the local load sharing rule having identical safety functions of the form

$$S(t,\cdot)=[1,S(t,1),\ldots,S(t,z)],\ t\geq 0. \quad (13)$$

Then, we get a particular case of *Proposition 1* formulated below.

Proposition 2. If in a homogeneous multistate series system components are dependent according to the local load sharing rule and have safety functions (13), then its safety function is given by the vector

$$\boldsymbol{S}_{LLS}(t,\cdot)=[1,\boldsymbol{S}_{LLS}(t,1),\ldots,\boldsymbol{S}_{LLS}(t,z)],\ t\geq 0, \quad (14)$$

with the coordinates

$$\boldsymbol{S}_{LLS}(t,u)=[S(t,u+1)]^{n} + \int_{0}^{t}\sum_{j=1}^{n}[\tilde{f}(a,u+1)\cdot[S(a,u+1)]^{n-1}\cdot S(a,u) \cdot\prod_{i=1}^{n}S_{i/j}(t-a,u)da,\ u=1,2,\ldots,z-1, \quad (15)$$

$$\boldsymbol{S}_{LLS}(t,z)=[S(t,z)]^{n}, \quad (16)$$

where:
$S(t,u+1)$ − the safety function coordinate of a component,
$\tilde{f}(t,u+1)$ − the density function coordinate of a system component corresponding to the distribution function $\tilde{F}(t,u+1)$, given by

$$\tilde{F}(t,u+1)=1-\frac{S(t,u+1)}{S(t,u)},\ u=1,2,\ldots,z-1, \quad t\geq 0, \quad (17)$$

$S(t,u)$ − the safety function coordinate of a component,
$S_{i/j}(t,u)$ − the safety function coordinate of a component E_i, i = 1,…,n, after departure of the component E_j, j = 1,…,n, from the safety state subset {u+1,…,z}, u = 1,2,…,z-1, such that

$$S_{i/j}(t-a,u)=\frac{S_{i/j}(t,u)}{S(a,u)},\ u=1,2,\ldots,z-1, \quad 0<a<t,\ t\geq 0. \quad (18)$$

2.3. Multistate series systems with dependent components having exponential safety functions

We assume that components E_i, i = 1,…,n, have exponential safety functions

$$S_i(t,\cdot)=[1,S_i(t,1),\ldots,S_i(t,z)],\ t\geq 0, \quad (19)$$

with the coordinates

$$S_i(t,u)=\exp[-\lambda_i(u)t],\ u=1,2,\ldots,z, \quad (20)$$

where $\lambda_i(u)$, $\lambda_i(u)\geq 0$, i = 1,…,n, are components' intensities of departure from the safety state subset {u,u+1,…,z}, u = 1,2,…,z. Then, according to the well known relationship between the lifetime mean value in this safety state subset and the intensity of departure from this safety state subset of the form

$$E[T_i(u)]=\frac{1}{\lambda_i(u)},\ u=1,2,\ldots,z,$$

we get the formula for the intensities $\lambda_{i/j}(\upsilon)$, i = 1,…,n, j = 1,…,n, υ = u,u-1,…,1, of components' departure from this safety state subset after the departure of the jth component E_j, j = 1,…,n, from that safety state subset. Namely, from (5), we obtain

$$\lambda_{i/j}(\upsilon)=\frac{\lambda_i(\upsilon)}{q(\upsilon,d_{ij})},\ \upsilon=u,u-1,\ldots,1. \quad (21)$$

Thus, considering (19)-(20) and (21), the components E_i, i = 1,…,n, after the departure of the jth component E_j, j = 1,…,n, from that safety state subset {u,u+1,…,z}, u = 1,2,…,z, have the safety functions

$$S_{i/j}(t,\cdot)=[1,S_{i/j}(t,1),\ldots,S_{i/j}(t,u)],\ t\geq 0, \quad i=1,\ldots,n, j=1,\ldots,n, \quad (22)$$

with the coordinates

$$S_{i/j}(t,\upsilon)=\exp[-\frac{\lambda_i(\upsilon)}{q(\upsilon,d_{ij})}t],\ \upsilon=u,u-1,\ldots,1, \quad u=1,2,\ldots,z-1, \quad (23)$$

$$S_{i/j}(t,\upsilon)=\exp[-\lambda_i(\upsilon)t],\ i=1,\ldots,n, j=1,\ldots,n, \quad \upsilon=u+1,\ldots,z,\ u=1,2,\ldots,z-1.$$

Further for the exponential multistate series system with dependent components, the distribution function corresponding to the system component E_j , given by (11), takes form

$$\tilde{F}_j(t,u+1)=1-\frac{\exp[-\lambda_j(u+1)t]}{\exp[-\lambda_j(u)t]}$$

$$= 1 - \exp[-(\lambda_j(u+1) - \lambda_j(u))t], \tag{24}$$

and its corresponding density function is

$$\tilde{f}_j(t, u+1) = (\lambda_j(u+1) - \lambda_j(u))$$
$$\cdot \exp[-(\lambda_j(u+1) - \lambda_j(u))t],$$
$$u = 1,2,\ldots,z-1, \ t \geq 0. \tag{25}$$

Considering (22)-(25), in case the system components have exponential safety functions from *Proposition 1* we can obtain the following result.

Proposition If in a multistate series system components are dependent according to the local load sharing rule and have exponential safety functions (19)-(20), then its safety function is given by the vector

$$\boldsymbol{S}_{LLS}(t,\cdot) = [1, \boldsymbol{S}_{LLS}(t,1), \ldots, \boldsymbol{S}_{LLS}(t,z)], \ t \geq 0, \tag{26}$$

with the coordinates

$$\boldsymbol{S}_{LLS}(t,u) = \exp[-\sum_{i=1}^{n} \lambda_i(u+1)t]$$
$$+ \sum_{j=1}^{n} \frac{\lambda_j(u+1) - \lambda_j(u)}{\sum_{i=1}^{n} \lambda_i(u+1) - \sum_{i=1}^{n} \lambda_i(u)}$$
$$\cdot [\exp[-\sum_{i=1}^{n} \frac{\lambda_i(u)}{q(u,d_{ij})} t]$$
$$- \exp[-(\sum_{i=1}^{n} \lambda_i(u+1) - \sum_{i=1}^{n} \lambda_i(u) + \sum_{i=1}^{n} \frac{\lambda_i(u)}{q(u,d_{ij})})t]],$$
$$u = 1,2,\ldots,z-1, \tag{27}$$

$$\boldsymbol{S}_{LLS}(t,z) = \exp[-\sum_{i=1}^{n} \lambda_i(z)t]. \tag{28}$$

Corollary 1. If in a multistate series system components are dependent according to the local load sharing rule and have exponential safety functions given by (19)-(20), then its mean lifetime in the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, is given by

$$\mu_{LLS}(u) = \frac{1}{\sum_{i=1}^{n} \lambda_i(u+1)} + \sum_{j=1}^{n} \frac{\lambda_j(u+1) - \lambda_j(u)}{\sum_{i=1}^{n} \lambda_i(u+1) - \sum_{i=1}^{n} \lambda_i(u)}$$
$$\cdot [\frac{1}{\sum_{i=1}^{n} \frac{\lambda_i(u)}{q(u,d_{ij})}} - \frac{1}{\sum_{i=1}^{n} \lambda_i(u+1) - \sum_{i=1}^{n} \lambda_i(u) + \sum_{i=1}^{n} \frac{\lambda_i(u)}{q(u,d_{ij})}}],$$
$$u = 1,2,\ldots,z-1, \tag{29}$$

$$\mu_{LLS}(z) = \frac{1}{\sum_{i=1}^{n} \lambda_i(z)}, \tag{30}$$

and the standard deviation of the system sojourn time in the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, is given by

$$\sigma_{LLS}(u) = \sqrt{n_{LLS}(u) - [\mu_{LLS}(u)]^2}, \ u = 1,2,\ldots,z-1, \tag{31}$$

where

$$n_{LLS}(u) = \frac{2}{[\sum_{i=1}^{n} \lambda_i(u+1)]^2} + 2\sum_{j=1}^{n} \frac{\lambda_j(u+1) - \lambda_j(u)}{\sum_{i=1}^{n} \lambda_i(u+1) - \sum_{i=1}^{n} \lambda_i(u)}$$
$$\cdot [\frac{1}{[\sum_{i=1}^{n} \frac{\lambda_i(u)}{q(u,d_{ij})}]^2}$$
$$- \frac{1}{[\sum_{i=1}^{n} \lambda_i(u+1) - \sum_{i=1}^{n} \lambda_i(u) + \sum_{i=1}^{n} \frac{\lambda_i(u)}{q(u,d_{ij})}]^2}],$$
$$u = 1,2,\ldots,z-1, \tag{32}$$

and

$$\sigma_{LLS}(z) = \frac{1}{\sum_{i=1}^{n} \lambda_i(z)}. \tag{33}$$

Next, we assume that the safety functions of system components (13) have exponential coordinates

$$S(t,u) = \exp[-\lambda(u)t], t \geq 0, \lambda(u) \geq 0,$$
$$u = 1,2,\ldots,z, \tag{34}$$

where $\lambda(u)$, $u = 1,2,\ldots,z$, are components' intensities of departure from the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$. Then, according to the relationship between the lifetime mean value in this safety state subset and the intensity of departure from this safety state subset, we get the formula for the intensities $\lambda_{i/j}(\upsilon)$, $i = 1,\ldots,n$, $j = 1,\ldots,n$, $\upsilon = u,u\text{-}1,\ldots,1$, of components' departure from this safety state subset after the departure of the jth component E_j, $j = 1,\ldots,n$. Namely, from (5), we obtain

$$\lambda_{i/j}(\upsilon) = \frac{\lambda(\upsilon)}{q(\upsilon,d_{ij})}, \ \upsilon = u, u-1,\ldots,1. \tag{35}$$

In this case *Proposition 3* takes the form presented below.

