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# ON PREPONDERANTLY CONTINUOUS FUNCTIONS

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**Abstract**
In the present paper, a few different notions of preponderant continuity of a real function are discussed. We study the relationship between them and give some properties of preponderant continuity.

## 1. Preliminaries

Let $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}^+$, $\mathbb{Z}$ be the sets of real numbers, rational numbers, positive rational numbers and integer numbers, respectively. Next, let $I$ denote a closed interval, $U$ any open subset of $\mathbb{R}$ and $\mathrm{Int}(A)$ be an interior of a set $A \subset \mathbb{R}$ in the natural metric. Let $\lambda$ stand for Lebesgue measure in $\mathbb{R}$. If $E \subset \mathbb{R}$ is a measurable set, we define the lower and upper densitis of $E$ at $x_0 \in \mathbb{R}$ by:

$$\underline{d}\,(E, x_0) = \liminf_{\lambda(I)\to 0, x_0\in I} \frac{\lambda(I\cap E)}{\lambda(I)} \quad \text{and} \quad \overline{d}\,(E, x_0) = \limsup_{\lambda(I)\to 0, x_0\in I} \frac{\lambda(I\cap E)}{\lambda(I)}.$$

If $\underline{d}\,(E, x_0) = \overline{d}\,(E, x_0)$, we denote this common value by $d(E, x_0)$ and call it the density of $E$ at $x_0$. In a similar way, we also define the one-sided lower and upper densities of the set $E$ at the point $x_0$: $\underline{d}^+\,(E, x_0)$, $\underline{d}^-\,(E, x_0)$, $\overline{d}^+\,(E, x_0)$ and $\overline{d}^-\,(E, x_0)$. It is easy to verify that $\underline{d}\,(E, x_0) = \min\{\,\underline{d}^+\,(E, x_0)\,, \underline{d}^-\,(E, x_0)\}$ and $\overline{d}\,(E, x_0) = \max\{\,\overline{d}^+\,(E, x_0)\,, \overline{d}^-\,(E, x_0)\}$.

**Definition 1.** [1, 3, 4] *A point $x_0 \in \mathbb{R}$ is said to be the point of preponderant density in Denjoy sense of a measurable set $E \subset \mathbb{R}$ if $\underline{d}\,(E, x_0) > \frac{1}{2}$.*

**Definition 2.** *A point $x_0 \in \mathbb{R}$ is said to be the point of preponderant density in Denjoy sense of a measurable set $E \subset \mathbb{R}$ at right (at left, respectively) if $\underline{d}^+\,(E, x_0) > \frac{1}{2}$ ($\underline{d}^-\,(E, x_0) > \frac{1}{2}$, respectively).*

**Corollary 1.** *A point $x_0 \in \mathbb{R}$ is the point of preponderant density in Denjoy sense of a measurable set $E \subset \mathbb{R}$ iff it is the point of preponderant density in Denjoy sense of the measurable set $E$ at right and at left.*

**Definition 3.** [5] *A point $x_0 \in \mathbb{R}$ is said to be the point of preponderant density in O'Malley sense of a measurable set $E \subset \mathbb{R}$ if there exists $\varepsilon > 0$ such that for every closed interval $I$ satisfying conditions $x_0 \in I$ and $I \subset [x_0 - \varepsilon, x_0 + \varepsilon]$, the inequality $\frac{\lambda(E \cap I)}{\lambda(I)} > \frac{1}{2}$ holds.*

**Definition 4.** *A point $x_0 \in \mathbb{R}$ is said to be the point of preponderant density in O'Malley sense of a measurable set $E \subset \mathbb{R}$ at right (at left, respectively) if there exists $\varepsilon > 0$ such that for every $\delta < \varepsilon$ the inequality $\frac{\lambda(E \cap [x_0, x_0 + \delta])}{\delta} > \frac{1}{2}$ ( $\frac{\lambda(E \cap [x_0 - \delta, x_0])}{\delta} > \frac{1}{2}$, respectively) holds.*

**Corollary 2.** *A point $x_0 \in \mathbb{R}$ is the point of preponderant density in O'Malley sense of a measurable set $E \subset \mathbb{R}$ iff it is the point of preponderant density in O'Malley sense of the measurable set $E$ at right and at left.*

**Corollary 3.** *Let $E$ be a measurable subset of $\mathbb{R}$ and $x_0 \in \mathbb{R}$. If $x_0$ is the point of preponderant density in Denjoy sense of the set $E$, then $x_0$ is the point of preponderant density in O'Malley sense of the set $E$.*

**Definition 5.** [1, 3, 4] *A function $f\colon U \to \mathbb{R}$ is said to be preponderantly continuous in Denjoy sense at $x_0 \in U$ if there exists a measurable set $E \subset U$ containing $x_0$ such that $\underline{d}\,(E, x_0) > \frac{1}{2}$ and $f_{|E}$ is continuous at $x_0$. A function $f\colon U \to \mathbb{R}$ is said to be preponderantly continuous in Denjoy sense if it is preponderantly continuous in Denjoy sense at each point $x_0 \in U$. The class of all functions which are preponderantly continuous in Denjoy sense will be denoted by $\mathcal{PD}$.*

**Definition 6.** [6] *A function $f\colon U \to \mathbb{R}$ is said to be preponderantly continuous in O'Malley sense at $x_0 \in U$ if there exists a measurable set $E \subset U$ containing $x_0$ such that $x_0$ is the point of preponderant density in O'Malley sense of the set $E$ and $f_{|E}$ is continuous at $x_0$. A function $f\colon U \to \mathbb{R}$ is said to be preponderantly continuous in O'Malley sense if it is preponderantly* *continuous in O'Malley sense at each $x_0 \in U$. The class of all functions which are preponderantly continuous in O'Malley sense will be denoted by $\mathcal{PO}$.*

**Corollary 4.**

$$\mathcal{PD} \subset \mathcal{PO}.$$

Grande [5] defined a property of real functions, called the $A_1$ property. Based on this result, we can define a similar property, which extends the notion of preponderant continuity.

