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THE STRONG 3-RAINBOW INDEX OF SOME CERTAIN GRAPHS AND ITS AMALGAMATION

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Abstract. We introduce a strong k-rainbow index of graphs as modification of well-known k-rainbow index of graphs. A tree in an edge-colored connected graph G, where adjacent edge may be colored the same, is a rainbow tree if all of its edges have distinct colors. Let k be an integer with $2 \leq k \leq n$. The strong k-rainbow index of G, denoted by $srx_k(G)$, is the minimum number of colors needed in an edge-coloring of G so that every k vertices of G is connected by a rainbow tree with minimum size. We focus on $k = 3$. We determine the strong 3-rainbow index of some certain graphs. We also provide a sharp upper bound for the strong 3-rainbow index of amalgamation of graphs. Additionally, we determine the exact values of the strong 3-rainbow index of amalgamation of some graphs.

Keywords: amalgamation, rainbow coloring, rainbow Steiner tree, strong k-rainbow index.

Mathematics Subject Classification: 05C05, 05C15, 05C40.

1. INTRODUCTION

All graphs considered in this paper are simple, finite, and connected. We follow the terminology and notation of Diestel [7]. For simplifying, we define $[a, b]$ as a set of all integers x with $a \leq x \leq b$. Let G be an edge-colored graph of order $n \geq 3$, where adjacent edges may be colored the same. A path P in G is a *rainbow path*, if no two edges of P are colored the same. The minimum number of colors needed in an edge-coloring of G such that there exists a rainbow $u - v$ path for each pair u and v of distinct vertices of G is the *rainbow connection number* $rc(G)$ of G. It was first introduced by Chartrand *et al.* in 2008 [4].

In [2], Caro *et al.* conjectured that deciding whether a graph G has $rc(G) = 2$ is NP-Complete, in particular, computing $rc(G)$ is NP-Hard. This conjecture then was proved by Chakraborty *et al.* [3]. They also proved that to decide whether a given edge-colored graph is rainbow-connected (i.e., the graph contains a rainbow

$u-v$ path for every two vertices u and v of graph [4]) is NP-Complete. Many authors also investigated bounds, algorithms, and computational complexity of the rainbow connection number of graphs (see [11–13]). Other known results about rainbow connection number of graphs can be found in [4, 9, 10, 14–16]. The rainbow connection number of amalgamation of some graphs were done in [9].

Chartrand *et al.* in 2010 [4] introduced the concept of *strong rainbow connection number* $src(G)$ of G, that is, the minimum number of colors needed in an edge-coloring of G such that for each pair u and v of distinct vertices of G, there exists a rainbow $u-v$ path of length $d(u,v)$, where $d(u,v)$ is the *distance* between u and v (the length of a shortest $u-v$ path in G). Such a rainbow $u-v$ path is called a *rainbow* $u-v$ *geodesic*. They provided lower and upper bounds for the strong rainbow connection number, that is $diam(G) \leq rc(G) \leq src(G) \leq |E(G)|$, where $diam(G)$ is the diameter of G (the largest distance between two vertices of G).

Rainbow connection has an interesting application for the secure transfer of classified information between agencies. Ericksen [8] stated that the attacks on September 11, 2001, was the realization that intelligence agencies were not able to communicate with each other through their regular channels, from radio systems to databases. There was no way for them to cross check information between various organizations. Although the information needed to be protected since it is vital to national security, procedures must be in place so that the appropriate parties can access the information and communicate with each other. This can be addressed by assigning a large enough number of passwords and firewalls to the information transfer paths between agencies (which may have other agencies as intermediaries) so that any path between agencies has no password repeated. An immediate question arises: What is the minimum number of passwords or firewalls needed that allows one or more secure paths between every two agencies so that the passwords along each path are distinct? This situation can be modeled by a graph and studied by the means of rainbow coloring.

In real life, it is possible to have more than two agencies to communicate with each other. Therefore, Chartrand *et al.* [5] introduced the generalization of the rainbow connection number called the *k-rainbow index* $rx_k(G)$ of G, where k denote the number of agencies that communicate with each other. A tree T in G is called a *rainbow tree*, if no two edges of T are colored the same. For $S \subseteq V(G)$, a *rainbow S-tree* is a rainbow tree containing S. Let k be an integer with $k \in [2,n]$. A *k-rainbow coloring* of G is an edge-coloring of G having property that for every set S of k vertices of G, there exists a rainbow S-tree in G. The k-rainbow index $rx_k(G)$ of G is the minimum number of colors needed in a k-rainbow coloring of G. It is obvious that $rx_2(G) = rc(G)$. It follows, for every nontrivial connected graph G of order n, that $rx_2(G) \leq rx_3(G) \leq \ldots \leq rx_n(G)$.

The *Steiner distance* $d(S)$ of a set S of vertices in G is the minimum size of a tree in G containing S. Such a tree is called a *Steiner S-tree*. The *k-Steiner diameter* $sdiam_k(G)$ of G is the maximum Steiner distance of S among all sets of S with k vertices in G. Thus, if $k=2$ and $S=\{u,v\}$, then $d(S)=d(u,v)$ and $sdiam_2(G)=diam(G)$. It is easy to see that $diam(G) \leq sdiam_3(G) \leq \ldots \leq sdiam_n(G)$. Chartrand *et al.* stated that for every connected graph G of order $n \geq 3$ and each integer k with $k \in [3,n]$, $k-1 \leq sdiam_k(G) \leq rx_k(G) \leq n-1$ [5].

In [5], Chartrand *et al.* showed that trees are composed of a class of graphs whose k-rainbow index attains the upper bound for $rx_k(G)$. They also determined the k-rainbow index of cycles for general k. Chen *et al.* [6] determined the 3-rainbow index of regular complete bipartite graphs and wheels. We determined the 3-rainbow index of amalgamation of some graphs with diameter 2 [1].

Theorem 1.1 ([5])**.** *Let T_n be a tree of order $n \geq 3$. For each integer k with $k \in [3, n]$, $rx_k(T_n) = n - 1$.*

Theorem 1.2 ([5])**.** *Let C_n be a cycle of order $n \geq 3$. For each integer k with $k \in [3, n]$,*

$$rx_k(C_n) = \begin{cases} n-2 & \text{if } k = 3 \text{ and } n \geq 4, \\ n-1 & \text{if } k = n = 3 \text{ or } k \in [4, n]. \end{cases}$$

Theorem 1.3 ([6])**.** *For $n \geq 3$, let $K_{n,n}$ be a regular complete bipartite graph of order $2n$. Then $rx_3(K_{n,n}) = 3$.*

Theorem 1.4 ([6])**.** *For $n \geq 3$, let W_n be a wheel of order $n + 1$. Then*

$$rx_3(W_n) = \begin{cases} 2, & n = 3, \\ 3, & n \in [4, 6], \\ 4, & n \in [7, 16], \\ 5, & n \geq 17. \end{cases}$$

One of the things that is also considered in making a secure communication network is the time needed so that every k agencies can access the information and communicate with each other safely. In order to model this problem, we introduce a generalization of the k-rainbow index of G called the *strong k-rainbow index* of G, denoted by $srx_k(G)$. A *rainbow Steiner S-tree* is a rainbow S-tree of size $d(S)$. An edge-coloring of G is called a *strong k-rainbow coloring* of G, if for every set S of k vertices of G, there exists a rainbow Steiner S-tree in G. The minimum number of colors needed in a strong k-rainbow coloring of G is the strong k-rainbow index $srx_k(G)$ of G. Thus, we have $rx_k(G) \leq srx_k(G)$ for every connected graph G. This concept is useful when agencies want to communicate with each other or transfer information as quickly as possible.

Note that every coloring that assigns distinct colors to all edges of a connected graph is a strong k-rainbow coloring. Thus, the strong k-rainbow index is defined for every connected graph G. Furthermore, if G is a nontrivial connected graph of size $|E(G)|$ whose k-Steiner diameter is $sdiam_k(G)$, then it is easy to check that

$$sdiam_k(G) \leq rx_k(G) \leq srx_k(G) \leq |E(G)|. \tag{1.1}$$

To illustrate these concepts, consider a graph G as shown in Figure 1. We show that $srx_3(G) = 4$. Given a tree T of size m, we define $T = \{e_1, e_2, \ldots, e_m\}$ as a tree with edge set $\{e_1, e_2, \ldots, e_m\}$. Since $sdiam_3(G) = 4$, we have $srx_3(G) \geq 4$ by (1.1). Next, we show that an edge-coloring shown in Figure 1 is a strong 3-rainbow coloring of G. It suffices to show that for every set S of three vertices of G, there exists a rainbow Steiner S-tree. Note that there are $\binom{8}{3}$ sets S of three vertices of G, where $\binom{8}{3}$ denotes the number of combinations of 8 vertices taken 3 at a time. For instances, if $S = \{v_2, v_6, v_8\}$, then $T = \{v_1v_2, v_1v_6, v_1v_7, v_7v_8\}$ is a rainbow Steiner S-tree. If $S = \{v_1, v_3, v_8\}$, then $T = \{v_1v_7, v_7v_8, v_8v_4, v_4v_3\}$ is a rainbow Steiner S-tree. If $S = \{v_1, v_5, v_8\}$, then $T = \{v_1v_6, v_6v_5, v_5v_4, v_4v_8\}$ is a rainbow Steiner S-tree. By considering other sets S, it is easy to find a rainbow Steiner S-tree. Hence, $srx_3(G) = 4$.