Proposition 4. If in a homogeneous multistate series system components are dependent according to the local load sharing rule and have exponential safety functions with the coordinates (34), then its safety function is given by the vector

$$\boldsymbol{S}_{LLS}(t,\cdot)=[1,\boldsymbol{S}_{LLS}(t,1),\ldots,\boldsymbol{S}_{LLS}(t,z)],\ t\geq 0, \quad (36)$$

with the coordinates

$$\boldsymbol{S}_{LLS}(t,u)=\exp[-n\lambda(u+1)t]$$
$$+\frac{1}{n}\sum_{j=1}^{n}[\exp[-\lambda(u)\sum_{i=1}^{n}\frac{1}{q(u,d_{ij})}t]$$
$$-\exp[-(n\lambda(u+1)-n\lambda(u)+\lambda(u)\sum_{i=1}^{n}\frac{1}{q(u,d_{ij})})t]],$$
$$u=1,2,\ldots,z-1, \quad (37)$$

$$\boldsymbol{S}_{LLS}(t,z)=\exp[-n\lambda(z)t]. \quad (38)$$

From *Proposition 4*, we immediately obtain a corollary concerned with the mean values and standard deviations of the lifetimes in the safety state subsets of a homogeneous multistate series system.

Corollary 2. If in a homogeneous multistate series system components are dependent according to the local load sharing rule and have exponential safety functions with the coordinates (34), then its mean lifetime in the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, is given by

$$\mu_{LLS}(u)=\frac{1}{n\lambda(u+1)}+\sum_{j=1}^{n}\frac{1}{n}$$
$$\cdot[\frac{1}{\lambda(u)\sum_{i=1}^{n}\frac{1}{q(u,d_{ij})}}$$
$$-\frac{1}{n\lambda(u+1)-n\lambda(u)+\lambda(u)\sum_{i=1}^{n}\frac{1}{q(u,d_{ij})}}],$$
$$u=1,2,\ldots,z-1, \quad (39)$$

$$\mu_{LLS}(z)=\frac{1}{n\lambda(z)}, \quad (40)$$

and the standard deviation of the system sojourn time in the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, is given by

$$\sigma_{LLS}(u)=\sqrt{n_{LLS}(u)-[\mu_{LLS}(u)]^2},$$
$$u=1,2,\ldots,z-1, \quad (41)$$

where

$$n_{LLS}(u)=\frac{2}{[n\lambda(u+1)]^2}+2\sum_{j=1}^{n}\frac{1}{n}\cdot[\frac{1}{[\lambda(u)\sum_{i=1}^{n}\frac{1}{q(u,d_{ij})}]^2}$$
$$-\frac{1}{[n\lambda(u+1)-n\lambda(u)+\lambda(u)\sum_{i=1}^{n}\frac{1}{q(u,d_{ij})}]^2}], \quad (42)$$

and

$$\sigma_{LLS}(z)=\frac{1}{n\lambda(z)}. \quad (43)$$

Using *Proposition 4*, we can find another, practically very important, safety characteristics of the considered homogeneous multistate system with dependent components, namely the system intensities of departures from the safety state subset.

Corollary 3. If in a homogeneous multistate series system components are dependent according to the local load sharing rule and have exponential safety functions with the coordinates (34), then the intensities of its departures from the safety state subsets $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, are given by

$$\boldsymbol{\lambda}_{LLS}(t,u)=\{n\lambda(u+1)+\sum_{j=1}^{n}\frac{1}{n}$$
$$\cdot[\lambda(u)\sum_{i=1}^{n}\frac{1}{q(u,d_{ij})}\cdot\exp[n\lambda(u+1)t-\lambda(u)\sum_{i=1}^{n}\frac{1}{q(u,d_{ij})}t]$$
$$-(n\lambda(u+1)-n\lambda(u)+\lambda(u)\sum_{i=1}^{n}\frac{1}{q(u,d_{ij})})$$
$$\cdot\exp[n\lambda(u)t-\lambda(u)\sum_{i=1}^{n}\frac{1}{q(u,d_{ij})}t]]\}$$

$$/\{1+\sum_{j=1}^{n}\frac{1}{n}\cdot[\exp[n\lambda(u+1)t-\lambda(u)\sum_{i=1}^{n}\frac{1}{q(u,d_{ij})}t]$$
$$-\exp[n\lambda(u)t-\lambda(u)\sum_{i=1}^{n}\frac{1}{q(u,d_{ij})}t]]\},$$
$$u=1,2,\ldots,z-1, \quad (44)$$

$$\boldsymbol{\lambda}_{LLS}(t,z)=n\lambda(z). \quad (45)$$

2.4. Multistate series systems with dependent components having Rayleigh safety functions

We assume that components E_i, $i = 1,\dots,n$, have Rayleigh safety functions

$$S_i(t,\cdot) = [1, S_i(t,1),\dots,S_i(t,z)],\ t \ge 0, \quad (46)$$

with the coordinates

$$S_i(t,u) = \exp[-\lambda_i(u)t^2],\ u = 1,2,\dots,z, \quad (47)$$

where $\lambda_i(u)$, $\lambda_i(u) \ge 0$, $i = 1,\dots,n$, are components' intensities of departure from the safety state subset $\{u,u+1,\dots,z\}$, $u = 1,2,\dots,z$.

Then, the distribution function $\tilde{F}_j(t,u+1)$, defined by (11), is given by

$$\tilde{F}_j(t,u+1) = 1 - \frac{\exp[-\lambda_j(u+1)t^2]}{\exp[-\lambda_j(u)t^2]}$$
$$= 1 - \exp[-(\lambda_j(u+1) - \lambda_j(u))t^2],\ u = 1,2,\dots,z-1,$$
$$t \ge 0,$$

and the corresponding density function is

$$\tilde{f}_j(t,u+1) = 2(\lambda_j(u+1) - \lambda_j(u))t$$
$$\cdot \exp[-(\lambda_j(u+1) - \lambda_j(u))t^2],\ u = 1,2,\dots,z-1,$$
$$t \ge 0. \quad (48)$$

According to the relationship between the lifetime mean value in this safety state subset and the intensity of departure from this safety state subset of the form

$$E[T_i(u)] = \frac{1}{2}\sqrt{\frac{\pi}{\lambda_i(u)}},\ u = 1,2,\dots,z,$$

we get the formula for the intensities $\lambda_{i/j}(\upsilon)$, $i = 1,\dots,n$, $j = 1,\dots,n$, $\upsilon = u,u-1,\dots,1$, of components' departure from this safety state subset after the departure of the jth component E_j, $j = 1,\dots,n$, from that safety state subset. Namely, from (5), assuming local sharing rule we obtain

$$\lambda_{i/j}(\upsilon) = \frac{\lambda_i(\upsilon)}{[q(\upsilon,d_{ij})]^2},\ \upsilon = u,u-1,\dots,1. \quad (49)$$

Thus, considering (46)-(47) and (49), the components E_i, $i = 1,\dots,n$, after the departure of the jth component E_j, $j = 1,\dots,n$, from that safety state subset $\{u,u+1,\dots,z\}$, $u = 1,2,\dots,z$, have the safety functions

$$S_{i/j}(t,\cdot) = [1, S_{i/j}(t,1),\dots,S_{i/j}(t,u)],\ t \ge 0,$$
$$i = 1,\dots,n,\ j = 1,\dots,n, \quad (50)$$

with the coordinates

$$S_{i/j}(t,\upsilon) = \exp[-\lambda_{i/j}(\upsilon)t^2] = \exp[-\frac{\lambda_i(\upsilon)}{[q(\upsilon,d_{ij})]^2}t^2],$$
$$\upsilon = u,u-1,\dots,1,\ u = 1,2,\dots,z-1, \quad (51)$$

$$S_{i/j}(t,\upsilon) = \exp[-\lambda_i(\upsilon)t^2],\ \upsilon = u+1,\dots,z,$$
$$u = 1,2,\dots,z-1.$$

Considering (48) and (50)-(51), in case the system components have Rayleigh distribution from *Proposition 1* we can obtain the following result. More specifically, substituting the safety function coordinates of components (47), the density function coordinates (48) and the safety function coordinates of components after the departure one of components from the safety state subset given by (51) into (8)-(10), we obtain the thesis of the *Proposition 5*.