**Definition 7.** *A function $f\colon U \to \mathbb{R}$ is said to have $A_1$ property in Denjoy sense at $x_0 \in U$ if there exist measurable sets $E_1 \subset U$ and $E_2 \subset U$ containing $x_0$ such that $x_0$ is the point of preponderant density in Denjoy sense of both sets $E_1$ and $E_2$, $f_{|E_1}$ is upper semicontinuous at $x_0$ and $f_{|E_2}$ is lower semicontinuous at $x_0$. A function $f\colon U \to \mathbb{R}$ has $A_1$ property in Denjoy sense if it has $A_1$ property in Denjoy sense at each $x_0 \in U$. The class of all functions which have $A_1$ property in Denjoy sense will be denoted by $\mathcal{GPD}$.*

**Definition 8.** *A function $f\colon U \to \mathbb{R}$ is said to have $A_1$ property in O'Malley sense at $x_0 \in U$ if there exist measurable sets $E_1 \subset U$ and $E_2 \subset U$ containing $x_0$ such that $x_0$ is the point of preponderant density in O'Malley sense of both sets $E_1$ and $E_2$, $f_{|E_1}$ is upper semicontinuous at $x_0$ and $f_{|E_2}$ is lower semicontinuous at $x_0$. A function $f\colon U \to \mathbb{R}$ has $A_1$ property in O'Malley sense if it has $A_1$ property in O'Malley sense at every $x_0$. The class of all functions which have $A_1$ property in O'Malley sense will be denoted by $\mathcal{GPO}$.*

**Corollary 5.**

$$\mathcal{GPD} \subset \mathcal{GPO}.$$

In the sequel we will often consider "interval sets" at the point $x_0$, that is sets of the particular form $E = \bigcup_{n=1}^{\infty} [a_n, b_n]$, where $a_{n+1} < b_{n+1} < a_n$ for every $n$ and $x_0 = \lim_{n\to\infty} a_n$.

**Proposition 1.** *Let $E = \bigcup_{n=1}^{\infty} [a_n, b_n]$, where $a_{n+1} < b_{n+1} < a_n$ for every $n$ and $x_0 = \lim_{n\to\infty} a_n$. Then*

$$\frac{\lambda([x_0, x] \cap E)}{\lambda([x_0, x])} \geq \frac{\lambda([x_0, a_n] \cap E)}{\lambda([x_0, a_n])} \quad \textit{for} \quad x \in [b_{n+1}, b_n]$$

*and*

$$\frac{\lambda([x_0, x] \cap E)}{\lambda([x_0, x])} \leq \frac{\lambda([x_0, b_{n+1}] \cap E)}{\lambda([x_0, b_{n+1}])} \quad \textit{for} \quad x \in [a_{n+1}, a_n].$$

**Proof:** It is easy to verify that the following inequality $\frac{a}{c} < \frac{a+b}{c+b}$ holds for positive reals $0 < c < a$ and $0 < b$. Moreover, if $x \in [b_{n+1}, a_n]$, then $\lambda([x_0, x] \cap E) = \lambda([x_0, b_{n+1}] \cap E) = \lambda([x_0, a_n] \cap E)$, and if $x \in [a_n, b_n]$, then $\lambda([x_0, x] \cap E) = \lambda([x_0, a_n] \cap E) + \lambda([a_n, x])$. Hence, for $x \in [b_{n+1}, a_n]$ we get

$$\frac{\lambda([x_0, b_{n+1}] \cap E)}{\lambda([x_0, b_{n+1}])} \geq \frac{\lambda([x_0, b_{n+1}] \cap E)}{\lambda([x_0, x])} = \\ = \frac{\lambda([x_0, x] \cap E)}{\lambda([x_0, x])} = \frac{\lambda([x_0, a_n] \cap E)}{\lambda([x_0, x])} \geq \frac{\lambda([x_0, a_n] \cap E)}{\lambda([x_0, a_n])}.$$

Similarly, for each $x \in [a_{n+1}, b_{n+1}]$ it is true that

$$\frac{\lambda([x_0, b_{n+1}] \cap E)}{\lambda([x_0, b_{n+1}])} = \frac{\lambda([x_0, x] \cap E) + (b_{n+1} - x)}{\lambda([x_0, x]) + \lambda([x, b_{n+1}])} \geq \frac{\lambda([x_0, x] \cap E)}{\lambda([x_0, x])}$$

and for each $x \in [a_n, b_n]$ we get

$$\frac{\lambda([x_0, x] \cap E)}{\lambda([x_0, x])} = \frac{\lambda([x_0, a_n] \cap E) + (x - a_n)}{\lambda([x_0, a_n]) + \lambda([a_n, x])} \geq \frac{\lambda([x_0, a_n] \cap E}{\lambda([x_0, a_n])}.$$