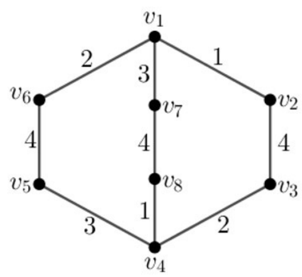

Fig. 1. A strong 3-rainbow coloring of G

Next, we consider a connected graph G which contains some bridges. Let $e = uv$ and $f = xy$ be two bridges of G. Then $G - e - f$ contains three components G_1, G_2, and G_3. Without loss of generality, let $u \in V(G_1)$, $y \in V(G_2)$, and $v, x \in V(G_3)$. If S is a set of k vertices containing u and y, then every rainbow Steiner S-tree must contains bridges e and f. This gives us the following observation.

Observation 1.5. *Let G be a connected graph of order n. Let $e, f \in E(G)$, where e and f are the bridges of G. For each integer k with $k \in [2, n]$, every strong k-rainbow coloring must assign distinct colors to e and f.*

The following theorem is an immediate consequence of Observation 1.5. We show that the strong k-rainbow index of trees attains the upper bound in (1.1) for every k with $k \in [3, n]$.

Theorem 1.6. *Let T_n be a tree of order $n \geq 3$. For each integer k with $k \in [3, n]$, $srx_k(T_n) = |E(T_n)| = n - 1$.*

One of the ways that can be done to make a larger and complex communication network is by extending the previous networks. In other words, it can be modeled by doing some operation on the graphs. In this paper, we study about the amalgamation of graphs. For an integer $t \geq 2$, let $\{G_1, G_2, \ldots, G_t\}$ be a collection of finite, simple, and connected graphs and each G_i has a fixed vertex v_{o_i} called a *terminal vertex*. The *amalgamation* of $G_1, G_2, \ldots, G_t$, denoted by $Amal\{G_i; v_{o_i}\}$, is a graph obtained by taking all the G_i's and identifying their terminal vertices. If for each $i \in [1, t]$, $G_i \cong G$ and $v_{o_i} = v$, then $Amal\{G_i; v_{o_i}\}$ is denoted by $Amal(G, v, t)$. The study of amalgamation of graphs is needed when we want to make a larger and complex communication network and some agencies must pass through the center in order to communicate with each other.

This paper is organized as follows. In Section 2, we determine the strong 3-rainbow index of some certain graphs, such as ladders, regular complete bipartite graphs, cycles, and wheels. In Section 3, we provide a sharp upper bound for the strong 3-rainbow index of $Amal(G, v, t)$. We also determine the exact values of the strong 3-rainbow index of $Amal(G, v, t)$ for some connected graphs G.

2. THE STRONG 3-RAINBOW INDEX OF SOME CERTAIN GRAPHS

In this section, we determine the exact values of the strong 3-rainbow index of some certain graphs. First, we consider ladder graphs L_n. A *ladder* graph L_n is the *Cartesian product* of a P_n and a P_2, where P_n is a path of order n. In the following theorem, we show that the strong 3-rainbow index of L_n attains the lower bound in (1.1).

Theorem 2.1. *For $n \geq 3$, let L_n be a ladder graph of order $2n$. Then $srx_3(L_n) = n$.*

Proof. Let
$$V(L_n) = \{v_i \colon i \in [1, 2n]\}$$
be such that
$$E(L_n) = \{v_i v_{i+1} \colon i \in [1, n-1] \cup [n+1, 2n-1]\} \cup \{v_i v_{i+n} \colon i \in [1, n]\}.$$

It is easy to check that $sdiam_3(L_n) = n$, thus $srx_3(L_n) \geq n$ by (1.1). Next, we show that $srx_3(L_n) \leq n$. We define an edge-coloring $c : E(L_n) \to [1, n]$ which can be obtained by assigning colors i to the edges $v_i v_{i+1}$ and $v_{i+n} v_{i+n+1}$ for all $i \in [1, n-1]$ and color n to the edges $v_i v_{i+n}$ for all $i \in [1, n]$.

Now, we show that c is a strong 3-rainbow coloring of L_n. It suffices to show that for every set S of three vertices of L_n, there exists a rainbow Steiner S-tree. By symmetry, we consider two cases.

Case 1. $S = \{v_i, v_j, v_k\}$ with $i, j, k \in [1, n]$ and $i < j < k$.

Then $T = \{v_l v_{l+1} \colon l \in [i, k-1]\}$ is a rainbow Steiner S-tree.

Case 2. $S = \{v_i, v_j, v_k\}$ with $i, j \in [1, n]$, $i < j$, and $k \in [n+1, 2n]$.

If $k \leq i + n$, then $T = \{v_k v_{k-n}\} \cup \{v_l v_{l+1} \colon l \in [k-n, j-1]\}$ is a rainbow Steiner S-tree. If $i + n < k < j + n$, then $T = \{v_k v_{k-n}\} \cup \{v_l v_{l+1} \colon l \in [i, j-1]\}$ is a rainbow Steiner S-tree. If $k \geq j + n$, then $T = \{v_l v_{l+1} \colon l \in [i, k-n-1]\} \cup \{v_k v_{k-n}\}$ is a rainbow Steiner S-tree. □

Next, we determine the strong 3-rainbow index of regular complete bipartite graphs $K_{n,n}$.

Theorem 2.2. *For $n \geq 3$, let $K_{n,n}$ be a regular complete bipartite graph of order $2n$. Then $srx_3(K_{n,n}) = n$.*

Proof. Let $U = \{u_i \colon i \in [1, n]\}$ and $W = \{w_i \colon i \in [1, n]\}$ be the partite sets of $K_{n,n}$. Let c be a strong 3-rainbow coloring of $K_{n,n}$. For $i, j, k \in [1, n]$ with $j \neq k$, tree $T = \{u_i w_j, u_i w_k\}$ is the only possible rainbow Steiner $\{u_i, w_j, w_k\}$-tree. Hence, $c(u_i w_j) \neq c(u_i w_k)$. Since $d(u_i) = n$, $srx_3(K_{n,n}) \geq n$.

Next, we show that $srx_3(K_{n,n}) \leq n$ by defining a strong 3-rainbow coloring $c : E(K_{n,n}) \to [1, n]$ as follows:

$$c(u_i w_j) = \begin{cases} j - i + 1 & \text{if } i \leq j, \\ n + j - i + 1 & \text{if } i > j. \end{cases}$$

Now, we show that c is a strong 3-rainbow coloring of $K_{n,n}$. Let S be a set of three vertices of $K_{n,n}$. We consider two cases.

Case 1. The vertices of S belong to the same partition of $K_{n,n}$.

Without loss of generality, let $S = \{u_i, u_j, u_k\}$ with $i, j, k \in [1, n]$ and $i < j < k$. Then $T = \{u_i w_i, u_j w_i, u_k w_i\}$ is a rainbow Steiner S-tree.

Case 2. The vertices of S belong to different partitions of $K_{n,n}$.

Without loss of generality, let $S = \{u_i, u_j, w_k\}$ with $i, j, k \in [1, n]$ and $i < j$. Then $T = \{u_i w_k, u_j w_k\}$ is a rainbow Steiner S-tree. □

In the following theorem, we determine the strong 3-rainbow index of cycles.

Theorem 2.3. *Let C_n be a cycle of order $n \geq 3$. Then*

$$srx_3(C_n) = \begin{cases} 2 & \text{for } n = 3, \\ n - 2 & \text{for } n \in [4, 6] \text{ or } n = 8, \\ n & \text{for } n = 7 \text{ or } n \geq 9. \end{cases}$$

Proof. Let

$$V(C_n) = \{v_i : i \in [1, n]\}$$

be such that

$$E(C_n) = \{v_i v_{i+1} : i \in [1, n]\},$$

where $v_{n+1} = v_1$.

For $n = 3$, since $sdiam_3(C_3) = 2$, we have $srx_3(C_3) \geq 2$ by (1.1). Next, we show that $srx_3(C_3) \leq 2$ by defining a strong 3-rainbow coloring of C_3 as shown in Figure 2.

For $n \in [4, 6]$ or $n = 8$, it follows by Theorem 1.2 and (1.1) that $srx_3(C_n) \geq rx_3(C_n) = n - 2$. Next, we show that $srx_3(C_n) \leq n - 2$ by defining a strong 3-rainbow coloring of C_n as shown in Figure 2.