Proposition 5. If in a multistate series system components are dependent according to the local load sharing rule and have Rayleigh safety functions (46)-(47), then its safety function is given by the vector

$$\boldsymbol{S}_{LLS}(t,\cdot) = [1, \boldsymbol{S}_{LLS}(t,1),\dots,\boldsymbol{S}_{LLS}(t,z)],\ t \ge 0, \quad (52)$$

with the coordinates

$$\boldsymbol{S}_{LLS}(t,u) = \exp[-\sum_{i=1}^{n}\lambda_i(u+1)t^2]$$
$$+ \sum_{j=1}^{n}\frac{\lambda_j(u+1) - \lambda_j(u)}{\sum_{i=1}^{n}(\lambda_i(u+1) - \lambda_i(u))}] \cdot [\exp[-\sum_{i=1}^{n}\frac{\lambda_i(u)}{[q(u,d_{ij})]^2}t^2]$$
$$- \exp[-\sum_{i=1}^{n}(\lambda_i(u+1) - \lambda_i(u) + \frac{\lambda_i(u)}{[q(u,d_{ij})]^2})t^2]]$$
$$u = 1,2,\dots,z-1, \quad (53)$$

$$\boldsymbol{S}_{LLS}(t,z) = \exp[-\sum_{i=1}^{n}\lambda_i(z)t^2]. \quad (54)$$

In case of a homogeneous multistate series system with components having Rayleigh safety functions with the intensity parameter $\lambda(u)$, $u = 1,2,\dots,z$, given by (13) with the coordinates

$$S(t,u) = \exp[-\lambda(u)t^2],\ t \ge 0,\ \lambda(u) \ge 0,$$
$$u = 1,2,\dots,z, \quad (55)$$

Proposition 5 takes the form presented below.

Proposition 6. If in a homogeneous multistate series system components are dependent according to the local load sharing rule and have Rayleigh safety functions with the coordinates (55), then its safety function is given by the vector

$$\boldsymbol{S}_{LLS}(t,\cdot) = [1, \boldsymbol{S}_{LLS}(t,1), \dots, \boldsymbol{S}_{LLS}(t,z)],\ t \ge 0, \qquad (56)$$

with the coordinates

$$\boldsymbol{S}_{LLS}(t,u) = \exp[-n\lambda(u+1)t^2] + \frac{1}{n}\sum_{j=1}^{n}[\exp[-\lambda(u)\sum_{i=1}^{n}\frac{1}{[q(u,d_{ij})]^2}t^2] - \exp[-(n\lambda(u+1) - n\lambda(u) + \lambda(u)\sum_{i=1}^{n}\frac{1}{[q(u,d_{ij})]^2})t^2]],$$
$$u = 1,2,\dots,z-1, \qquad (57)$$

$$\boldsymbol{S}_{LLS}(t,z) = \exp[-n\lambda(z)t^2]. \qquad (58)$$

2.5. Multistate series systems with dependent components having Erlang safety functions

We assume that components E_i, $i = 1,\dots,n$, have Erlang-l safety functions with the intensity parameter $\lambda(u)$, $u = 1,2,\dots,z$, given by

$$S(t,\cdot) = [1, S(t,1), \dots, S(t,z)],\ t \ge 0, \qquad (59)$$

with the coordinates

$$S(t,u) = \sum_{\omega=0}^{l-1}\frac{[\lambda(u)t]^{\omega}}{\omega!}\exp[-\lambda(u)t],\ t \ge 0, \lambda(u) \ge 0,$$
$$u = 1,2,\dots,z. \qquad (60)$$

Then, the distribution function $\tilde{F}(t,u+1)$ defined by (17), is given by

$$\tilde{F}(t,u+1) = 1 - \frac{\sum_{\omega=0}^{l-1}\frac{[\lambda(u+1)t]^{\omega}}{\omega!}}{\sum_{\omega=0}^{l-1}\frac{[\lambda(u)t]^{\omega}}{\omega!}}\exp[-(\lambda(u+1) - \lambda(u))t],$$
$$u = 1,2,\dots,z-1,\ t \ge 0, \qquad (61)$$

and the corresponding density function is

$$\tilde{f}(t,u+1) = \frac{d}{dt}(1 - \frac{\sum_{\omega=0}^{l-1}\frac{[\lambda(u+1)t]^{\omega}}{\omega!}}{\sum_{\omega=0}^{l-1}\frac{[\lambda(u)t]^{\omega}}{\omega!}}\exp[-(\lambda(u+1) - \lambda(u))t])$$
$$= [\frac{\sum_{\omega=0}^{l-1}\frac{[\lambda(u+1)t]^{\omega}}{\omega!}\cdot\sum_{\omega=0}^{l-2}\frac{[\lambda(u)]^{\omega+1}t^{\omega}}{\omega!}}{[\sum_{\omega=0}^{l-1}\frac{[\lambda(u)t]^{\omega}}{\omega!}]^2} - \frac{\sum_{\omega=0}^{l-2}\frac{[\lambda(u+1)]^{\omega+1}t^{\omega}}{\omega!}\cdot\sum_{\omega=0}^{l-1}\frac{[\lambda(u)t]^{\omega}}{\omega!}}{[\sum_{\omega=0}^{l-1}\frac{[\lambda(u)t]^{\omega}}{\omega!}]^2} + \frac{\sum_{\omega=0}^{l-1}\frac{[\lambda(u+1)t]^{\omega}}{\omega!}(\lambda(u+1) - \lambda(u))}{\sum_{\omega=0}^{l-1}\frac{[\lambda(u)t]^{\omega}}{\omega!}}]$$
$$\cdot\exp[-(\lambda(u+1) - \lambda(u))t],$$
$$u = 1,2,\dots,z-1,\ t \ge 0. \qquad (62)$$

According to the relationship between the lifetime mean value in this safety state subset and the intensity of departure from this safety state subset of the form

$$E[T_i(u)] = \frac{l}{\lambda(u)},\ u = 1,2,\dots,z,$$

we get the formula for the intensities $\lambda_{i/j}(\upsilon)$, $i = 1,\dots,n$, $j = 1,\dots,n$, $\upsilon = u,u-1,\dots,1$, of components' departure from this safety state subset after the departure of the jth component E_j, $j = 1,\dots,n$, from that safety state subset. Namely, from (5), assuming local sharing rule we obtain

$$\lambda_{i/j}(\upsilon) = \frac{\lambda(\upsilon)}{q(\upsilon,d_{ij})},\ \ \upsilon = u,u-1,\dots,1. \qquad (63)$$

Thus, considering (59)-(60) and (63), the components E_i, $i = 1,\dots,n$, after the departure of the jth component E_j, $j = 1,\dots,n$, from that safety state subset $\{u,u+1,\dots,z\}$, $u = 1,2,\dots,z$, have the safety functions

$$S_{i/j}(t,\cdot) = [1, S_{i/j}(t,1), \dots, S_{i/j}(t,u)],\ t \ge 0,$$
$$i = 1,\dots,n,\ j = 1,\dots,n, \qquad (64)$$

with the coordinates

$$S_{i/j}(t,\upsilon) = \sum_{\omega=0}^{l-1}\frac{[\lambda_{i/j}(\upsilon)t]^{\omega}}{\omega!}\exp[-\lambda_{i/j}(\upsilon)t]$$

$$= \sum_{\omega=0}^{l-1} \frac{[\frac{\lambda(\upsilon)}{q(\upsilon,d_{ij})} t]^{\omega}}{\omega!} \exp[-\frac{\lambda(\upsilon)}{q(\upsilon,d_{ij})} t],$$

$$\upsilon = u, u-1, \ldots, 1, \tag{65}$$

$$S_{i/j}(t,\upsilon) = \sum_{\omega=0}^{l-1} \frac{[\lambda_i(\upsilon)t]^{\omega}}{\omega!} \exp[-\lambda_i(\upsilon)t],$$

$$\upsilon = u+1, \ldots, z, \text{ for } u = 1,2,\ldots, z-1.$$

Considering (64)-(65), in case the system components have Erlang safety functions from *Proposition 2* we can obtain the following result. More specifically, substituting the safety function coordinates of components (60) and the safety function coordinates of components after the departure one of components from the safety state subset given by (65) into (14)-(16), we obtain the thesis of the *Proposition 7*.

Proposition 7. If in a homogeneous multistate series system components are dependent according to the local load sharing rule and have Erlang safety functions (59)-(60), then its safety function is given by the vector

$$\boldsymbol{S}_{LLS}(t,\cdot) = [1, \boldsymbol{S}_{LLS}(t,1), \ldots, \boldsymbol{S}_{LLS}(t,z)],\ t \geq 0, \tag{66}$$

with the coordinates

$$\boldsymbol{S}_{LLS}(t,u) = [\sum_{\omega=0}^{l-1} \frac{[\lambda(u+1)t]^{\omega}}{\omega!}]^{n} \cdot \exp[-n\lambda(u+1)t]$$
$$+ \sum_{j=1}^{n} \int_{0}^{t} \tilde{f}(a,u+1) [\sum_{\omega=0}^{l-1} \frac{[\lambda(u+1)a]^{\omega}}{\omega!}]^{n-1} \cdot \sum_{\omega=0}^{l-1} \frac{[\lambda(u)a]^{\omega}}{\omega!}$$
$$\cdot \exp[-((n-1)\lambda(u+1) + \lambda(u))a]$$
$$\cdot [\prod_{i=1}^{n} \frac{\sum_{\omega=0}^{l-1} \frac{[\frac{\lambda(u)}{q(u,d_{ij})} t]^{\omega}}{\omega!}}{\sum_{\omega=0}^{l-1} \frac{[\lambda(u)a]^{\omega}}{\omega!}} \cdot \exp[-\frac{\lambda(u)}{q(u,d_{ij})} t + \lambda(u)a]] da,$$

$$u = 1,2,\ldots, z-1, \tag{67}$$

and after substituting the density function coordinate $\tilde{f}(a,u+1)$ given by (62) into (67) the coordinates take form