It implies required inequalities. $\square$

**Corollary 6.** *Let $E = \bigcup_{n=1}^{\infty} [a_n, b_n]$, where $b_{n+1} < a_n < a_n$ for every $n$ and $x_0 = \lim_{n\to\infty} a_n$. Then*

*[1]* $\underline{d}^+ (E, x_0) = \liminf_{n\to\infty} \frac{\lambda([x_0, a_n] \cap E)}{\lambda([x_0, a_n])}$ *and* $\overline{d}^+ (E, x_0) = \limsup_{n\to\infty} \frac{\lambda([x_0, b_n] \cap E)}{\lambda([x_0, b_n])}$.

*[2] The point $x_0$ is the point of preponderant density in Denjoy sense of the set $E$ iff there exists $n_0 \in \mathbb{N}$ such that $\dfrac{\lambda(E \cap [x_0, a_n])}{\lambda([x_0, a_n])} > \dfrac{1}{2}$ for every $n \geq n_0$.*

Similar formulas hold for sets of the form $E = \bigcup_{n=1}^{\infty} [c_n, d_n]$, where $d_n < c_{n+1} < d_{n+1}$ for every $n$ and $x_0 = \lim_{n\to\infty} c_n$.

For a function $f \colon U \to \mathbb{R}$, $x_0 \in U$ and $\varepsilon > 0$ we will use the following notation
$E_{f,x_0,\varepsilon} = \{x \in U \colon |f(x) - f(x_0)| < \varepsilon\}$, $E^+_{f,x_0,\varepsilon} = \{x \in U \colon f(x) < f(x_0) + \varepsilon\}$ and $E^-_{f,x_0,\varepsilon} = \{x \in U \colon f(x) > f(x_0) - \varepsilon\}$.

## 2. Main results

**Proposition 2.** *Let $f \colon U \to \mathbb{R}$ and $x_0 \in U$. Then*

1. *If $f$ is preponderantly continuous in Denjoy sense at $x_0$, then $\underline{d} (E_{f,x_0,\varepsilon}, x_0) > \frac{1}{2}$ for every $\varepsilon > 0$.*

2. *If $f$ is preponderantly continuous in O'Malley sense at $x_0$, then $x_0$ is the point of preponderant density in O'Malley sense of the set $E_{f,x_0,\varepsilon}$ for every $\varepsilon > 0$.*
3. *If $f$ has $A_1$ property in Denjoy sense at $x_0$, then $\underline{d}\left(E^+_{f,x_0,\varepsilon}, x_0\right) > \frac{1}{2}$ and $\underline{d}\left(E^-_{f,x_0,\varepsilon}, x_0\right) > \frac{1}{2}$ for every $\varepsilon > 0$.*
4. *If $f$ has $A_1$ property in O'Malley sense at $x_0$, then $x_0$ is the point of preponderant density in O'Malley sense of the sets $E^+_{f,x_0,\varepsilon}$ and $E^-_{f,x_0,\varepsilon}$ for every $\varepsilon > 0$.*

**Proof:** 1) Let $E \subset U$ be a measurable set such that $x_0 \in E$, $\underline{d}(E, x_0) > \frac{1}{2}$ and $f_{|E}$ is continuous at $x_0$. Fix $\varepsilon > 0$. There exists $\delta > 0$ such that $E \cap [x_0 - \delta, x_0 + \delta] \subset E_{f,x_0,\varepsilon}$. Hence

$$\underline{d}\left(E_{f,x_0,\varepsilon}, x_0\right) \geq \underline{d}(E, x_0) > \tfrac{1}{2}.$$

2) proof of this fact is similar to the proof of 1) and we omit it.

3) Let $E_1$, $E_2$ be measurable subsets of $U$ such that $x_0 \in E_1 \cap E_2$, $\underline{d}(E_1, x_0) > \frac{1}{2}$, $\underline{d}(E_2, x_0) > \frac{1}{2}$, $f_{|E_1}$ is upper semicontinuous at $x_0$ and $f_{|E_2}$ is lower semicontinuous at $x_0$. Fix $\varepsilon > 0$. There exists $\delta > 0$ such that $E_1 \cap [x_0 - \delta, x_0 + \delta] \subset E^+_{f,x_0,\varepsilon}$ and $E_2 \cap [x_0 - \delta, x_0 + \delta] \subset E^-_{f,x_0,\varepsilon}$. Hence

$$\underline{d}\left(E^+_{f,x_0,\varepsilon}, x_0\right) \geq \underline{d}(E_1, x_0) > \tfrac{1}{2}$$

and

$$\underline{d}\left(E^-_{f,x_0,\varepsilon}, x_0\right) \geq \underline{d}(E_2, x_0) > \tfrac{1}{2}.$$

4) proof of this fact is similar to the proof of 3) and we omit it. □

**Example 1.** *We can construct a sequence of closed intervals $\{I_n = [a_n, b_n] \colon n \geq 1\}$ such that $0 < a_{n+1} < b_{n+1} < a_n$, $n \in \mathbb{N}$ and*

$$d^+\left(\bigcup_{n=1}^{\infty} I_{3n}, 0\right) = d^+\left(\bigcup_{n=0}^{\infty} I_{3n+1}, 0\right) = d^+\left(\bigcup_{n=0}^{\infty} I_{3n+2}, 0\right) = \frac{1}{3}.$$