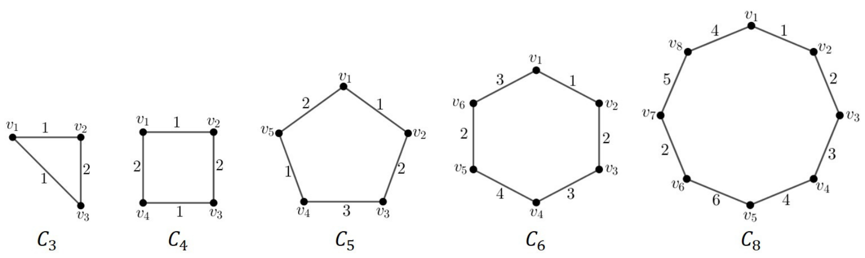

Fig. 2. Strong 3-rainbow colorings of C_3, C_4, C_5, C_6, and C_8

For $n = 7$ or $n \geq 9$, suppose that $srx_3(C_n) \leq n - 1$. Then there exists a strong 3-rainbow coloring $c : E(C_n) \to [1, n-1]$. Since $|E(C_n)| = n$, there are two edges of C_n, say v_1v_2 and v_pv_{p+1} for some $p \in [2, n]$, which have the same color. Suppose, as we proceed cyclically about C_n, that these two edges are encountered in the order v_1v_2, v_pv_{p+1}. Note that $d(v_1, v_p) \leq \lfloor \frac{n}{2} \rfloor$. By symmetry, we only consider when $1 \leq p-1 \leq \lfloor \frac{n}{2} \rfloor$. Observe that the only possible rainbow Steiner $\{v_1, v_{\lceil \frac{p+1}{2} \rceil}, v_{p+1}\}$-tree is $T = \{v_lv_{l+1} \colon l \in [1, p]\}$, where no edge of the tree is colored the same, but $c(v_1v_2) = c(v_pv_{p+1})$, a contradiction. Thus, $srx_3(C_n) \geq n$. Since $|E(C_n)| = n$, it follows by (1.1) that $srx_3(C_n) = n$. □

Let n be an integer with $n \geq 3$. A *wheel* W_n is a graph constructed by joining a vertex v to every vertex of a cycle $C_n = v_1v_2 \ldots v_nv_1$. The vertex v is called the *center vertex* of W_n. For each $i \in [1, n]$, edge vv_i is called the *spokes* of W_n.

In the next theorem, we determine the strong 3-rainbow index of wheels. First, we verify this lemma.

Lemma 2.4. *For $n \geq 4$, let W_n be a wheel of order $n+1$. If c is a strong 3-rainbow coloring of W_n, then at most two spokes vv_i and vv_j of W_n may be colored the same where $d(v_i, v_j) = 1$.*

Proof. Suppose that there are three spokes vv_i, vv_j, and vv_k of W_n with $c(vv_i) = c(vv_j) = c(vv_k)$. Since at least two of the three vertices v_i, v_j, and v_j are not adjacent, this means the distance between these two vertices equals 2. Without loss of generality, let $d(v_i, v_k) = 2$. Observe that $T = \{vv_i, vv_k\}$ is the only rainbow Steiner $\{v, v_i, v_k\}$-tree, but $c(vv_i) = c(vv_k)$, a contradiction. Thus, at most two spokes of W_n, say vv_i and vv_j, may be colored the same where $d(v_i, v_j) = 1$. □

Theorem 2.5. *For $n \geq 3$, let W_n be a wheel of order $n+1$. Then*

$$srx_3(W_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{for } n = 3 \text{ or } n \geq 5, \\ 3 & \text{for } n = 4. \end{cases}$$

Proof. Let

$$V(W_n) = \{v\} \cup \{v_i \colon i \in [1, n]\}$$

be such that

$$E(W_n) = \{vv_i \colon i \in [1, n]\} \cup \{v_iv_{i+1} \colon i \in [1, n]\},$$

where $v_{n+1} = v_1$.

For $n = 3$, since $sdiam_3(W_3) = 2$, $srx_3(W_3) \geq 2$ by (1.1). Next, we show that $srx_3(W_3) \leq 2$ by defining a strong 3-rainbow coloring of W_3 as shown in Figure 3.

For $n = 4$, suppose that $srx_3(W_4) \leq 2$. Then there exists a strong 3-rainbow coloring $c : E(W_4) \to [1, 2]$. By Lemma 2.4, we need at least two distinct colors assigned to all spokes of W_4. Without loss of generality, let $c(vv_1) = c(vv_2) = 1$ and $c(vv_3) = c(vv_4) = 2$. By considering $\{v, v_1, v_2\}$, $\{v_1, v_2, v_3\}$, and $\{v_2, v_3, v_4\}$ successively, we have $c(v_1v_2) = 2$, $c(v_2v_3) = 1$, and $c(v_3v_4) = 2$. However, there is no rainbow Steiner $\{v, v_3, v_4\}$-tree, a contradiction. Thus, $srx_3(W_4) \geq 3$. Next, we show that $srx_3(W_4) \leq 3$ by defining a strong 3-rainbow coloring of W_4 as shown in Figure 3.

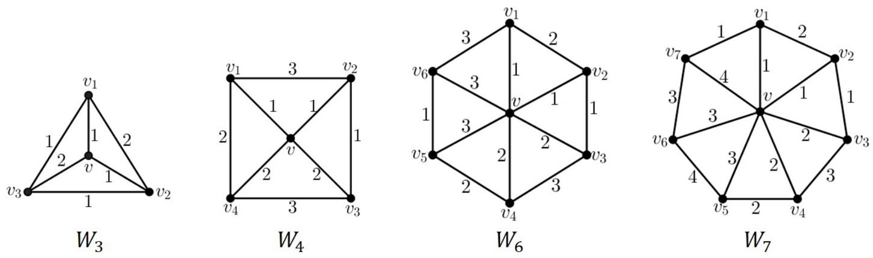

Fig. 3. Strong 3-rainbow colorings of W_3, W_4, W_6, and W_7

For $n \geq 5$, let c be a strong 3-rainbow coloring of W_n. Thus, $srx_3(W_n) \geq \lceil \frac{n}{2} \rceil$ by Lemma 2.4. Next, we show that $srx_3(W_n) \leq \lceil \frac{n}{2} \rceil$ by defining a strong 3-rainbow coloring $c : E(W_n) \to \left[1, \lceil \frac{n}{2} \rceil\right]$ as follows:

$$c(vv_i) = \left\lceil \frac{i}{2} \right\rceil \quad \text{for } i \in [1, n],$$

for odd n,

$$c(v_i v_{i+1}) = \begin{cases} \lceil \frac{i}{2} \rceil + 1 & \text{for odd } i \in [1, n-2], \\ 1 & \text{for } i = n, \\ \frac{i}{2} & \text{for even } i \in [1, n-1], \end{cases}$$

for even n,

$$c(v_i v_{i+1}) = \begin{cases} \lceil \frac{i}{2} \rceil + 1 & \text{for odd } i \in [1, n-3], \\ 1 & \text{for } i = n-1, \\ \frac{i}{2} & \text{for even } i \in [1, n]. \end{cases}$$

Now, we show that c is a strong 3-rainbow coloring of W_n. Let S be a set of three vertices of W_n and let $i, j, k \in [1, n]$ with $i \neq j$, $i \neq k$, and $j \neq k$. We consider two cases.

Case 1. The vertices of S belong to the cycle C_n.

Without loss of generality, let $S = \{v_i, v_j, v_k\}$. If $j = i + 1$ and $k = i + 2$, then $T = \{v_i v_{i+1}, v_{i+1} v_{i+2}\}$ is a rainbow Steiner S-tree. If i is odd ($i \neq n$ if n is odd), $j = i + 1$, and $k = i + 3$, then $T = \{v_i v_{i+1}, v_{i+1} v_{i+2}, v_{i+2} v_{i+3}\}$ is a rainbow Steiner S-tree. If i is odd, $j = i + 1$, and $k \geq i + 4$ or $k \leq i - 2$, then $T = \{vv_i, v_i v_{i+1}, vv_k\}$ is a rainbow Steiner S-tree. For others i, j, and k, $T = \{vv_i, vv_j, vv_k\}$ is a rainbow Steiner S-tree.

Case 2. Two vertices of S belong to the cycle C_n.

Without loss of generality, let $S = \{v, v_i, v_j\}$. If i is odd and $j = i + 1$, then $T = \{vv_i, v_i v_{i+1}\}$ is a rainbow Steiner S-tree. For others i and j, $T = \{vv_i, vv_j\}$ is a rainbow Steiner S-tree. $\square$

3. THE STRONG 3-RAINBOW INDEX OF AMALGAMATION OF SOME GRAPHS

Let G be a connected graph of order $n \geq 3$. For $t \geq 2$, let

$$V\left(Amal(G, v, t)\right) = \{v\} \cup \{v_i^p : i \in [1, t], p \in [1, n-1]\},$$

where v denote the identified vertex. In this section, we provide an upper bound for the strong 3-rainbow index of $Amal(G, v, t)$ and we show that the upper bound is sharp. We also determine the exact values of the strong 3-rainbow index of $Amal(G, v, t)$ for some connected graphs G.