$$\boldsymbol{S}_{LLS}(t,u) = [\sum_{\omega=0}^{l-1} \frac{[\lambda(u+1)t]^{\omega}}{\omega!}]^{n} \cdot \exp[-n\lambda(u+1)t]$$
$$+ \sum_{j=1}^{n} \int_{0}^{t} [\sum_{\omega=0}^{l-1} \frac{[\lambda(u+1)a]^{\omega}}{\omega!}]^{n-1} \cdot \sum_{\omega=0}^{l-1} \frac{[\lambda(u)a]^{\omega}}{\omega!}$$
$$[\frac{\sum_{\omega=0}^{l-1} \frac{[\lambda(u+1)a]^{\omega}}{\omega!} \cdot \sum_{\omega=0}^{l-2} \frac{[\lambda(u)]^{\omega+1} a^{\omega}}{\omega!}}{[\sum_{\omega=0}^{l-1} \frac{[\lambda(u)a]^{\omega}}{\omega!}]^{2}}$$
$$- \frac{\sum_{\omega=0}^{l-2} \frac{[\lambda(u+1)]^{\omega+1} a^{\omega}}{\omega!} \cdot \sum_{\omega=0}^{l-1} \frac{[\lambda(u)a]^{\omega}}{\omega!}}{[\sum_{\omega=0}^{l-1} \frac{[\lambda(u)a]^{\omega}}{\omega!}]^{2}}$$
$$+ \frac{\sum_{\omega=0}^{l-1} \frac{[\lambda(u+1)a]^{\omega}}{\omega!} (\lambda(u+1) - \lambda(u))}{\sum_{\omega=0}^{l-1} \frac{[\lambda(u)a]^{\omega}}{\omega!}}]$$
$$\cdot \exp[-n\lambda(u+1)a]$$
$$\cdot [\prod_{i=1}^{n} \frac{\sum_{\omega=0}^{l-1} \frac{[\frac{\lambda(u)}{q(u,d_{ij})} t]^{\omega}}{\omega!}}{\sum_{\omega=0}^{l-1} \frac{[\lambda(u)a]^{\omega}}{\omega!}} \cdot \exp[-\frac{\lambda(u)}{q(u,d_{ij})} t + \lambda(u)a]] da,$$

$$u = 1,2,\ldots, z-1, \tag{68}$$

$$\boldsymbol{S}_{LLS}(t,z) = [\sum_{\omega=0}^{l-1} \frac{[\lambda(z)t]^{\omega}}{\omega!}]^{n} \cdot \exp[-n\lambda(z)t]. \tag{69}$$

2.6. Safety of a multistate series-parallel system with dependent components of its subsystems

In this section we consider a multistate regular series-parallel system, with a scheme given in Figure 2, as a parallel system of series subsystems with dependent components. We assume there are k series subsystems working independently linked parallel and in each subsystem there are l dependent components linked in series. We denote by E_{ij}, $i = 1,2,...,k$, $j = 1,2,...,l$, k, $l \in N$, components of a system and assume that all components E_{ij} have the same safety state set as before $\{0,1,...,z\}$. Then, $T_{ij}(u)$, $i = 1,2,...,k$, $j = 1,2,...,l$, k, $l \in N$, are random variables representing lifetimes of components E_{ij} in the safety state subset $\{u,u+1,...,z\}$, while they were in the safety state z at the moment $t = 0$. In each series subsystem we assume local load sharing model of components dependency described in Section 2.1.

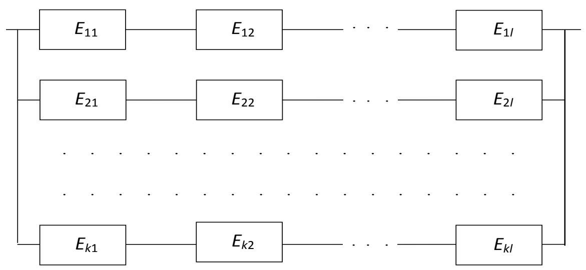

Figure 2. The scheme of a regular series-parallel system.

Then linking the results for a multistate series system assuming its components' dependency with the safety function of a parallel system with independent components [Kołowrocki, 2014], we obtain following proposition.

Proposition 8. If in a homogeneous multistate regular series-parallel system, its subsystems are working independently and components of these series subsystems are dependent according to the local load sharing rule and have safety functions (13), then its safety function is given by the vector

$$\bar{\boldsymbol{S}}_{LLS}(t,\cdot) = [1, \bar{\boldsymbol{S}}_{LLS}(t,1), \ldots, \bar{\boldsymbol{S}}_{LLS}(t,z)], \; t \geq 0, \tag{70}$$

with the coordinates

$$\bar{\boldsymbol{S}}_{LLS}(t,u) = 1 - [1 - \boldsymbol{S}_{LLS}(t,u)]^k, \; u = 1,2,\ldots,z, \tag{71}$$

where the safety function coordinates of series subsystems with local load sharing model of dependency $\boldsymbol{S}_{LLS}(t,u)$, $u = 1,2,\ldots,z$, are given by (15)-(16).
Next, considering (70)-(71), in case the system components have exponential safety functions from *Proposition 4* we can obtain the following result.

Proposition 9. If in a homogeneous multistate regular series-parallel system, its subsystems are working independently and components of these series subsystems are dependent according to the local load sharing rule and have exponential safety functions with the coordinates (34), then its safety function is given by the vector

$$\bar{\boldsymbol{S}}_{LLS}(t,\cdot) = [1, \bar{\boldsymbol{S}}_{LLS}(t,1), \ldots, \bar{\boldsymbol{S}}_{LLS}(t,z)], \; t \geq 0, \tag{72}$$

with the coordinates

$$\bar{\boldsymbol{S}}_{LLS}(t,u) = 1 - [1 - \exp[-l\lambda(u+1)t]$$

$$-\frac{1}{l}\sum_{j=1}^{l}[\exp[-\lambda(u)\sum_{i=1}^{l}\frac{1}{q(u,d_{ij})}t]$$

$$-\exp[-(l\lambda(u+1) - l\lambda(u) + \lambda(u)\sum_{i=1}^{l}\frac{1}{q(u,d_{ij})})t]]]^k,$$

$$u = 1,2,\ldots,z-1, \tag{73}$$

$$\bar{\boldsymbol{S}}_{LLS}(t,z) = 1 - [1 - \exp[-l\lambda(z)t]]^k. \tag{74}$$

2.7. Safety of a multistate series-"m out of k" system with dependent components of its subsystems

Similarly as in Section 2.6, we consider a multistate regular series-"*m* out of *k*" system, with a scheme given in Figure 3, as a "*m* out of *k*" system composed of series subsystems with dependent components. We assume that *k* is a number of series subsystems working independently and *l* is a number of dependent components of these series subsystems. In each series subsystem we assume local load sharing model of components dependency described in Section 2.1.

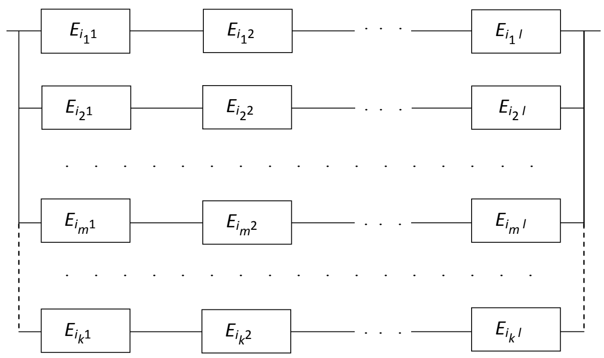

Figure 3. The scheme of a regular series-"*m* out of *k*" system.

Then linking the results for a multistate series system assuming its components' dependency with the safety function of a "*m* out of *k*" system with independent components [Kołowrocki, 2014], we obtain following proposition.

Proposition 10. If in a homogeneous multistate regular series-"*m* out of *k*" system, its subsystems are working independently and components of these series subsystems are dependent according to the local load sharing rule and have safety functions (13), then its safety function is given by the vector

$$\bar{\boldsymbol{S}}^{(m)}_{LLS}(t,\cdot) = [1, \bar{\boldsymbol{S}}^{(m)}_{LLS}(t,1), \ldots, \bar{\boldsymbol{S}}^{(m)}_{LLS}(t,z)], \; t \geq 0, \tag{75}$$

with the coordinates

$$\overline{\boldsymbol{S}}_{LLS}^{(m)}(t,u) = 1 - \sum_{r=0}^{m-1} \binom{k}{r} [\boldsymbol{S}_{LLS}(t,u)]^r [1 - \boldsymbol{S}_{LLS}(t,u)]^{k-r},$$
$$u = 1,2,\ldots,z, \qquad (76)$$

or by the vector

$$\overline{\boldsymbol{S}}_{LLS}^{(\overline{m})}(t,\cdot) = [1, \overline{\boldsymbol{S}}_{LLS}^{(\overline{m})}(t,1),\ldots,\overline{\boldsymbol{S}}_{LLS}^{(\overline{m})}(t,z)],\ t \geq 0, \qquad (77)$$

with the coordinates

$$\overline{\boldsymbol{S}}_{LLS}^{(\overline{m})}(t,u) = \sum_{r=0}^{\overline{m}} \binom{k}{r} [1 - \boldsymbol{S}_{LLS}(t,u)]^r [\boldsymbol{S}_{LLS}(t,u)]^{k-r},$$
$$\overline{m} = k - m,\ u = 1,2,\ldots,z, \qquad (78)$$

where the safety function coordinates of series subsystems with local load sharing model of dependency $\boldsymbol{S}_{LLS}(t,u)$, $u = 1,2,\ldots,z$, are given by (15)-(16).

Next, in case the system components have exponential safety functions, considering safety function for a homogeneous multistate series system given by (37)-(38) and applying (75)-(76) or (77)-(78) respectively, from *Proposition 10* we can obtain immediately the following result.