*Define a function $f \colon \mathbb{R} \to \mathbb{R}$ as follows:*

$$f(x) = \begin{cases} 1 & \textit{if} \quad x \in \bigcup_{n=1}^{\infty} I_{3n}, \\ 0 & \textit{if} \quad x \in \bigcup_{n=0}^{\infty} I_{3n+1} \cup (-\infty, 0] \cup [b_1, \infty), \\ -1 & \textit{if} \quad x \in \bigcup_{n=0}^{\infty} I_{3n+2}, \\ \multicolumn{2}{l}{\textit{linear on the intervals of the set } [0, a_1) \setminus \bigcup_{n=1}^{\infty} I_n.} \end{cases}$$

*The function $f$ is continuous at every point except the point $0$. Let*

$$E_1 = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} I_{3n+1} \cup \bigcup_{n=1}^{\infty} I_{3n}, \quad E_2 = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} I_{3n+1} \cup \bigcup_{n=1}^{\infty} I_{3n+2}.$$

*Then $\underline{d}(E_1, 0) = \frac{2}{3}$, $\underline{d}(E_2, 0) = \frac{2}{3}$, $f_{|E_1}$ is lower semi-continuous and $f_{|E_2}$ is upper semi-continuous. Hence $f \in \mathcal{GPO}$. On the other hand,*

$$\underline{d}^+ (\{x\colon |f(x)| < 1\}, 0) \le \underline{d}^+ \left( \mathbb{R} \setminus \bigcup_{n=1}^{\infty} (I_{3n} \cup I_{3n+2}), 0 \right) = \frac{1}{3} < \frac{1}{2}.$$

*Thus $f \notin \mathcal{PO}$.*

**Corollary 7.**

$$\mathcal{GPO} \not\subset \mathcal{PO} \quad \text{and} \quad \mathcal{GPD} \not\subset \mathcal{PD}.$$

**Example 2.** *Let $I_n = [3^{-n} - 8^{-n}, 2\cdot 3^{-n} + 8^{-n}]$, $J_n = [2\cdot 3^{-n} + 6^{-n}, 3^{-n+1} - 6^{-n}]$ for $n = 1, 2, \ldots$. For every $n \ge 1$ we have*

$$\frac{\lambda(\bigcup_{i=1}^{\infty} I_i \cap [0, 3^{-n} - 8^{-n}])}{3^{-n} - 8^{-n}} = \frac{\lambda(\bigcup_{i=n+1}^{\infty} I_i \cap [0, 3^{-i} - 8^{-i}])}{3^{-n} - 8^{-n}} =$$

$$= \frac{\sum_{i=n+1}^{\infty} (3^{-i} + 2\cdot 8^{-i})}{3^{-n} - 8^{-n}} = \frac{\frac{3}{2} 3^{-n-1} + \frac{16}{7} 8^{-n-1}}{3^{-n} - 8^{-n}} = \frac{1}{2} \cdot \frac{3^{-n} + \frac{4}{7} 8^{-n}}{3^{-n} - 8^{-n}} > \frac{1}{2}.$$

*By Corollary 6, $0$ is the point of preponderant density of the set $(-\infty, 0] \cup \bigcup_{n=1}^{\infty} I_n$. On the other hand, $\bigcup_{n=1}^{\infty} I_n \cap \bigcup_{n=1}^{\infty} J_n = \emptyset$ and*

$$\frac{\lambda(\bigcup_{i=1}^{\infty} J_i \cap [0, 3^{-n+1} - 6^{-n}])}{3^{-n+1} - 6^{-n}} = \frac{\lambda(\bigcup_{i=n}^{\infty} J_i \cap [0, 3^{-n+1} - 6^{-n}])}{3^{-n+1} - 6^{-n}} =$$

$$= \frac{\sum_{i=n}^{\infty} (3^{-i} - 2\cdot 6^{-i})}{3^{-n+1} - 6^{-n}} = \frac{\frac{3}{2} 3^{-n} - \frac{12}{5} 6^{-n}}{3^{-n+1} - 6^{-n}} = \frac{1}{2} \cdot \frac{3^{-n+1} - \frac{24}{5} 6^{-n}}{3^{-n+1} - 6^{-n}}$$

*for every $n \ge 1$. Hence $\overline{d}^+ (\bigcup_{n=1}^{\infty} J_n, 0) = \frac{1}{2}$. Define a function $f\colon \mathbb{R} \to \mathbb{R}$,*

$$f(x) = \begin{cases} 0 \quad \text{for} \quad x \in (-\infty, 0] \cup [1, \infty) \cup \bigcup_{n=1}^{\infty} I_n, \\ 1 \quad \text{for} \quad x \in \bigcup_{n=1}^{\infty} J_n, \\ \text{linear on the intervals } [2\cdot 3^{-n} + 8^{-n}, 2\cdot 3^{-n} + 6^{-n}] \\ \qquad \text{and } [3^{-n} - 6^{-n}, 3^{-n} - 8^{-n}], \qquad n = 1, 2, \ldots. \end{cases}$$