3.1. SHARP UPPER BOUND FOR $srx_3(Amal(G, v, t))$

The following theorem provides a sharp upper bound for $srx_3(Amal(G, v, t))$.

Theorem 3.1. *Let t and n be two integers with $t \geq 2$ and $n \geq 3$. Let G be a connected graph of order n and v be a terminal vertex of G. Then*

$$srx_3(Amal(G, v, t)) \leq srx_3(G)\, t.$$

Moreover, the upper bound is sharp.

Proof. We show that $srx_3(Amal(G, v, t)) \leq srx_3(G)\, t$ by defining a strong 3-rainbow coloring of $Amal(G, v, t)$. Let c' be a strong 3-rainbow coloring of G. We define an edge-coloring $c : E(Amal(G, v, t)) \to [1, srx_3(G)\, t]$ as follows:

$$c(e) = \begin{cases} c'(e) & e \in E(G_1), \\ c'(e) + srx_3(G)\,(q-1) & e \in E(G_q) \text{ for each } q \in [2, t]. \end{cases}$$

Now, we show that c is a strong 3-rainbow coloring of $Amal(G, v, t)$. It suffices to show that for every set S of three vertices of $Amal(G, v, t)$, there exists a rainbow Steiner S-tree. We consider three cases.

Case 1. The vertices of S belong to the same graph G_i for some $i \in [1, t]$.

There exists a rainbow Steiner S-tree by coloring c corresponding to coloring c'.

Case 2. Two vertices of S belong to the same graph G_i for some $i \in [1, t]$.

Without loss of generality, let $S = \{v_i^p, v_i^q, v_j^r\}$ with $j \in [1, t]$, $j \neq i$, $p, q, r \in [1, n-1]$, and $p \neq q$. Note that if an edge-coloring is a strong 3-rainbow coloring, then it is also a strong rainbow coloring, that is, there exists a rainbow geodesic for any two vertices of graphs. Thus, there exist a rainbow Steiner $\{v, v_i^p, v_i^q\}$-tree and a rainbow $v - v_j^r$ geodesic by coloring c corresponding to coloring c'. We can find a rainbow Steiner S-tree by identifying vertex v in a rainbow Steiner $\{v, v_i^p, v_i^q\}$-tree and a rainbow $v - v_j^r$ geodesic since we use distinct colors in $E(G_i)$ and $E(G_j)$ by coloring c.

Case 3. Each vertex of S belongs to different graphs G_i, G_j, and G_k for some $i, j, k \in [1, t]$ with $i < j < k$.

Without loss of generality, let $S = \{v_i^p, v_j^q, v_k^r\}$ with $p, q, r \in [1, n-1]$. There exist a rainbow $v - v_i^p$ geodesic, a rainbow $v - v_j^q$ geodesic, and a rainbow $v - v_k^r$ geodesic by coloring c corresponding to coloring c'. We can find a rainbow Steiner S-tree by identifying vertex v in a rainbow $v - v_i^p$ geodesic, a rainbow $v - v_j^q$ geodesic, and a rainbow $v - v_k^r$ geodesic since we use distinct colors in $E(G_i)$, $E(G_j)$, and $E(G_k)$ by coloring c.

Now, let us prove the sharpness of the upper bound. Consider graphs $Amal(T_n, v, t)$, where T_n is a tree of order $n \geq 3$ and v is an arbitrary vertex of T_n. Since the amalgamation of trees is also a tree with $|E(Amal(T_n, v, t))| = |E(T_n)|\, t$, then

$$srx_3(Amal(T_n, v, t)) = |E(Amal(T_n, v, t))| = |E(T_n)|\, t = (n-1)\, t = srx_3(T_n)\, t$$

by Theorem 1.6, which attains the upper bound. □

Following Theorem 3.1, we have that $srx_3(Amal(T_n, v, t))$ attains the upper bound with $srx_3(Amal(T_n, v, t)) = |E(Amal(T_n, v, t))|$. A natural thought is like this: Is there any connected graph G of order n except a tree such that $srx_3(Amal(G, v, t))$ also attains the upper bound in Theorem 3.1 but $srx_3(Amal(G, v, t)) \neq |E(Amal(G, v, t))|$? The following theorem shows that such a graph G exists.

Theorem 3.2. *Let t and n be two integers with $t \geq 2$ and $n \geq 3$. Let L_n be a ladder of order $2n$ and $v \in V(L_n)$ with $d(v) = 2$. Then $srx_3(Amal(L_n, v, t)) = n\, t$.*

Proof. By Theorem 2.1, $srx_3(L_n) = n$, thus $srx_3(Amal(L_n, v, t)) \leq n\, t$ by Theorem 3.1.

Now, we prove the lower bound. Let

$$V(Amal(L_n, v, t)) = \{v\} \cup \{v_i^p : i \in [1, t], p \in [1, 2n-1]\}$$

be such that

$$\begin{aligned} E(Amal(L_n, v, t)) = \{&vv_i^1, vv_i^n : i \in [1, t]\} \\ &\cup \{v_i^p v_i^{p+1} : i \in [1, t], p \in [1, n-2] \cup p \in [n, 2n-2]\} \\ &\cup \{v_i^p v_i^{p+n} : i \in [1, t], p \in [1, n-1]\}. \end{aligned}$$

Let c be a strong 3-rainbow coloring of $Amal(L_n, v, t)$. For each $i \in [1, t]$, let A_i be a set of colors assigned to the edges of path $v_i^n v v_i^1 v_i^2 \ldots v_i^{n-2} v_i^{n-1}$. By considering $\{v, v_i^{n-1}, v_i^n\}$, we have $|A_i| \geq n$. For $i, j \in [1, t]$ with $i \neq j$, by considering $\{v, v_i^{n-1}, v_j^{n-1}\}$, $\{v, v_i^{n-1}, v_j^n\}$, and $\{v, v_i^n, v_j^n\}$, we have $A_i \cap A_j = \emptyset$. Hence, $srx_3(Amal(L_n, v, t)) \geq n\, t$. □

Note that $sdiam_3(G)$ is the natural lower bound for $srx_3(G)$ by (1.1). Consider the amalgamation of ladders shown in Theorem 3.2. It is easy to check that $sdiam_3(Amal(L_n, v, 2)) = 2n$ and $sdiam_3(Amal(L_n, v, t)) = 3n$ for $t \geq 3$. Hence, $srx_3(Amal(L_n, v, t)) = sdiam_3(Amal(L_n, v, t))$ for $t \in [2, 3]$.

The following theorem shows that $srx_3(Amal(K_{n,n}, v, t))$ also attains the upper bound in Theorem 3.1.

Theorem 3.3. *Let t and n be two integers with $t \geq 2$ and $n \geq 3$. Let $K_{n,n}$ be a regular complete bipartite graph of order $2n$ and v be an arbitrary vertex of $K_{n,n}$. Then $srx_3(Amal(K_{n,n}, v, t)) = n\, t$.*

Proof. Let

$$V(Amal(K_{n,n},v,t)) = \{v\} \cup \{u_i^p : i \in [1,t], p \in [1,n-1]\} \\ \cup \{w_i^p : i \in [1,t], p \in [1,n]\}$$

be such that

$$E(Amal(K_{n,n},v,t)) = \{vw_i^p : i \in [1,t], p \in [1,n]\} \\ \cup \{u_i^p w_i^q : i \in [1,t], p \in [1,n-1], q \in [1,n]\}.$$

Since $srx_3(K_{n,n}) = n$ by Theorem 2.2, we have $srx_3(Amal(K_{n,n},v,t)) \leq n\,t$ by Theorem 3.1. Now, we show that $srx_3(Amal(K_{n,n},v,t)) \geq n\,t$. Let c be a strong 3-rainbow coloring of $Amal(K_{n,n},v,t)$. By considering $\{v, w_i^p, w_j^q\}$ for all $i,j \in [1,t]$ and $p,q \in [1,n]$, we have $c(vw_i^p) \neq c(vw_j^q)$. Since $d(v) = n\,t$, $srx_3(Amal(K_{n,n},v,t)) \geq n\,t$. □

3.2. THE STRONG 3-RAINBOW INDEX OF $Amal(C_n,v,t)$

Based on the definition of amalgamation of graphs, it is natural to have a thought whether the selection of identified vertex $v \in V(G)$ affects the value of $srx_3(Amal(G,v,t))$ or not. Following Theorems 3.1 and 3.3, we obtain that the values of $srx_3(Amal(T_n,v,t))$ and $srx_3(Amal(K_{n,n},v,t))$ are both not affected by the selection of vertex v. In this subsection, we provide another graph $Amal(G,v,t)$ whose srx_3 is also not affected by the selection of vertex v.