Proposition 11. If in a homogeneous multistate regular series-"*m* out of *k*" system, its subsystems are working independently and components of these series subsystems are dependent according to the local load sharing rule and have exponential safety functions with the coordinates (34), then its safety function is given by the vector

$$\overline{\boldsymbol{S}}_{LLS}^{(m)}(t,\cdot) = [1, \overline{\boldsymbol{S}}_{LLS}^{(m)}(t,1),\ldots,\overline{\boldsymbol{S}}_{LLS}^{(m)}(t,z)],\ t \geq 0, \qquad (79)$$

with the coordinates

$$\overline{\boldsymbol{S}}_{LLS}^{(m)}(t,u) = 1 - \sum_{r=0}^{m-1} \binom{k}{r} [\exp[-l\lambda(u+1)t]$$
$$+ \frac{1}{l} \sum_{j=1}^{l} [\exp[-\lambda(u) \sum_{i=1}^{l} \frac{1}{q(u,d_{ij})} t]$$
$$- \exp[-(l\lambda(u+1) - l\lambda(u) + \lambda(u) \sum_{i=1}^{l} \frac{1}{q(u,d_{ij})})t]]]^r$$
$$\cdot [1 - \exp[-l\lambda(u+1)t] - \frac{1}{l} \sum_{j=1}^{l} [\exp[-\lambda(u) \sum_{i=1}^{l} \frac{1}{q(u,d_{ij})} t]$$
$$- \exp[-(l\lambda(u+1) - l\lambda(u) + \lambda(u) \sum_{i=1}^{l} \frac{1}{q(u,d_{ij})})t]]]^{k-r},$$
$$u = 1,2,\ldots,z-1, \qquad (80)$$

$$\overline{\boldsymbol{S}}_{LLS}^{(m)}(t,z)$$
$$= 1 - \sum_{r=0}^{m-1} \binom{k}{r} [\exp[-l\lambda(z)t]]^r [1 - \exp[-l\lambda(z)t]]^{k-r}, \qquad (81)$$

or by the vector

$$\overline{\boldsymbol{S}}_{LLS}^{(\overline{m})}(t,\cdot) = [1, \overline{\boldsymbol{S}}_{LLS}^{(\overline{m})}(t,1),\ldots,\overline{\boldsymbol{S}}_{LLS}^{(\overline{m})}(t,z)],\ t \geq 0, \qquad (82)$$

with the coordinates

$$\overline{\boldsymbol{S}}_{LLS}^{(\overline{m})}(t,u) = \sum_{r=0}^{\overline{m}} \binom{k}{r} [1 - \exp[-l\lambda(u+1)t]$$
$$- \frac{1}{l} \sum_{j=1}^{l} [\exp[-\lambda(u) \sum_{i=1}^{l} \frac{1}{q(u,d_{ij})} t]$$
$$- \exp[-(l\lambda(u+1) - l\lambda(u) + \lambda(u) \sum_{i=1}^{l} \frac{1}{q(u,d_{ij})})t]]]^r$$
$$\cdot [\exp[-l\lambda(u+1)t] + \frac{1}{l} \sum_{j=1}^{l} [\exp[-\lambda(u) \sum_{i=1}^{l} \frac{1}{q(u,d_{ij})} t]$$
$$- \exp[-(l\lambda(u+1) - l\lambda(u) + \lambda(u) \sum_{i=1}^{l} \frac{1}{q(u,d_{ij})})t]]]^{k-r},$$
$$\overline{m} = k - m,\ u = 1,2,\ldots,z-1, \qquad (83)$$

$$\overline{\boldsymbol{S}}_{LLS}^{(\overline{m})}(t,z)$$
$$= \sum_{r=0}^{\overline{m}} \binom{k}{r} [1 - \exp[-l\lambda(z)t]]^r [\exp[-l\lambda(z)t]]^{k-r},$$
$$\overline{m} = k - m. \qquad (84)$$

3. Equal load sharing model of components dependency

3.1. Approach description

Depending on the structure of a system and behaviour of the system components we can consider different types of inside systems dependencies. We assume that after decreasing the safety state by one of the parallel or "*m* out of *n*" system components the increased load can be shared equally among the remaining components. Then, the inside interactions between remaining components may cause the decrease of these components lifetimes in the safety state subset equally. More exactly, we assume that if anyone of system components gets out of the safety state subset $\{u,u+1,...,z\}$, then the safety of remaining ones is getting worse so that their mean values of lifetimes

$T_i'(u)$ in safety state subset $\{u,u+1,...,z\}$ become less according to the formula

$$E[T_i'(u)] = c(u)\frac{n-1}{n}E[T_i(u)],\ i = 1,\dots,n,$$
$$u = 1,\dots,z,$$

where $c(u)$ is the component stress proportionality correction coefficient for each u, $u = 1,2,\dots,z$, [Kołowrocki, 2013]. This model of equal load sharing (ELS) is often applied to parallel or "*m* out of *n*" systems and has been analyzed in [Blokus-Roszkowska, Kołowrocki, 2014a,b].

We assume these lifetimes decrease uniformly depending on the number of components that have left the safety state subset. Additionally these changes are influenced by the component stress proportionality correction coefficient, concerned with particular components' features. The value of this coefficient can be estimated on the basis of behaviour of the component safety state changing dynamics or assumed a priori. However, in both cases, it should be verified by the actual safety data analysis and experts' judgment.

Generalizing, if $\omega, \omega = 0,1,2,\dots,n-1$, components of a system are out of the safety state subset $\{u,u+1,...,z\}$, the mean values of the lifetimes $T_i'(u)$ in the safety state subset $\{u,u+1,...,z\}$ of the system remaining components are given by

$$E[T_i'(u)] = c(u)\frac{n-\omega}{n}E[T_i(u)],\ i = 1,2,\dots,n,$$
$$u = 1,2,\dots,z, \tag{85}$$

where $c(u)$ is the component stress proportionality correction coefficient for each u, $u = 1,2,\dots,z$.

Hence, for case when considered system is homogeneous with components having exponential safety functions of the form

$$S_i(t,\cdot) = [1, S_i(t,1),\dots,S_i(t,z)],\ t \ge 0,$$
$$i = 1,2,\dots,n, \tag{86}$$

with the coordinates

$$S_i(t,u) = \exp[-\lambda(u)t],\ t \ge 0,\ i = 1,2,\dots,n,$$
$$u = 1,2,\dots,z, \tag{87}$$

with intensity of departure $\lambda(u) \ge 0$ from the safety state subset $\{u,u+1,...,z\}$, according to the well known relationship between the lifetime mean value in this safety state subset and intensity of departure from this safety state subset of the form

$$E[T_i(u)] = \frac{1}{\lambda(u)},\ i = 1,2,\dots,n,\ u = 1,2,\dots,z,$$

we get following formula for intensities of departure from this safety state subset of remaining components

$$\lambda^{(\omega)}(u) = \frac{n}{n-\omega}\frac{\lambda(u)}{c(u)},\ \omega = 0,1,\dots,n-1,$$
$$u = 1,2,\dots,z. \tag{88}$$

3.2. Safety of a multistate parallel system with dependent components

With this simple approach to inside dependencies of parallel systems with homogeneous components we can find analytical solutions of their safety characteristics.

Proposition 12. If in a homogeneous multistate parallel system components are dependent according to the equal load sharing rule and have exponential safety function given by (86)-(87), then its safety function is given by the vector

$$\boldsymbol{S}_{ELS}(t,\cdot) = [1, \boldsymbol{S}_{ELS}(t,1),\dots,\boldsymbol{S}_{ELS}(t,z)],\ t \ge 0, \tag{89}$$

with the coordinates

$$\boldsymbol{S}_{ELS}(t,u) = \sum_{\omega=0}^{n-1}\frac{\left[\frac{n\lambda(u)}{c(u)}t\right]^{\omega}}{\omega!}\exp\left[-\frac{n\lambda(u)}{c(u)}t\right],$$
$$u = 1,2,\dots,z. \tag{90}$$

Corollary 4. If in a homogeneous multistate parallel system components are dependent according to the equal load sharing rule and have exponential safety function given by (86)-(87), then the system lifetime in the safety state subset $\{u,u+1,\dots,z\}$, $u = 1,2,\dots,z$, has Erlang distribution with the shape parameter n and the intensity parameter $n\lambda(u)/c(u)$, $u = 1,2,\dots,z$, and its distribution function is given by the vector

$$\boldsymbol{F}_{ELS}(t,\cdot) = [0, \boldsymbol{F}_{ELS}(t,1),\dots,\boldsymbol{F}_{ELS}(t,z)],\ t \ge 0,$$

with the coordinates

$$\boldsymbol{F}_{ELS}(t,u) = 1 - \sum_{\omega=0}^{n-1} \frac{\left[\frac{n\lambda(u)}{c(u)}t\right]^{\omega}}{\omega!} \exp[-\frac{n\lambda(u)}{c(u)}t],$$
$u = 1,2,\ldots,z.$

3.3. Safety of a multistate "m out of n" system with dependent components

The definition of the multistate "m out of n" system means that it is in the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, if and only if at least m out of its n components are in this safety state subset. Thus, assuming that components of a multistate "m out of n" system are dependent according to the equal load sharing rule, if $\omega, \omega = 0,1,2,\ldots,n-m$, components of a system are out of the safety state subset $\{u,u+1,...,z\}$, $u = 1, 2,\ldots,z$, the mean values of the lifetimes in this safety state subset of the system remaining components are given by (85). Then, in case components have identical exponential safety functions given by (86)-(87), the intensities of departure from the safety state subset $\{u,u+1,...,z\}$, $u = 1,2,\ldots,z$, of remaining components are given by

$$\lambda^{(\omega)}(u) = \frac{n}{n-\omega}\frac{\lambda(u)}{c(u)}, \ \omega = 0,1,\ldots,n-m,$$
$$u = 1,2,\ldots,z. \qquad (91)$$

A similar result to the result presented below in the form of Proposition 13, concerned with reliability of a "m out of n" system with dependent components, has been formulated and proved in [Blokus-Roszkowska, Kołowrocki, 2014b].