*Certainly, $f$ is continuous at every point except the point $0$. Since $f(x) = 0$ for each $x \in (-\infty, 0] \cup \bigcup_{n=1}^{\infty} I_n$, we conclude that $f$ is preponderantly* *continuous in O'Malley sense at* 0. *Hence* $f \in \mathcal{PO} \cap \mathcal{GPO}$. *On the other hand,* $\bigcup_{n=1}^{\infty} J_n \subset \{x \in \mathbb{R} \colon f(x) > \frac{1}{2}\}$. *Thus*

$$\underline{d}\left(\{x \in \mathbb{R} \colon f(x) < f(0) + \tfrac{1}{2}\}, 0\right) \le 1 - \overline{d}\left(\bigcup_{n=1}^{\infty} J_n, 0\right) = \tfrac{1}{2}.$$

*Therefore,* $f$ *does not have the* $A_1$ *property in Denjoy sense at* 0 *and* $f \notin \mathcal{PD} \cup \mathcal{GPD}$.

**Corollary 8.**

$$\mathcal{PO} \not\subset \mathcal{PD} \quad \textit{and} \quad \mathcal{GPO} \not\subset \mathcal{GPD}.$$

All proven inclusions between defined classes of functions are presented in the following diagram:

$$\begin{array}{ccc} \mathcal{PD} & \longrightarrow & \mathcal{PO} \\ \downarrow & & \downarrow \\ \mathcal{GPD} & \longrightarrow & \mathcal{GPO} \end{array}$$

We proved that no other inclusion holds.

**Theorem 1.** *If* $f \in \mathcal{GPO}$ *then* $f$ *is the Baire class 1 function.*

**Proof:** (This proof is based on the proof of Theorem 1 in [5].)

Suppose that there exists $f \in \mathcal{GPO}$ which is not the Baire class 1 function. Then we can find a perfect set $P$ such that $f_{|P}$ is discontinuous at every point. Let $\omega(f, P, x)$ be an oscillation of $f_{|P}$ at $x$ and let

$$P_n = \left\{x \in P \colon \omega(f, P, x) \ge \tfrac{1}{n}\right\}$$

for $n \ge 1$. Then every set $P_n$ is closed and $P = \bigcup_{n=1}^{\infty} P_n$. Since $P$ is a complete space, there exist $k \in \mathbb{N}$ and a closed non-degenerate interval $I_1$ such that $\emptyset \ne P \cap \mathrm{Int}(I_1)$ and $P_k \cap I_1 = P \cap I_2$. Next, let

$$P_k^m = \left\{x \in P \cap I_1 \colon f(x) \in \left[\tfrac{m}{3k}, \tfrac{m+1}{3k}\right)\right\}$$

for $m \in \mathbb{Z}$. Again, using completeness of the set $P \cap I_1$ we can find $l \in \mathbb{N}$ and a closed non-degenerate interval $I_2 \subset I_1$ for which $P_k^l$ is a dense subset of $P \cap \mathrm{Int}(I_2) \ne \emptyset$. Since $\omega(f, P, x) \ge \frac{1}{k}$ for each $x \in P_k^l \subset P_k$, we get that

$$\left\{x \in P \cap I_2 \colon f(x) \le \tfrac{l-1}{3k}\right\} \cup \left\{x \in P \cap I_2 \colon f(x) \ge \tfrac{l+2}{3k}\right\}$$

is a dense subset of $P \cap I_2$. Hence there exists a closed non-degenerate interval $I_3 \subset I_2$ such that one of the components of this sum is dense in $P \cap \mathrm{Int}(I_3) \neq \emptyset$. Since

$$P_k^l \cap I_3 \subset \left\{x \in P \cap I_3 \colon f(x) \le \tfrac{l+1}{3k}\right\} \quad \text{and} \quad P_k^l \cap I_3 \subset \left\{x \in P \cap I_3 \colon f(x) \ge \tfrac{l}{3k}\right\},$$

we get that the sets $\{x \in P \cap I_3 \colon f(x) \le \frac{l+1}{3k}\}$ and $\{x \in P \cap I_3 \colon f(x) \ge \frac{l}{3k}\}$ are dense in $P \cap I_3$.

Thus we have proved that there exist a perfect set $F = P \cap I_3$ and $\alpha, \beta \in \mathbb{R}$, ($\alpha = \frac{l-1}{3k}$ and $\beta = \frac{l}{3k}$ or $\alpha = \frac{l+1}{3k}$ and $\beta = \frac{l+2}{3k}$ ), $\alpha < \beta$ such that the sets $\{x \colon f(x) \ge \beta\}$ and $\{x \colon f(x) \le \alpha\}$ are dense in $F$.

Let

$$B_1 = F \cap \{x \colon f(x) \le \alpha\}, \quad B_2 = F \cap \{x \colon f(x) \ge \beta\},$$

$$B_3 = F \cap \left\{x \colon f(x) \le \alpha + \tfrac{2(\beta-\alpha)}{3}\right\} \quad \text{and} \quad B_4 = F \cap \left\{x \colon f(x) \ge \alpha + \tfrac{\beta-\alpha}{3}\right\}.$$

Now, we use the assumption that $f \in \mathcal{GPO}$. Let $x \in B_i$, $i = 1,2,3,4$ and let $E_1$, $E_2$ be the sets satisfying the conditions of Definition 8. Then there exists $r > 0$ such that

$$[x-r, x+r] \cap E_1 \subset B_i \quad \text{if} \quad i = 1,3 \quad \text{and} \quad [x-r, x+r] \cap E_2 \subset B_i \quad \text{if} \quad i = 2,4.$$