For $t \geq 2$, let

$$V(Amal(C_n,v,t)) = \{v\} \cup \{v_i^p : i \in [1,t], p \in [1,n-1]\}$$

be such that

$$E(Amal(C_n,v,t)) = \{vv_i^1, vv_i^{n-1} : i \in [1,t]\} \\ \cup \{v_i^p v_i^{p+1} : i \in [1,t], p \in [1,n-2]\}.$$

For each $i \in [1,t]$, let C_n^i denote the i-th cycle in $Amal(C_n,v,t)$.

First, we verify the following observations which will be used to prove the lower bound for $srx_3(Amal(C_n,v,t))$.

Observation 3.4. *Let t be an integer at least 2 and n be an odd integer at least 3. Let c be a strong 3-rainbow coloring of $Amal(C_n,v,t)$. For each $i \in [1,t]$, let A_i be a set of colors assigned to all edges of C_n^i except edge $v_i^{\lfloor \frac{n}{2} \rfloor} v_i^{\lfloor \frac{n}{2} \rfloor + 1}$. Then $A_i \cap A_j = \emptyset$ for all distinct $i,j \in [1,t]$.*

Proof. Let $i,j \in [1,t]$ with $i \neq j$. By considering $\{v, v_i^{\lfloor \frac{n}{2} \rfloor}, v_j^{\lfloor \frac{n}{2} \rfloor}\}$, we obtain that no edge of path $v_i^{\lfloor \frac{n}{2} \rfloor} v_i^{\lfloor \frac{n}{2} \rfloor - 1} \ldots v_i^1 v v_j^1 \ldots v_j^{\lfloor \frac{n}{2} \rfloor - 1} v_j^{\lfloor \frac{n}{2} \rfloor}$ is colored the same. Similarly, by considering $\{v, v_i^{\lfloor \frac{n}{2} \rfloor}, v_j^{\lfloor \frac{n}{2} \rfloor + 1}\}$ and $\{v, v_i^{\lfloor \frac{n}{2} \rfloor + 1}, v_j^{\lfloor \frac{n}{2} \rfloor + 1}\}$, we have $A_i \cap A_j = \emptyset$. □

Observation 3.5. *Let t be an integer at least 2 and n be an even integer at least 4. Let c be a strong 3-rainbow coloring of $Amal(C_n, v, t)$. For each $i \in [1, t]$, let A_i be a set of colors assigned to all edges of C_n^i except edges $v_i^{\frac{n}{2}-1} v_i^{\frac{n}{2}}$ and $v_i^{\frac{n}{2}} v_i^{\frac{n}{2}+1}$. Then $A_i \cap A_j = \emptyset$ for all distinct $i, j \in [1, t]$.*

Proof. By using a similar argument as Observation 3.4 and considering $\{v, v_i^p, v_j^q\}$ for all $i, j \in [1, t]$ with $i \neq j$ and $p, q \in \{\frac{n}{2} - 1, \frac{n}{2} + 1\}$, we have $A_i \cap A_j = \emptyset$. □

Observation 3.6. *Let n be an even integer at least 10. Let c be a strong 3-rainbow coloring of $Amal(C_n, v, 2)$. Then at least three colors are needed to color edges $v_1^{\frac{n}{2}-1} v_1^{\frac{n}{2}}$, $v_1^{\frac{n}{2}} v_1^{\frac{n}{2}+1}$, $v_2^{\frac{n}{2}-1} v_2^{\frac{n}{2}}$, and $v_2^{\frac{n}{2}} v_2^{\frac{n}{2}+1}$ in $Amal(C_n, v, 2)$.*

Proof. Observe that the rainbow Steiner $\left\{v_1^{\frac{n}{2}-1}, v_1^{\frac{n}{2}+1}, v_2^{\frac{n}{2}}\right\}$-tree must contains edges $v_1^{\frac{n}{2}-1} v_1^{\frac{n}{2}}$, $v_1^{\frac{n}{2}} v_1^{\frac{n}{2}+1}$, and either $v_2^{\frac{n}{2}-1} v_2^{\frac{n}{2}}$ or $v_2^{\frac{n}{2}} v_2^{\frac{n}{2}+1}$. This implies we need at least three colors assigned to these four edges in $Amal(C_n, v, 2)$. □

Observation 3.7. *Let t be an integer at least 2 and n be an even integer at least 10. Let c be a strong 3-rainbow coloring of $Amal(C_n, v, t)$. For each $i \in [1, t]$, let $c(v_i^{\frac{n}{2}-1} v_i^{\frac{n}{2}}) = a_i$ and $c(v_i^{\frac{n}{2}} v_i^{\frac{n}{2}+1}) = b_i$. Then $\{a_i, b_i\} \neq \{a_j, b_j\}$ for all distinct $i, j \in [1, t]$.*

Proof. An argument similar to that used in the proof of Observation 3.6 will verify that $\{a_i, b_i\} \neq \{a_j, b_j\}$ for all distinct $i, j \in [1, t]$. □

Observation 3.8. *Let t and r be two integers with $t \geq 2$ and $r \geq 3$, and n be an even integer at least 10. Let r be the minimum number such that $t \leq \frac{r(r-1)}{2}$. If c is a strong 3-rainbow coloring of $Amal(C_n, v, t)$, then r is the minimum number of colors needed to color edges $v_i^{\frac{n}{2}-1} v_i^{\frac{n}{2}}$ and $v_i^{\frac{n}{2}} v_i^{\frac{n}{2}+1}$ for all $i \in [1, t]$.*

Proof. Suppose that $r - 1$ is the maximum number of colors needed to color edges $v_i^{\frac{n}{2}-1} v_i^{\frac{n}{2}}$ and $v_i^{\frac{n}{2}} v_i^{\frac{n}{2}+1}$ for all $i \in [1, t]$. By Observation 3.7, we have at most $\binom{r-1}{2}$ color pairs to color all pairs of two edges $\{v_i^{\frac{n}{2}-1} v_i^{\frac{n}{2}}, v_i^{\frac{n}{2}} v_i^{\frac{n}{2}+1}\}$ for all $i \in [1, t]$, where $\binom{r-1}{2}$ denote the number of combinations of $r - 1$ colors taken 2 at a time. Note that $\binom{r-1}{2} = \frac{(r-1)!}{2!(r-3)!} = \frac{(r-1)(r-2)}{2}$. However, $t > \frac{(r-1)(r-2)}{2}$, which implies there are at least two pairs of two edges $\{v_i^{\frac{n}{2}-1} v_i^{\frac{n}{2}}, v_i^{\frac{n}{2}} v_i^{\frac{n}{2}+1}\}$ and $\{v_j^{\frac{n}{2}-1} v_j^{\frac{n}{2}}, v_j^{\frac{n}{2}} v_j^{\frac{n}{2}+1}\}$ for some $i, j \in [1, t]$, $i \neq j$, such that $\{c(v_i^{\frac{n}{2}-1} v_i^{\frac{n}{2}}), c(v_i^{\frac{n}{2}} v_i^{\frac{n}{2}+1})\} = \{c(v_j^{\frac{n}{2-1}} v_j^{\frac{n}{2}}), c(v_j^{\frac{n}{2}} v_j^{\frac{n}{2}+1})\}$, contradicts Observation 3.7. □

Now, we determine the strong 3-rainbow index of $Amal(C_n, v, t)$.

Theorem 3.9. *Let t, n, and r be three integers with $t \geq 2$ and $n, r \geq 3$. Let C_n be a cycle of order n, v be an arbitrary vertex of C_n, and r be the minimum number such that $t \leq \frac{r(r-1)}{2}$. Then*

$$srx_3(Amal(C_n, v, t)) = \begin{cases} (srx_3(C_n) - 1)\, t + 1 & \text{for odd } n \geq 3 \text{ or } n \in \{6, 8\}, \\ 2t & \text{for } n = 4, \\ (n-2)\, t + r & \text{for even } n \geq 10. \end{cases}$$

Proof. For each $i \in [1,t]$, let C_n^i denote the i-th cycle in $Amal(C_n, v, t)$.

Case 1. n is odd.

For each $i \in [1,t]$, let A_i be a set of colors assigned to all edges of C_n^i except edge $v_i^{\lfloor \frac{n}{2} \rfloor} v_i^{\lfloor \frac{n}{2} \rfloor + 1}$. We distinguish three subcases.

Subcase 1.1. $n = 3$.

By Theorem 2.3, $srx_3(C_3) = 2$. Suppose that $srx_3(Amal(C_3, v, t)) \leq t$. Then there exists a strong 3-rainbow coloring of $Amal(C_3, v, t)$ using t colors. By Observation 3.4, we need at least t distinct colors to color edges vv_i^1 for all $i \in [1,t]$. Now, observe that the rainbow Steiner tree connecting $\{v_1^1, v_1^2, v_i^1\}$ for all $i \in [2,t]$ can be obtained identifying vertex v in a rainbow Steiner $\{v, v_1^1, v_1^2\}$-tree and an edge vv_i^1. Hence, no edge of Steiner $\{v, v_1^1, v_1^2\}$-tree is colored with $c(vv_i^1)$, which means we only have one color, that is $c(vv_1^1)$, to color two edges in Steiner $\{v, v_1^1, v_1^2\}$-tree, which is impossible. Thus, $srx_3(Amal(C_3, v, t)) \geq t + 1$.