Proposition 13. If in a homogeneous multistate "m out of n" system components are dependent according to the equal load sharing rule and have exponential safety function given by (86)-(87), then its safety function is given by the vector

$$\boldsymbol{S}_{ELS}(t,\cdot) = [1, \boldsymbol{S}_{ELS}(t,1),\ldots,\boldsymbol{S}_{ELS}(t,z)], \qquad (92)$$

with the coordinates

$$\boldsymbol{S}_{ELS}(t,u) = \sum_{\omega=0}^{n-m} \frac{\left[\frac{n\lambda(u)}{c(u)}t\right]^{\omega}}{\omega!} \exp[-\frac{n\lambda(u)}{c(u)}t], \ t \geq 0,$$
$$u = 1,2,\ldots,z. \qquad (93)$$

Corollary 5. If in a homogeneous multistate "m out of n" system components are dependent according to the equal load sharing rule and have exponential safety function given by (86)-(87), then the system lifetime in the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, has Erlang distribution with the shape parameter $n - m + 1$ and the intensity parameter $n\lambda(u)/c(u)$, $u = 1,2,\ldots,z$, and its distribution function is given by the vector

$$\boldsymbol{F}_{ELS}(t,\cdot) = [0, \boldsymbol{F}_{ELS}(t,1),\ldots,\boldsymbol{F}_{ELS}(t,z)], \ t \geq 0,$$

with the coordinates

$$\boldsymbol{F}_{ELS}(t,u) = 1 - \sum_{\omega=0}^{n-m} \frac{\left[\frac{n\lambda(u)}{c(u)}t\right]^{\omega}}{\omega!} \exp[-\frac{n\lambda(u)}{c(u)}t],$$
$u = 1,2,\ldots,z.$

3.4. Safety of a multistate parallel-series system with dependent components of its subsystems

In this section we consider a multistate regular parallel-series system with a scheme presented in Figure 4. We assume that k is a number of parallel subsystems working independently linked in series and l is a number of dependent components of these parallel subsystems. We denote by E_{ij}, $i = 1,2,...,k$, $j = 1,2,...,l$, k, $l \in N$, components of a system and assume that all components E_{ij} have the same safety state set as before $\{0,1,...,z\}$. Then, $T_{ij}(u)$, $i = 1,2,...,k$, $j = 1,2,...,l$, k, $l \in N$, are random variables representing lifetimes of components E_{ij} in the safety state subset $\{u,u+1,...,z\}$, $u = 1,2,\ldots,z$, while they were in the safety state z at the moment $t = 0$. In each parallel subsystem we assume equal load sharing model of components dependency described in Section 3.1.

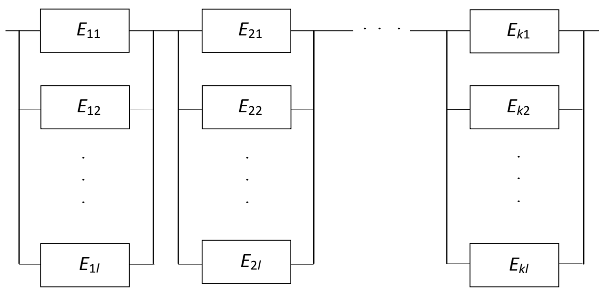

Figure 4. The scheme of a regular parallel-series system.

We assume similarly as in formula (85) for a multistate parallel system that if $\omega, \omega = 0,1,2,\ldots,l-1,$ components of each parallel subsystem of a system are out of the safety state subset $\{u,u+1,...,z\}$, $u = 1,2,\ldots,z$, the mean values of lifetimes $T_{ij}'(u)$ in the safety state subset $\{u,u+1,...,z\}$ of this subsystem remaining components are given by

$$E[T_{ij}'(u)] = c(u)\frac{l-\omega}{l}E[T_{ij}(u)],\ i = 1,2,\ldots,k,$$
$$j = 1,2,\ldots,l,\ u = 1,2,\ldots,z, \quad (94)$$

where $c(u)$, $u = 1,2,\ldots,z$, are component stress proportionality correction coefficients [Kołowrocki, 2013].

Then, in case when considered system is homogeneous with components having exponential safety functions of the form

$$S_{ij}(t,\cdot) = [1, S_{ij}(t,1),\ldots,S_{ij}(t,z)],\ t \geq 0,\ i = 1,2,\ldots,k,$$
$$j = 1,2,\ldots,l, \quad (95)$$

where

$$S_{ij}(t,u) = \exp[-\lambda(u)t],\ t \geq 0,\ i = 1,2,\ldots,k,$$
$$j = 1,2,\ldots,l,\ u = 1,2,\ldots,z, \quad (96)$$

with the intensity of departure $\lambda(u)$ from the safety state subset $\{u,u+1,...,z\}$, according to the well known relationship between the lifetime mean value in this safety state subset and the intensity of departure from this safety state subset we get following formula for intensities of departure from this safety state subset of subsystem remaining components

$$\lambda^{(\omega)}(u) = \frac{l}{l-\omega}\frac{\lambda(u)}{c(u)},\ \omega = 0,1,2,\ldots,l-1,$$
$$u = 1,2,\ldots,z.$$
(97)

Considering results, for a parallel system with components dependent according to the equal load sharing rule, given in *Proposition 12* and linking these results with the safety function of a series system with independent components, we can obtain the formula for the safety function of a parallel-series system in the form of following proposition.

Proposition 14. If in a homogeneous multistate regular parallel-series system, its subsystems are working independently and components of these parallel subsystems are dependent according to the equal load sharing rule and have exponential safety function given by (95)-(96), then its safety function is given by the vector

$$\boldsymbol{S}_{ELS}(t,\cdot) = [1, \boldsymbol{S}_{ELS}(t,1),\ldots,\boldsymbol{S}_{ELS}(t,z)], \quad (98)$$

with the coordinates

$$\boldsymbol{S}_{ELS}(t,u) = [\sum_{\omega=0}^{l-1}\frac{[\frac{l\lambda(u)}{c(u)}t]^{\omega}}{\omega!}\exp[-\frac{l\lambda(u)}{c(u)}t]]^{k},\ t \geq 0,$$
$$u = 1,2,\ldots,z. \quad (99)$$

3.5. Safety of a multistate "m out of l"-series system with dependent components of its subsystems

Here, we consider a multistate regular "*m* out of *l*"-series system with a scheme presented in *Figure 5* as a system composed of "*m* out of *l*" subsystems with components dependent according to the equal load sharing rule. We assume that *k* is a number of "*m* out of *l*" subsystems working independently linked in series and *l* is a number of components of these "*m* out of *l*" subsystems.

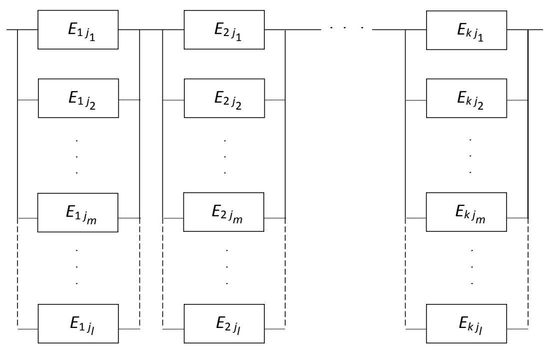

Figure 5. The scheme of a regular "*m* out of *l*"-series system.

In each "*m* out of *l*" subsystem we assume equal load sharing model of components' dependency, described in Section 3.1, and assume that if $\omega, \omega = 0,1,2,\ldots,l-m,$ components of each "*m* out of *l*" subsystem are out of the safety state subset $\{u,u+1,...,z\}$, $u = 1,2,\ldots,z$, the mean values of lifetimes in the safety state subset $\{u,u+1,...,z\}$ of this subsystem remaining components are given by (94).

Further, we assume that a system is homogeneous with components having exponential safety functions given by (95)-(96) and, similarly as for a parallel-series system in Section 3.4, the intensities of departure from the safety state subset $\{u,u+1,...,z\}$, $u =$

1,2,…,*z*, of subsystem remaining components are given by

$$\lambda^{(\omega)}(u) = \frac{l}{l-\omega}\frac{\lambda(u)}{c(u)},\ \omega = 0,1,2,\ldots,l-m,$$
$$u = 1,2,\ldots,z. \quad (100)$$

Proposition 13 slight extension yields the following result.