Thus for each $x \in B_i$, $i = 1,2,3,4$ we can find a positive rational number $r_i(x)$ for which, if $0 < \delta < r_i(x)$, then $\lambda(B_i \cap [x, x+\delta]) > \frac{1}{2}\delta$ and $\lambda(B_i \cap [x-\delta, x]) > \frac{1}{2}\delta$. Let

$$D_i^r = \{x \in B_i \colon r_i(x) = r\} \quad \text{for} \quad i = 3,4 \quad \text{and} \quad r \in \mathbb{Q}^+.$$

Since $B_3 \cup B_4 = F$, $B_3 = \bigcup_{r \in \mathbb{Q}^+} D_3^r$ and $B_4 = \bigcup_{r \in \mathbb{Q}^+} D_4^r$, we get that there exist $q \in \mathbb{Q}^+$, $j \in \{3,4\}$ and a closed interval $I_4$ such that the set $D_j^q$ is dense in $F \cap \mathrm{Int}(I_4) \neq \emptyset$. Without loss of generality, we can assume that $j = 3$. Let $y \in B_2 \cap I_4$. Since the set $B_2$ is dense in $F$, such a point exists. Let $r = \min\{r_2(y), q\}$ and $z \in D_3^q \cap I_4 \cap [y-r, y+r]$. Then $\lambda(B_2 \cap [y,z]) > \frac{1}{2}\lambda([y,z])$, because $|z-y| < r_2(y)$, and $\lambda(B_3 \cap [y,z]) > \frac{1}{2}\lambda([y,z])$, because $|z-y| < r_3(z) = q$. Since $\lambda(B_2 \cap [y,z]) + \lambda(B_3 \cap [y,z]) > \lambda([y,z])$, we get that $B_2 \cap B_3 \neq \emptyset$. This contradicts the fact that $B_2 \cap B_3 = \emptyset$. The proof is complete. □

**Corollary 9.**

$$\mathcal{PO} \subset \mathcal{B}_1, \quad \mathcal{GPD} \subset \mathcal{B}_1 \quad \textit{and} \quad \mathcal{PD} \subset \mathcal{B}_1.$$

Now, we would like to show that inverse of the Propositinon 2 is true. We start from the preponderant continuity and $A_1$ condition in Denjoy sense.

**Lemma 1.** *If $F$ is a measurable subset of $\mathbb{R}$ and $\underline{d}^+(F, x_0) = q > 0$, then:*

$$\forall_{n\in\mathbb{N}}\ \exists_{\varepsilon>0}\ \forall_{0<a<b<\varepsilon}\quad \frac{\lambda\left(\left[x_0+\frac{a}{2n}, x_0+b\right]\cap F\right)}{\lambda\left([x_0, x_0+b]\right)} > q - \frac{1}{n}.$$

**Proof:** Take $n \in \mathbb{N}$. Since $\underline{d}^+(F, x_0) = q$, there exists $\varepsilon > 0$ such that

$$\frac{\lambda\left([x_0, x_0+c]\cap F\right)}{\lambda\left([x_0, x_0+c]\right)} > q - \frac{1}{4n}$$

for each $0 < c < \varepsilon$. If $0 < a < b < \varepsilon$, then

$$\lambda([x_0+\tfrac{a}{2n}, x_0+b]\cap F) = \lambda([x_0, x_0+b]\cap F) - \lambda([x_0, x_0+\tfrac{a}{2n}]\cap F) \geq$$
$$\geq b(q-\tfrac{1}{4n}) - \tfrac{a}{2n} > bq - \tfrac{3b}{4n}.$$

Therefore $\frac{\lambda\left(\left[x_0+\frac{a}{2n}, x_0+b\right]\cap F\right)}{\lambda([x_0, x_0+b])} \geq \frac{bq-\frac{3b}{4n}}{b} > q - \frac{1}{n}$. Thus the proof is completed. □

An analogous lemma holds for a left-sided neighbourhood of $x$.

**Lemma 2.** *Let $\{E_n : n \geq 1\}$ be a descending family of measurable sets, $x_0 \in \bigcap_{n=1}^{\infty} E_n$ and $\underline{d}(E_n, x_0) \geq q > 0$ for $n \in \mathbb{N}$. Then there exists a measurable set $E$ such that $\underline{d}(E, x_0) \geq q$, $x_0 \in E$ and for every $n \in \mathbb{N}$ one can find $\delta_n > 0$ for which $E \cap [x_0-\delta_n, x_0+\delta_n] \subset E_n$.*

**Proof:** By assumptions, $\underline{d}^+(E_n, x_0) \geq q > 0$ for every $n \in \mathbb{N}$. Let $\varepsilon_n$ fulfils Lemma 1 for $F = E_n$, $n \in \mathbb{N}$. By recursion, one can construct a sequence $\{a_n\}_{n\in\mathbb{N}}$ of positive reals such that $a_1 < \varepsilon_1$ and $a_{n+1} < \min\left\{\varepsilon_{n+1}, \frac{a_n}{2n}\right\}$ for $n \geq 1$. Let

$$H = \bigcup_{n=1}^{\infty}\left(E_n \cap [x_0+a_{n+2}, x_0+a_{n+1}]\right).$$

We shall show that $\underline{d}^+(H, x_0) \geq q$. Fix $n \in \mathbb{N}$ and take $c \in [a_{n+1}, a_n]$. Then

$$H \cap [x_0+a_{n+2}, x_0+c] \supset E_n \cap [x_0+a_{n+2}, x_0+c]$$

and applying Lemma 1, we get

$$\lambda\left(E_n \cap \left[x_0+\tfrac{a_{n+1}}{2n}, x_0+c\right]\right) > c\left(q-\tfrac{1}{n}\right).$$