Next, we show that $srx_3(Amal(C_3, v, t)) \leq t + 1$. We define an edge-coloring $c : E(Amal(C_3, v, t)) \to [1, t+1]$ which can be obtained by assigning colors i to the edges vv_i^1 and vv_i^2 and color $t+1$ to the edges $v_i^1 v_i^2$ for all $i \in [1,t]$. By this coloring, It is not hard to check that c is a strong 3-rainbow coloring of $Amal(C_3, v, t)$.

Subcase 1.2. $n = 5$.

Since $srx_3(C_5) = 3$ by Theorem 2.3, by using an argument similar to that used in the proof of lower bound for $n = 3$, it is not hard to show that $srx_3(Amal(C_5, v, t)) \geq 2t+1$.

Now, we show that $srx_3(Amal(C_5, v, t)) \leq 2t + 1$ by defining a strong 3-rainbow coloring $c : E(Amal(C_5, v, t)) \to [1, 2t+1]$. For each $i \in [1,t]$, assign colors i to the edges vv_i^1 and $v_i^3 v_i^4$, colors $t+i$ to the edges $v_i^1 v_i^2$ and vv_i^4, and color $2t+1$ to the edges $v_i^2 v_i^3$. By this coloring, it is not hard to find a rainbow Steiner S-tree for every set S of three vertices of $Amal(C_5, v, t)$.

Subcase 1.3. $n \geq 7$.

By Theorem 2.3, $srx_3(C_n) = n$. Suppose that $srx_3\,(Amal(C_n, v, t)) \leq (n-1)\,t$. Then there exists a a strong 3-rainbow coloring of $Amal(C_n, v, t)$ using $(n-1)\,t$ colors. Since no edge of C_n is colored the same, it follows by Observation 3.4 that we need at least $(n-1)\,t$ distinct colors to color all edges of $Amal(C_n, v, t)$ except edges $v_i^{\lfloor \frac{n}{2} \rfloor} v_i^{\lfloor \frac{n}{2} \rfloor + 1}$ for all $i \in [1,t]$. This means we have used all available colors. Now, consider $\{v_1^{\lfloor \frac{n}{2} \rfloor}, v_1^{\lfloor \frac{n}{2} \rfloor + 1}, v_i^p\}$ for all $i \in [2,t]$ and $p \in \{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1\}$. We obtain that edge $v_1^{\lfloor \frac{n}{2} \rfloor} v_1^{\lfloor \frac{n}{2} \rfloor + 1}$ should be contained in the rainbow Steiner tree connecting those three vertices, which means this edge can not be colored with colors from A_i. By Theorem 2.3, edge $v_1^{\lfloor \frac{n}{2} \rfloor} v_1^{\lfloor \frac{n}{2} \rfloor + 1}$ also can not be colored with colors from A_1. Hence, we need one new color to color edge $v_1^{\lfloor \frac{n}{2} \rfloor} v_1^{\lfloor \frac{n}{2} \rfloor + 1}$, which is impossible. Thus, $srx_3(Amal(C_n, v, t)) \geq (n-1)\,t + 1$.

Next, we show that $srx_3\,(Amal(C_n, v, t)) \leq (n-1)\,t+1$. We define an edge-coloring $c : E(Amal(C_n, v, t)) \to [1, (n-1)\,t+1]$ which can be obtained by assigning color 1 to the edges $v_i^{\lfloor \frac{n}{2} \rfloor} v_i^{\lfloor \frac{n}{2} \rfloor + 1}$ for all $i \in [1,t]$ and colors $2, 3, 4, \ldots, (n-1)\,t+1$ to the remaining $(n-1)\,t$ edges in $Amal(C_n, v, t)$. By this coloring, we obtain that all edges of $Amal(C_n, v, t)$ have distinct colors except edges $v_i^{\lfloor \frac{n}{2} \rfloor} v_i^{\lfloor \frac{n}{2} \rfloor + 1}$ for all $i \in [1,t]$, where

$c(v_i^{\lfloor \frac{n}{2} \rfloor} v_i^{\lfloor \frac{n}{2} \rfloor + 1}) = c(v_i^{\lfloor \frac{n}{2} \rfloor} v_i^{\lfloor \frac{n}{2} \rfloor + 1})$ for all distinct $i, j \in [1, t]$. Hence, it is not hard to find a rainbow Steiner S-tree for every set S of three vertices of $Amal(C_n, v, t)$.

Case 2. n is even.

For each $i \in [1, t]$, let A_i be a set of colors assigned to all edges of C_n^i except edges $v_i^{\frac{n}{2}-1} v_i^{\frac{n}{2}}$ and $v_i^{\frac{n}{2}} v_i^{\frac{n}{2}+1}$. We distinguish four subcases.

Subcase 2.1. $n = 4$.

Let c be a strong 3-rainbow coloring of $Amal(C_4, v, t)$. It is clear that $c(vv_i^1) \neq c(vv_i^3)$ for all $i \in [1, t]$. Hence, $srx_3\left(Amal(C_4, v, t)\right) \geq 2t$ by Observation 3.5.

Next, we show that $srx_3\left(Amal(C_4, v, t)\right) \leq 2t$ by defining a strong 3-rainbow coloring $c : E(Amal(C_4, v, t)) \to [1, 2t]$. This coloring can be obtained by assigning colors $1, 2, 3, \ldots, 2t$ to all edges of $Amal(C_4, v, t)$ where $c(vv_i^1) = c(v_i^2 v_i^3)$ and $c(v_i^1 v_i^2) = c(vv_i^3)$ for all $i \in [1, t]$. By this coloring, it is not hard to find a rainbow Steiner S-tree for every set S of three vertices of $Amal(C_4, v, t)$.

Subcase 2.2. $n = 6$.

By Theorem 2.3, $srx_3(C_6) = 4$. Suppose that $srx_3(Amal(C_6, v, t)) \leq 3t$. Then there exists a strong 3-rainbow coloring of $Amal(C_6, v, t)$ using $3t$ colors. For each $i \in [1, t]$, consider $\{v, v_i^2, v_i^5\}$. It is clearly that no edge of path $v_i^2 v_i^1 v v_i^5$ is colored the same. It follows by Observation 3.5 that we need at least $3t$ distinct colors to color edges vv_i^1, $v_i^1 v_i^2$, and vv_i^5 for all $i \in [1, t]$. Now, for all $i \in [2, t]$ and $p \in \{2, 5\}$, consider $\{v_1^2, v_1^4, v_i^p\}$. By identifying the vertex v in a rainbow Steiner $\{v, v_1^2, v_1^4\}$-tree and a rainbow $v - v_i^p$ geodesic, we obtain a rainbow Steiner tree connecting $\{v_1^2, v_1^4, v_i^p\}$. This implies we only have three colors, which are $c(vv_1^1)$, $c(v_1^1 v_1^2)$, and $c(vv_1^5)$, to color four edges in a Steiner $\{v, v_1^2, v_1^4\}$-tree, which is impossible. Thus, $srx_3(Amal(C_6, v, t)) \geq 3t + 1$.

Next, we show that $srx_3(Amal(C_6, v, t)) \leq 3t + 1$ by defining a strong 3-rainbow coloring $c : E(Amal(C_6, v, t)) \to [1, 3t + 1]$. For each $i \in [1, t]$, assign colors i to the edges vv_i^1, colors $t + i$ to the edges $v_i^1 v_i^2$ and $v_i^4 v_i^5$, colors $2t + i$ to the edegs $v_i^2 v_i^3$ and vv_i^5, and color $3t + 1$ to the edges $v_i^3 v_i^4$. it is not hard to find a rainbow Steiner S-tree for every set S of three vertices of $Amal(C_6, v, t)$.

Subcase 2.3. $n = 8$.