Proposition 15. If in a homogeneous multistate regular "*m* out of *l*"-series system, its subsystems are working independently and components of these "*m* out of *l*" subsystems are dependent according to the equal load sharing rule and have exponential safety function given by (95)-(96), then its safety function is given by the vector

$$\boldsymbol{S}_{ELS}(t,\cdot) = [1, \boldsymbol{S}_{ELS}(t,1),\ldots,\boldsymbol{S}_{ELS}(t,z)], \quad (101)$$

with the coordinates

$$\boldsymbol{S}_{ELS}(t,u) = [\sum_{\omega=0}^{l-m}\frac{[\frac{l\lambda(u)}{c(u)}t]^{\omega}}{\omega!}\exp[-\frac{l\lambda(u)}{c(u)}t]]^{k},\ t \ge 0,$$
$$u = 1,2,\ldots,z. \quad (102)$$

4. Mixed load sharing model of components and subsystems dependency

4.1. Approach description

In the safety analysis of various system structures we can link the results for previously described models of dependency between their components and subsystems. For instance, the obtained results for a parallel-series and "*m* out of *l*"-series system with independent subsystems and their components dependent according to the equal load sharing rule, have been presented in Section 3.4 and in Section 3.5, respectively. The results for a series-parallel and series-"*m* out of *k*" system composed of independent subsystems with their components dependent according to the local load sharing rule can be found in this report in Section 2.6 and 2.7 and in [Blokus-Roszkowska, 2015].

In more complex models of dependency, apart from the dependency of subsystems' departures from the safety states subsets we can take into account the dependencies between components in subsystems. This way we can proceed with parallel-series and "*m* out of *l*"-series systems assuming the dependence between their parallel, respectively "*m* out of *l*", subsystems according to the local load sharing rule and the dependence between their components in subsystems according to the equal load sharing rule. Further, such model of dependency we will call a mixed load sharing (MLS) model. The MLS model for a multistate regular "*m* out of *l*"-series system has been presented and applied to reliability evaluation of the shipyard ship-rope elevator, in case its components have piecewise exponential reliability functions with interdependent departures rates from the subsets of their reliability states, in [Blokus-Roszkowska, Kołowrocki, 2015d].

4.2. Safety of a multistate parallel-series system

In this section, we apply a mixed load sharing model of components and subsystems dependency to safety analysis of a multistate regular parallel-series system. We consider a multistate regular parallel-series system composed of *k* parallel subsystems S_i, *i* = 1,2,…,*k*, linked in series, illustrated in Figure 6. Further, by E_{ij}, *i* = 1,2,…,*k*, *j* = 1,2,…,*l*, we denote the *j*th component being in the *i*th subsystem S_i, and we assume all system components have identical exponential safety functions, given by (95)-(96).

In each parallel subsystem of such model we consider dependency of its *l* components according to the equal load sharing model, presented in Section 3. Then, after departure from the safety state subset {*u*,*u*+1,…,*z*}, *u* = 1,2,…,*z*, by $\omega, \omega = 0,1,2,\ldots,l-1,$ components of a subsystem, the intensities of departure from this safety state subset of the subsystem remaining components are given by (97). From Corollary 4, we conclude that the lifetime in the safety state subset {*u*,*u*+1,…,*z*}, *u* = 1,2,…,*z*, of each parallel subsystem, assuming that its components are dependent according to the equal load sharing rule, has Erlang distribution with the shape parameter *l* and the intensity parameter *l*λ(*u*)/*c*(*u*), *u* = 1,2,…,*z*.

Further, between these subsystems, linked in series, we assume the local load sharing model of dependency, presented in Section 2. Then, we assume that after departure from the safety state subset {*u*,*u*+1,…,*z*}, *u* = 1,2,…,*z*, by the subsystem S_g, *g* = 1,2,…,*k*, the safety parameters of components of remaining subsystems are changing dependently of the distance from the subsystem that has got out of the safety state subset {*u*,*u*+1,…,*z*}, *u* = 1,2,…,*z*, expressed by index *d*. However, within a single subsystem the changes of the safety parameters for all of its components are on the same level according to the equal load sharing rule. The meaning of the distance *d* in mixed load sharing model is illustrated in *Figure 6.*

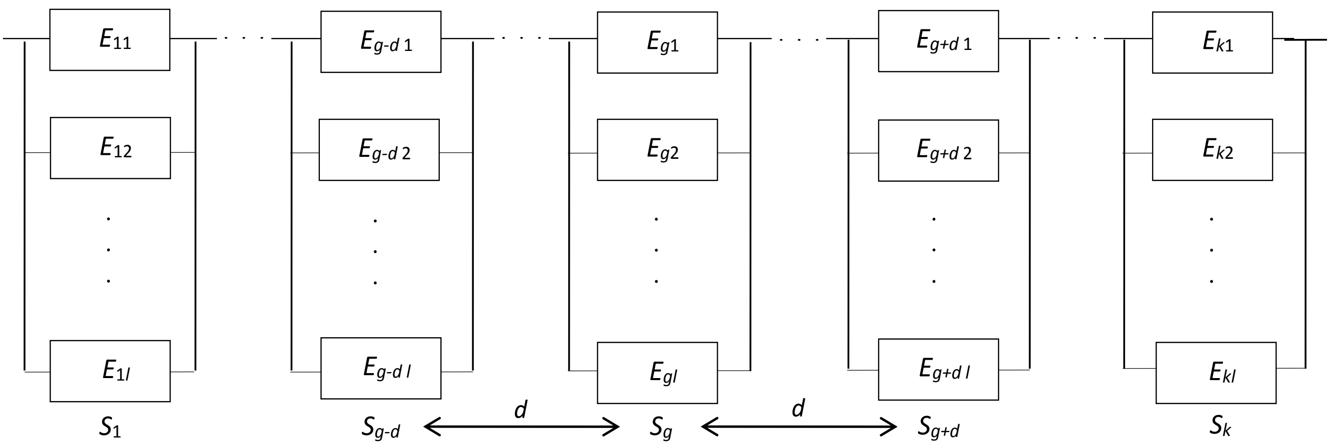

Figure 6. The scheme of a regular parallel-series system.

We denote by $E[T_{i,j}(u)]$ and $E[T_{i/g,j}(u)]$, $i = 1,2,...,k$, $g = 1,2,...,k$, $j = 1,2,...,l$, $u = 1,2,\ldots,z$, the mean values of the lifetimes of ith subsystem components $T_{i,j}(u)$ and $T_{i/g,j}(u)$, respectively, before and after departure of one fixed subsystem S_g, $g = 1,\ldots,k$, from the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$. With this notation, in the local load sharing model used between subsystems, the mean values of their components lifetimes in the safety state subset $\{\upsilon,\upsilon+1,\ldots,z\}$, $\upsilon = u,u-1,\ldots,1$, $u = 1,2,\ldots,z$, are decreasing, using (89), according to the following formula:

$$E[T_{i/g,j}(\upsilon)] = q(\upsilon,d_{ig}) \cdot E[T_{i,j}(\upsilon)], i = 1,2,...,k,$$
$$g = 1,2,...,k, j = 1,2,...,l,\ \upsilon = u,u-1,\ldots,1, \tag{103}$$

where the coefficients $q(\upsilon,d_{ig})$, $0 < q(\upsilon,d_{ig}) \le 1$ for $i = 1,2,...,k$, $g = 1,2,...,k$, and $q(\upsilon,0) = 1$ for $\upsilon = u,u-1,\ldots,1$, $u = 1,2,\ldots,z-1$, are non-increasing functions of subsystems' distance $d_{ig} = |i - g|$ from the subsystem that has got out of the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$.

Considering Corollary 4 concerned with Erlang distribution of system lifetime in the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, in case components of parallel system are dependent according to the equal load sharing rule, and linking this result with the safety function of a series system with components dependent according to the local load sharing rule and having Erlang safety functions, presented in Section 2.5, we can obtain the safety function of a homogeneous multistate regular parallel-series system with mixed model of dependency. Then, applying Proposition 7 for series system composed of k subsystems, and using fact that these subsystems have Erlang safety functions with the shape parameter l and with the intensity parameter $l\lambda(u)/c(u)$, $u = 1,2,\ldots,z$, we immediately get the following result.

Proposition 16. If in a homogeneous multistate regular parallel-series system, its subsystems are dependent according to the local load sharing rule and components of these parallel subsystems are dependent according to the equal load sharing rule and have exponential safety functions (95)-(96), then its safety function is given by the vector

$$\boldsymbol{S}_{MLS}(t,\cdot) = [1, \boldsymbol{S}_{MLS}(t,1),\ldots,\boldsymbol{S}_{MLS}(t,z)],\ t \ge 0, \tag{104}$$

with the coordinates

$$\boldsymbol{S}_{MLS}(t,u) = [\sum_{\omega=0}^{l-1} \frac{[\frac{l\lambda(u+1)}{c(u+1)}t]^{\omega}}{\omega!}]^{k} \cdot \exp[-\frac{kl\lambda(u+1)}{c(u+1)}t]$$
$$+ \sum_{g=1}^{k} \int_0^t \tilde{f}(a,u+1)[\sum_{\omega=0}^{l-1} \frac{[\frac{l\lambda(u+1)}{c(u+1)}a]^{\omega}}{\omega!}]^{k-1}$$
$$\cdot \sum_{\omega=0}^{l-1} \frac{[\frac{l\lambda(u)}{c(u)}a]^{\omega}}{\omega!} \cdot \exp[-((k-1)\frac{l\lambda(u+1)}{c(u+1)} + \frac{l\lambda(u)}{c(u)})a]$$
$$\cdot [\prod_{i=1}^{k} \frac{\sum_{\omega=0}^{l-1} \frac{[\frac{l\lambda(u)}{c(u)q(u,d_{ig})}t]^{\omega}}{\omega!}}{\sum_{\omega=0}^{l-1} \frac{[\frac{l\lambda(u)}{c(u)}a]^{\omega}}{\omega!}}$$
$$\cdot \exp[-\frac{l\lambda(u)}{c(u)q(u,d_{ig})}t + \frac{l\lambda(u)}{c(u)}a]]da,$$
$$u = 1,2,\ldots,z-1, \tag{105}$$