Since $\frac{a_{n+1}}{2n} > a_{n+2}$, we get

$$\lambda(H\cap[x_0, x_0+c]) > \lambda\left(H\cap\left[x_0+\tfrac{a_{n+1}}{2n}, x_0+c\right]\right) \geq$$
$$\geq \lambda\left(E_n\cap\left[x_0+\tfrac{a_{n+1}}{2n}, x_0+c\right]\right) > c\cdot(q-\frac{1}{n}).$$

It implies that $\underline{d}^+(H, x_0) \geq q$. Exactly from the definition of the set $H$ we get $H \cap [x_0, x_0 + a_{n+1}] \subset E_n$.

In a similar way, we can construct a measurable set $G \subset (-\infty, x_0)$ such that $\underline{d}^-(G, x_0) \geq q$ and for every $n \in \mathbb{N}$ there is $\delta_n > 0$ for which $G \cap [x_0 - \delta_n, x_0] \subset E_n$. The set $E = H \cup G \cup \{x_0\}$ has all required properties. $\square$

Using the last Lemma we can easily prove the following Theorem.

**Theorem 2.** *(i) A measurable function $f\colon U \to \mathbb{R}$ is preponderantly continuous in Denjoy sense at $x_0 \in U$ iff $\lim_{n\to\infty} \underline{d}\Big(E_{f,x_0,\frac{1}{n}}, x_0\Big) > \frac{1}{2}$,*

*(ii) A measurable function $f\colon U \to \mathbb{R}$ has $A_1$ property in Denjoy sense at $x_0 \in U$ iff*

$$\lim_{n\to\infty} \underline{d}\Big(E^+_{f,x_0,\frac{1}{n}}, x_0\Big) > \frac{1}{2} \quad \textit{and} \quad \lim_{n\to\infty} \underline{d}\Big(E^-_{f,x_0,\frac{1}{n}}, x_0\Big) > \frac{1}{2}.$$

**Proof:** $(i)$ If $f$ is preponderantly continuous in Denjoy sense at $x_0$, then there exists a measurable set $E$ containing $x_0$ such that $f_{|E}$ is continuous at $x_0$ and $\underline{d}(E, x_0) = q > \frac{1}{2}$. Then we get

$$\lim_{n\to\infty} \underline{d}\Big(E_{f,x_0,\frac{1}{n}}, x_0\Big) \geq \underline{d}(E, x_0) = q > \frac{1}{2}.$$

Now, suppose that $\lim_{n\to\infty} \underline{d}\left(E_{f,x_0,\frac{1}{n}}, x_0\right) = q > \frac{1}{2}$. Then $\underline{d}\left(E_{f,x_0,\frac{1}{n}}, x_0\right) \geq q > \frac{1}{2}$ for every $n \in \mathbb{N}$ and $x_0 \in \bigcap_{n=1}^{\infty} E_{f,x_0,\frac{1}{n}}$. Applying Lemma 2 with $E_n = E_{f,x_0,\frac{1}{n}}$ for $n \in \mathbb{N}$, one can construct a measurable set $E$ such that $x_0 \in E$, $\underline{d}(E, x_0) > \frac{1}{2}$ and for every $n$ there exists $\delta_n > 0$ for which $E \cap [x_0 - \delta_n, x_0 + \delta_n] \subset E_n$. The last condition implies that $f_{|E}$ is continuous at $x_0$. Hence $f$ is preponderantly continuous in Denjoy sense at $x_0$. (ii) the proof is similar and we omit it. $\square$

Now, we show equivalent conditions for prepondrant continuity and $A_1$ property in O'Malley sense.

**Lemma 3.** *If $F \subset \mathbb{R}$ is a measurable set and $x_0$ is a point of preponderant density in O'Malley sense of the set $F$, then:*

$$\exists_{\varepsilon>0} \forall_{0<b<a<\varepsilon} \exists_{0<c<b} \forall_{b<\eta<a} \frac{\lambda([x_0 + c, x_0 + \eta] \cap F)}{\lambda([x_0, x_0 + \eta])} > \frac{1}{2}.$$

**Proof:** There exists $\varepsilon > 0$ such that $\lambda([x_0, x_0 + a] \cap F) > \frac{1}{2} \cdot a$ for any $0 < a \leq \varepsilon$. Let us take any $a, b \in U$, $0 < b < a < \varepsilon$ and define a function $f\colon [b, a] \to \mathbb{R}$,

$$f(t) = \lambda\big([x_0, x_0 + t] \cap F\big) - \frac{t}{2}.$$

Since the function $f$ is continuous and $f(t) > 0$ for any $t \in [b, a]$, we conclude that $c = \frac{1}{2} \cdot \min\{f(t) \colon t \in [b, a]\} > 0$. Let $b < \eta < a$. Then $\lambda(F \cap [x_0, x_0 + \eta]) = f(\eta) + \frac{\eta}{2}$ and therefore

$$\lambda(F \cap [x_0 + c, x_0 + \eta]) = \lambda(F \cap [x_0, x_0 + \eta]) - \lambda(F \cap [x_0, x_0 + c]) \geq$$
$$\geq f(\eta) + \tfrac{\eta}{2} - c \geq 2c + \tfrac{\eta}{2} - c > \tfrac{\eta}{2}.$$

Thus the proof is completed. □

An analogous lemma holds for a left-sided neighbourhood of $x_0$.