By Theorem 2.3, $srx_3(C_8) = 6$. Suppose that $srx_3(Amal(C_8, v, t)) \leq 5t$. Then there exists a strong 3-rainbow coloring $c : E(Amal(C_8, v, t)) \to [1, 5t]$. For distinct $i, j \in [1, t]$ and $p \in \{2, 6\}$, consider $\{v_i^2, v_i^6, v_j^p\}$. It follows by Observation 3.5 that we need at least $4t$ distinct colors to color edges vv_i^1, $v_i^1 v_i^2$, $v_i^6 v_i^7$, and vv_i^7 for all $i \in [1, t]$. This implies we have at most t colors left. Let A be the set of these t colors. Now, for an arbitrary $i \in [1, t]$, consider edges $v_i^2 v_i^3$ and $v_i^5 v_i^6$. It is easy to check that $c(v_i^2 v_i^3) \notin \{c(vv_i^1), c(v_i^1 v_i^2), c(vv_i^7)\}$ and $c(v_i^5 v_i^6) \notin \{c(vv_i^1), c(v_i^6 v_i^7), c(vv_i^7)\}$. Hence, by considering $\{v, v_i^p, v_j^q\}$ for all $i, j \in [1, t]$ with $i \neq j$ and $p, q \in \{3, 5\}$, this forces $c(v_i^2 v_i^3) \in \{c(v_i^6 v_i^7)\} \cup A$ and $c(v_i^5 v_i^6) \in \{c(v_i^1 v_i^2)\} \cup A$. Observe that if $c(v_i^2 v_i^3) = c(v_i^6 v_i^7)$ and $c(v_i^5 v_i^6) = c(v_i^1 v_i^2)$, then there is no rainbow Steiner $\{v_i^1, v_i^3, v_i^6\}$-tree. Thus, for each $i \in [1, t]$, edge $v_i^2 v_i^3$ or $v_i^5 v_i^6$ should be colored with color from A. It means we need at least t new distinct colors to color edges $v_i^2 v_i^3$ and $v_i^5 v_i^6$ for all $i \in [1, t]$. Without loss of generality, consider $i = 1$. If $c(v_1^2 v_1^3) = c(v_1^6 v_1^7)$ and $c(v_1^5 v_1^6) \in A$, then consider $\{v_1^2, v_1^4, v_j^p\}$ for all $j \in [2, t]$ and $p \in \{3, 5\}$. This forces $c(v_1^3 v_1^4) \in \{c(v_1^5 v_1^6), c(vv_1^7)\}$.

If $c(v_1^3v_1^4) = c(v_1^5v_1^6)$, then there is no rainbow Steiner $\{v_1^3, v_1^4, v_1^6\}$-tree. If $c(v_1^3v_1^4) = c(vv_1^7)$, then there is no rainbow Steiner $\{v, v_1^3, v_1^6\}$-tree. Similarly, if $c(v_1^2v_1^3) \in A$ and $c(v_1^5v_1^6) = c(v_1^1v_1^2)$, then by considering $\{v_1^4, v_1^6, v_j^p\}$ for all $j \in [2,t]$ and $p \in \{3,5\}$, we will obtain a contradiction. Thus, $srx_3(Amal(C_8, v, t)) \geq 5t + 1$.

Next, we show that $srx_3(Amal(C_8, v, t)) \leq 5t + 1$ by defining a strong 3-rainbow coloring $c : E(Amal(C_8, v, t)) \to [1, 5t + 1]$. For each $i \in [1, t]$, assign colors i to the edges vv_i^1 and $v_i^4v_i^5$, colors $t + i$ to the edges $v_i^2v_i^3$ and $v_i^6v_i^7$, color $2t + 1$ to the edges $v_i^3v_i^4$, and colors $2t + 2, 2t + 3, \ldots, 5t, 5t + 1$ to the remaining $3t$ edges. It is not hard to find a rainbow Steiner S-tree for every set S of three vertices of $Amal(C_8, v, t)$.

Subcase 2.4. $n \geq 10$.

Let c be a strong 3-rainbow coloring of $Amal(C_n, v, t)$. By Theorem 2.3 and Observations 3.5 and 3.8, we need at least $(n-2)\,t + r$ distinct colors to color all edges of $Amal(C_n, v, t)$. Thus, $srx_3(Amal(C_n, v, t)) \geq (n-2)\,t + r$.

Next, we show that $srx_3(Amal(C_n, v, t)) \leq (n-2)\,t + r$. We define an edge-coloring $c : E(Amal(C_n, v, t) \to [1, (n-2)\,t + r]$ as follows.

(i) Assign a list of combinations of r colors taken 2 at a time to all pairs of two edges $v_i^{\frac{n}{2}-1}v_i^{\frac{n}{2}}$ and $v_i^{\frac{n}{2}}v_i^{\frac{n}{2}+1}$ for all $i \in [1, t]$, so that

$$\{c(v_i^{\frac{n}{2}-1}v_i^{\frac{n}{2}}), c(v_i^{\frac{n}{2}}v_i^{\frac{n}{2}+1})\} \neq \{c(v_j^{\frac{n}{2}-1}v_j^{\frac{n}{2}}), c(v_j^{\frac{n}{2}}v_j^{\frac{n}{2}+1})\}$$

for all distinct $i, j \in [1, t]$.

(ii) Assign colors $1 + r, 2 + r, 3 + r, \ldots, (n-2)\,t + r$ to the remaining $(n-2)\,t$ edges of $Amal(C_n, v, t)$.

By the coloring above, we can easily find a rainbow Steiner S-tree for every set S of three vertices of $Amal(C_n, v, t)$. □

Following Theorem 3.9, we obtain that $srx_3(Amal(C_4, v, t))$ attains the upper bound in Theorem 3.1.

3.3. THE STRONG 3-RAINBOW INDEX OF $Amal(W_n, v, t)$

We have shown that there are some graphs $Amal(G, v, t)$ whose srx_3 is not affected by the selection of vertex v. In this subsection, we show that the selection of vertex v affects the value of $srx_3(Amal(W_n, v, t))$.

First, consider graphs $Amal(W_n, v, t)$ where v is the center vertex of W_n. The following theorem provides the strong 3-rainbow index of $Amal(W_n, v, t)$.

Theorem 3.10. *Let t and n be two integers with $t \geq 2$ and $n \geq 3$. Let W_n be a wheel of order $n + 1$ and v be the center vertex of W_n. Then*

$$srx_3\,(Amal(W_n, v, t)) = \begin{cases} t + 2 & \text{for } n = 3, \\ \lceil \frac{n}{2} \rceil\, t & \text{for } n \geq 4. \end{cases}$$

Proof. Let

$$V\left(Amal(W_n, v, t)\right) = \{v\} \cup \{v_i^p : i \in [1,t], p \in [1,n]\}$$

be such that

$$E\left(Amal(W_n, v, t)\right) = \{vv_i^p, v_i^p v_i^{p+1} : i \in [1,t], p \in [1,n]\},$$

where $v_i^{n+1} = v_i^1$. For each $i \in [1,t]$, let W_n^i denote the i-th wheel in $Amal(W_n, v, t)$.

Case 1. $n = 3$.

Suppose that $srx_3(Amal(W_3, v, t)) \leq t + 1$. Then there exists a strong 3-rainbow coloring $c : E(Amal(W_3, v, t)) \to [1, t+1]$. Note that $c(vv_i^p) \neq c(vv_j^q)$ for all $i, j \in [1,t]$ with $i \neq j$ and $p, q \in [1,3]$. Since we have $t + 1$ colors, we consider two subcases.

Subcase 1.1. $c(vv_i^p) = i$ for all $i \in [1,t]$ and $p \in [1,3]$.

This implies we have one remaining color, say color a. By considering $\{v_1^1, v_1^2, v_i^1\}$ for all $i \in [2,t]$, we have $c(v_1^1 v_1^2) = a$. Similarly, by considering $\{v_1^2, v_1^3, v_i^1\}$ and $\{v_1^3, v_1^1, v_i^1\}$ for all $i \in [2,t]$, we have $c(v_1^2 v_1^3) = c(v_1^3 v_1^1) = a$. But, there is no rainbow Steiner $\{v_1^1, v_1^2, v_1^3\}$-tree, a contradiction.

Subcase 1.2. There exists $i \in [1,t]$ such that all spokes of W_3^i are colored with two colors.

Without loss of generality, let $i = 1$, $c(vv_1^1) = a$ and $c(vv_1^2) = b$. This implies for all $j \in [2,t]$, all spokes of W_3^j are not colored with a and b and $c(vv_j^1) = c(vv_j^2) = c(vv_j^3)$. Next, consider $\{v_2^1, v_2^2, v_j^1\}$ for all $j \in [3,t]$. This forces $c(v_2^1 v_2^2) \in \{a, b\}$. If $c(v_2^1 v_2^2) = a$, then there is no rainbow Steiner $\{v_2^1, v_2^2, v_1^1\}$-tree. Similarly, if $c(v_2^1 v_2^2) = b$, then there is no rainbow Steiner $\{v_2^1, v_2^2, v_1^2\}$-tree.

Next, we show that $srx_3(Amal(W_3, v, t)) \leq t + 2$ by defining a strong 3-rainbow coloring $c : E(Amal(W_3, v, t)) \to [1, t+2]$. For each $i \in [1,t]$ and $p \in [1,3]$, assign colors i to the spokes vv_i^p, color $t + 1$ to the edges $v_i^1 v_i^2$ and $v_i^2 v_i^3$, and color $t + 2$ to the edges $v_i^3 v_i^1$. It is not hard to find a rainbow Steiner S-tree for every set S of three vertices of $Amal(W_3, v, t)$.

Case 2. $n \geq 4$.

Let c be a strong 3-rainbow coloring of $Amal(W_n, v, t)$. For each $i \in [1,t]$, the minimum number of colors needed to color all spokes of W_n^i is $\lceil \frac{n}{2} \rceil$ by Lemma 2.4. Hence, by considering $\{v, v_i^p, v_j^q\}$ for all $i, j \in [1,t]$ with $i \neq j$ and $p, q \in [1,n]$, we have $srx_3\left(Amal(W_n, v, t)\right) \geq \lceil \frac{n}{2} \rceil t$.