where $\tilde{f}(a,u+1)$ is given by

$$\tilde{f}(a,u+1) = [\frac{\sum_{\omega=0}^{l-1}\frac{[\frac{l\lambda(u+1)}{c(u+1)}a]^{\omega}}{\omega!}\cdot\sum_{\omega=0}^{l-2}\frac{[\frac{l\lambda(u)}{c(u)}]^{\omega+1}a^{\omega}}{\omega!}}{[\sum_{\omega=0}^{l-1}\frac{[\frac{l\lambda(u)}{c(u)}a]^{\omega}}{\omega!}]^{2}}$$

$$-\frac{\sum_{\omega=0}^{l-2}\frac{[\frac{l\lambda(u+1)}{c(u+1)}]^{\omega+1}a^{\omega}}{\omega!}\cdot\sum_{\omega=0}^{l-1}\frac{[\frac{l\lambda(u)}{c(u)}a]^{\omega}}{\omega!}}{[\sum_{\omega=0}^{l-1}\frac{[\frac{l\lambda(u)}{c(u)}a]^{\omega}}{\omega!}]^{2}}$$

$$+\frac{\sum_{\omega=0}^{l-1}\frac{[\frac{l\lambda(u+1)}{c(u+1)}a]^{\omega}}{\omega!}(\frac{l\lambda(u+1)}{c(u+1)}-\frac{l\lambda(u)}{c(u)})}{\sum_{\omega=0}^{l-1}\frac{[\frac{l\lambda(u)}{c(u)}a]^{\omega}}{\omega!}}]$$

$$\cdot\exp[-(\frac{l\lambda(u+1)}{c(u+1)}-\frac{l\lambda(u)}{c(u)})a],\ u=1,2,\ldots,z-1, \quad (106)$$

and

$$\boldsymbol{S}_{MLS}(t,z) = [\sum_{\omega=0}^{l-1}\frac{[\frac{l\lambda(z)}{c(z)}t]^{\omega}}{\omega!}]^{k}\cdot\exp[-k\cdot\frac{l\lambda(z)}{c(z)}t]. \quad (107)$$

4.3. Safety of a multistate "m out of l"-series system

Next, we apply a mixed load sharing model of components and subsystems dependency, presented in Section 4.1 and 4.2, to the safety analysis of a multistate regular "*m* out of *l*"-series system. We consider a multistate regular "*m* out of *l*"-series system composed of *k* "*m* out of *l*" subsystems S_i, $i = 1,2,\ldots,k$, linked in series, illustrated in *Figure 7.* Further, similarly as for a parallel-series system, by E_{ij}, $i = 1,2,\ldots,k$, $j = 1,2,\ldots,l$ we denote the *j*th component being in the *i*th subsystem S_i, and we assume all system components have identical exponential safety functions, given by (95)-(96).

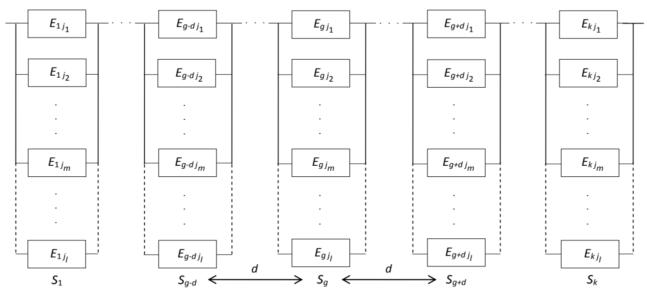

Figure 7. The scheme of a regular "*m* out of *l*"-series system.

In each "*m* out of *l*" subsystem of such model we consider dependency of its *l* components according to the equal load sharing model, presented in Section 3 Then, after departure from the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, by υ, $\upsilon = 0,1,\ldots,l-m$, components of a subsystem, the intensities of departure from this safety state subset of the subsystem remaining components are given by (100). From Corollary 5, we conclude that the lifetime in the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, of each "*m* out of *l*" subsystem, assuming that its components are dependent according to the equal load sharing rule, has Erlang distribution with the shape parameter $l - m + 1$ and the intensity parameter $l\lambda(u)/c(u)$, $u = 1,2,\ldots,z$.

Further, between these subsystems, linked in series, we assume the local load sharing model of dependency, presented in Section 3.1. Then, we assume that after departure from the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, by the subsystem S_g, $g = 1,2,\ldots,k$, the safety parameters of components of remaining subsystems are changing dependently of the distance from the subsystem that has got out of the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, and the mean values of these components lifetimes in the safety state subset $\{\upsilon,\upsilon+1,\ldots,z\}$, $\upsilon = u,u-1,\ldots,1$, $u = 1,2,\ldots,z$, are decreasing according to (103).

Considering Corollary 5 concerned with Erlang distribution of system lifetime in the safety state subset
$\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, in case components of "*m* out of *l*" system are dependent according to the equal load sharing rule and linking this result with the safety function of a series system with components dependent according to the local load sharing rule and having Erlang safety functions, presented in Section 2.5, we can obtain the safety function of a homogeneous multistate regular "*m* out of *l*"-series system with mixed model of dependency. Then, applying Proposition 7 for series system composed of *k* subsystems, and using fact that these subsystems have Erlang safety functions with the shape parameter $l - m + 1$ and with the intensity parameter

$l\lambda(u)/c(u)$, $u = 1,2,\ldots,z$, we immediately get the following result.

Proposition 17. If in a homogeneous multistate regular "*m* out of *l*"-series system, its subsystems are dependent according to the local load sharing rule and components of these "*m* out of *l*" subsystems are dependent according to the equal load sharing rule and have exponential safety functions (95)-(96), then its safety function is given by the vector

$$\boldsymbol{S}_{MLS}(t,\cdot) = [1, \boldsymbol{S}_{MLS}(t,1), \ldots, \boldsymbol{S}_{MLS}(t,z)], \quad t \geq 0, \quad (108)$$

with the coordinates

$$\boldsymbol{S}_{MLS}(t,u) = [\sum_{\omega=0}^{l-m} \frac{[\frac{l\lambda(u+1)}{c(u+1)}t]^{\omega}}{\omega!}]^{k} \cdot \exp[-\frac{kl\lambda(u+1)}{c(u+1)}t]$$
$$+ \sum_{g=1}^{k} \int_{0}^{t} \tilde{f}(a,u+1) [\sum_{\omega=0}^{l-m} \frac{[\frac{l\lambda(u+1)}{c(u+1)}a]^{\omega}}{\omega!}]^{k-1}$$
$$\cdot \sum_{\omega=0}^{l-m} \frac{[\frac{l\lambda(u)}{c(u)}a]^{\omega}}{\omega!} \cdot \exp[-((k-1)\frac{l\lambda(u+1)}{c(u+1)} + \frac{l\lambda(u)}{c(u)})a]$$
$$\cdot [\prod_{i=1}^{k} \frac{\sum_{\omega=0}^{l-m} \frac{[\frac{l\lambda(u)}{c(u)q(u,d_{ig})}t]^{\omega}}{\omega!}}{\sum_{\omega=0}^{l-m} \frac{[\frac{l\lambda(u)}{c(u)}a]^{\omega}}{\omega!}}$$
$$\cdot \exp[-\frac{l\lambda(u)}{c(u)q(u,d_{ig})}t + \frac{l\lambda(u)}{c(u)}a]]da,$$
$$u = 1,2,\ldots,z-1, \quad (109)$$

where $\tilde{f}(a,u+1)$ is given by

$$\tilde{f}(a,u+1) = [\frac{\sum_{\omega=0}^{l-m} \frac{[\frac{l\lambda(u+1)}{c(u+1)}a]^{\omega}}{\omega!} \cdot \sum_{\omega=0}^{l-m-1} \frac{[\frac{l\lambda(u)}{c(u)}]^{\omega+1} a^{\omega}}{\omega!}}{[\sum_{\omega=0}^{l-m} \frac{[\frac{l\lambda(u)}{c(u)}a]^{\omega}}{\omega!}]^{2}}$$
$$+ \frac{\sum_{\omega=0}^{l-m} \frac{[\frac{l\lambda(u+1)}{c(u+1)}a]^{\omega}}{\omega!} (\frac{l\lambda(u+1)}{c(u+1)} - \frac{l\lambda(u)}{c(u)})}{\sum_{\omega=0}^{l-m} \frac{[\frac{l\lambda(u)}{c(u)}a]^{\omega}}{\omega!}}]$$
$$\cdot \exp[-(\frac{l\lambda(u+1)}{c(u+1)} - \frac{l\lambda(u)}{c(u)})t], \quad u = 1,2,\ldots,z-1,$$

(110)

and

$$\boldsymbol{S}_{MLS}(t,z) = [\sum_{\omega=0}^{l-m} \frac{[\frac{l\lambda(z)}{c(z)}t]^{\omega}}{\omega!}]^{k}$$
$$\cdot \exp[-k \cdot \frac{l\lambda(z)}{c(z)}t]. \quad (111)$$

5. Conclusions

The proposed in this report models for safety evaluation and prediction of the considered systems with independent and dependent components are the basic backgrounds for the considerations in further Tasks of the EU-CIRCLE Project. These system safety models, together with the models of the system operation process presented in [EU-CIRCLE Report D2.1-GMU2, 2016], are used for constructing the integrated joint general safety model of complex technical systems related to their operation processes [EU-CIRCLEReport D3.3-GMU3, 2016].

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