**Lemma 4.** *Let $\{E_n \colon n \geq 1\}$ be a descending family of measurable subsets of the real line, $x_0 \in \bigcap_{n=1}^{\infty} E_n$ and $x_0$ is a point of preponderant density in O'Malley sense of every set $E_n$ for $n \in \mathbb{N}$. Then there exists a measurable set $E$ such that $x_0$ is the point of preponderant density in O'Malley sense of the set $E$, $x_0 \in E$ and for every $n \in \mathbb{N}$ we can find $\delta_n > 0$ for which $E \cap [x_0 - \delta_n, x_0 + \delta_n] \subset E_n$.*

**Proof:** Let $\varepsilon_n$ fulfils Lemma 3 with $F = E_n$ for $n \geq 1$. We may assume that the sequence $\{\varepsilon_n \colon n \geq 1\}$ is strictly decreasing. We shall construct a decreasing sequence of positive reals $\{a_n \colon n \geq 1\}$. Let $a_1$ be any real number belonging to an open interval $(0, \varepsilon_1)$ and let $a_2$ be any real number belonging to an an open interval $(0, a_1\})$. If numbers $a_1, \ldots, a_n,\ 0 < a_n < \ldots < a_1$ are chosen and $c$ fulfils Lemma 3 with $F = E_{n-1}$, $\varepsilon = \varepsilon_{n-1}$, $a = a_{n-1}$ and $b = a_n$, then we take $a_{n+1} = \min\{c, \frac{\varepsilon_{n+1}}{2}\}$. Let

$$H = \bigcup_{n=1}^{\infty} \left(E_n \cap [x_0 + a_{n+2}, x_0 + a_{n+1}]\right).$$

It follows directly from the definition of the set $H$ that $H \cap [x_0, x_0 + a_{n+1}] \subset E_n$ and

$$E_n \cap [x_0 + a_{n+2}, x_0 + z] \subset H \cap [x_0, x_0 + z] \subset E_n$$

for any $n \in \mathbb{N}$ and $z \in [a_{n+2}, a_{n+1}]$. Hence, if $z \in [a_{n+2}, a_{n+1}]$, then applying Lemma 3, we get

$$\lambda(H \cap [x_0, x_0 + z]) \geq \lambda(E_n \cap [x_0 + a_{n+2}, x_0 + z]) > \frac{z}{2}.$$

In a similar way, we can construct a measurable set $G \subset (-\infty, x_0)$ such that $x_0$ is the point of preponderant density in O'Malley's sense of the set $G$ and for every $n \in \mathbb{N}$ we can find $\delta_n > 0$ for which $G \cap [x_0 - \delta_n, x_0] \subset E_n$. The set $E = H \cup G \cup \{x_0\}$ has all required properties. □

Using the last Lemma we can prove the following Theorem.

**Theorem 3.** *(i) A measurable function $f: U \to \mathbb{R}$ is preponderantly continuous in O'Malley sense at $x_0 \in U$ iff $x_0$ is the point of preponderant density in O'Malley sense of the set $E_{f,x_0,\varepsilon}$ for each $\varepsilon > 0$,*

*(ii) A measurable function $f: U \to \mathbb{R}$ has $A_1$ property in O'Malley sense at a point $x_0 \in U$ iff $x_0$ is a point of preponderant density in O'Malley sense of both sets $E^+_{f,x_0,\varepsilon}$ and $E^-_{f,x_0,\varepsilon}$ for each $\varepsilon > 0$.*

**Proof:** $(i)$ If $f$ is preponderantly continuous in O'Malley sense at $x_0$, then there exists a measurable set $E$ containing $x_0$ such that $f_{|E}$ is continuous at $x_0$ and $x_0$ is the point of preponderant density of the set $E$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$E \cap [x_0 - \delta, x_0 + \delta] \subset \{y: |f(x) - f(y)| < \varepsilon\} \cap [x_0 - \delta, x_0 + \delta].$$

Hence $x_0$ is the point of preponderant density of every set $E_{f,x_0,\varepsilon}$ for $\varepsilon > 0$.

Now, suppose that $x_0$ is the point of preponderant density of every set $E_{f,x_0,\varepsilon}$ for each $\varepsilon > 0$. Applying Lemma 4 with $E_n = E_{f,x_0,\frac{1}{n}}$ for $n \geq 1$, we can construct a measurable set $E$ such that $x_0 \in E$, $x_0$ is the point of preponderant density of the set $E$ and for every $n \in \mathbb{N}$ there exists $\delta_n > 0$ for which $E \cap [x_0 - \delta_n, x_0 + \delta_n] \subset E_n$. The last condition implies that $f_{|E}$ is continuous at $x_0$. Thus $f$ is preponderantly continuous in O'Malley sense at $x_0$.

(ii) the proof is analogous and we omit it. □

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