Next, we show that $srx_3(Amal(W_n, v, t)) \leq \lceil \frac{n}{2} \rceil t$. For $n \geq 5$, by Theorem 2.5, $srx_3(W_n) = \lceil \frac{n}{2} \rceil$. Hence, $srx_3(W_n) \leq \lceil \frac{n}{2} \rceil t$ by Theorem 3.1. For $n = 4$, we define a strong 3-rainbow coloring $c : E(Amal(W_4, v, t)) \to [1, 2t]$ as follows.

- (i) For each $i \in [1,t]$, assign colors $1 + 2(i-1)$ to the edges vv_i^1, vv_i^2, and $v_i^3 v_i^4$, and colors $2 + 2(i-1)$ to the edges vv_i^3, vv_i^4, and $v_i^1 v_i^2$.
- (ii) Assign colors $1 + 2i$ to the edges $v_i^2 v_i^3$ and $v_i^4 v_i^1$ for $i \in [1, t-1]$ and color 1 to the edges $v_t^2 v_t^3$ and $v_t^4 v_t^1$.

By the coloring above, it is not hard to find a rainbow Steiner S-tree for every set S of three vertices of $Amal(W_4, v, t)$. □

Figure 4 gives examples of strong 3-rainbow colorings of $Amal(W_5, v, 4)$ and $Amal(W_6, v, 4)$.

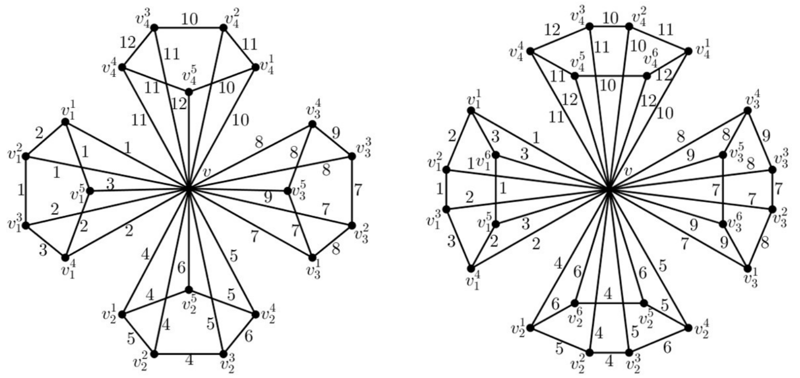

Fig. 4. Strong 3-rainbow colorings of $Amal(W_5, v, 4)$ and $Amal(W_6, v, 4)$ where v is the center vertex of W_n

For further discussion, consider graphs $Amal(W_n, v, t)$ where v is not the center vertex of W_n. Let

$$V\left(Amal(W_n, v, t)\right) = \{v\} \cup \{v_i^p \colon i \in [1, t], p \in [1, n]\}$$

be such that

$$E\left(Amal(W_n, v, t)\right) = \{vv_i^p \colon i \in [1, t], p \in \{1, 2, n\}\} \cup \{v_i^1 v_i^p \colon i \in [1, t], p \in [2, n]\} \cup \{v_i^p v_i^{p+1} \colon i \in [1, t], p \in [2, n-1]\}.$$

First, we verify the following observation.

Observation 3.11. *Let t and n be two integers with $t \geq 2$ and $n \geq 3$. Let W_n be a wheel of order $n+1$ and $v \in V(W_n)$ where v is not the center vertex. If c is a strong 3-rainbow coloring of $Amal(W_n, v, t)$, then:*

(i) *for each $i \in [1, t]$ and $n \geq 4$, $c(vv_i^2) \neq c(vv_i^n)$,*
(ii) *for each $i, j \in [1, t]$, $i \neq j$, and $p, q \in \{1, 2, n\}$, $c(vv_i^p) \neq c(vv_j^q)$,*
(iii) *for each $i, j \in [1, t]$, $i \neq j$, and $p, q \in [4, n-2]$, $c(v_i^1 v_i^p) \neq c(v_j^1 v_j^q)$,*
(iv) *for each $i, j \in [1, t]$ and $p \in [4, n-2]$, $c(vv_i^1) \neq c(v_j^1 v_j^p)$.*

Proof. We distinguish four cases.

(i) By considering $\{v, v_i^2, v_i^n\}$ for all $i \in [1, t]$, it is clear that $c(vv_i^2) \neq c(vv_i^n)$.
(ii) By considering $\{v, v_i^p, v_j^q\}$ for all $i, j \in [1, t]$, $i \neq j$, and $p, q \in \{1, 2, n\}$, $c(vv_i^p) \neq c(vv_j^q)$.
(iii) By considering $\{v, v_i^p, v_j^q\}$ for all $i, j \in [1, t]$, $i \neq j$, and $p, q \in [4, n-2]$, $c(v_i^1 v_i^p) \neq c(v_j^1 v_j^q)$.
(iv) By considering $\{v, v_i^1, v_j^p\}$ for all $i, j \in [1, t]$ and $p \in [4, n-2]$, $c(vv_i^1) \neq c(v_j^1 v_j^p)$.

$\square$

Now, we determine the strong 3-rainbow index of $Amal(W_n, v, t)$ where v is not the center vertex of W_n.

Theorem 3.12. *Let t and n be two integers with $t \geq 2$ and $n \geq 3$. Let W_n be a wheel of order $n+1$ and $v \in V(W_n)$ where v is not the center vertex. Then*

$$srx_3\left(Amal(W_n, v, t)\right) = \begin{cases} t+2 & \text{for } n = 3, \\ 2t+1 & \text{for } n \in [4,5], \\ \left(\left\lceil \frac{n-5}{2} \right\rceil + 1\right) t + 1 & \text{for even } n \geq 6, \\ \left(\left\lceil \frac{n-5}{2} \right\rceil + 1\right) t + 2 & \text{for odd } n \geq 6. \end{cases}$$

Proof. We consider three cases.

Case 1. $n = 3$.

Note that W_3 is a complete graph, thus W_3 is vertex-transitive. This means any vertex of W_3 can be thought of as the center vertex of W_3. Hence, the proof is the same as the proof of Case 1 in Theorem 3.10.

Case 2. $n \in [4, 5]$.

Suppose that $srx_3\left(Amal(W_n, v, t)\right) \leq 2t$. Then there exists a strong 3-rainbow coloring $c : E\left(Amal(W_n, v, t)\right) \to [1, 2t]$. By Observation 3.11(i)-(ii), we need at least $2t$ distinct colors assigned to the edges vv_i^2 and vv_i^n for all $i \in [1, t]$. For further steps, we always let $i \in [2, t]$, $p \in [3, n-1]$, and $q \in \{2, n\}$. Since $T = \{vv_1^1, v_1^1v_1^p, vv_i^q\}$ is the only rainbow Steiner $\{v_1^1, v_1^p, v_i^q\}$-tree, this forces $\{c(vv_1^1), c(v_1^1v_1^p)\} \subseteq \{c(vv_1^2), c(vv_1^n)\}$ where $c(vv_1^1) \neq c(v_1^1v_1^p)$. If $c(vv_1^1) = c(vv_1^2)$ and $c(v_1^1v_1^p) = c(vv_1^n)$, then by considering $\{v_1^1, v_1^2, v_i^q\}$ and $\{v_1^2, v_1^3, v_i^q\}$, we have $c(v_1^1v_1^2) = c(v_1^2v_1^3) = c(v_1^1v_1^p)$. However, there is no rainbow Steiner $\{v_1^1, v_1^2, v_1^3\}$-tree, a contradiction. Similarly, if $c(vv_1^1) = c(vv_1^n)$ and $c(v_1^1v_1^p) = c(vv_1^2)$, then there is no rainbow Steiner $\{v_1^1, v_1^{n-1}, v_i^n\}$-tree, a contradiction.

Next, we show that $srx_3\left(Amal(W_n, v, t)\right) \leq 2t+1$ by defining a strong 3-rainbow coloring $c : E\left(Amal(W_n, v, t)\right) \to [1, 2t+1]$. For $n = 4$ and each $i \in [1, t]$, assign colors $1 + 2(i-1)$ to the edges vv_i^1, vv_i^4, $v_i^1v_i^2$, and $v_i^2v_i^3$, colors $2 + 2(i-1)$ to the edges vv_i^2, $v_i^1v_i^3$, and $v_i^1v_i^4$, and color $2t+1$ to the edges $v_i^3v_i^4$. For $n = 5$ and each $i \in [1, t]$, assign colors $1 + 2(i-1)$ to the edges vv_i^1, vv_i^2, and $v_i^4v_i^5$, colors $2 + 2(i-1)$ to the edges vv_i^5, $v_i^1v_i^2$, $v_i^1v_i^3$, and $v_i^3v_i^4$, and color $2t+1$ to the edges $v_i^1v_i^4$, $v_i^1v_i^5$, and $v_1^2v_i^3$. By these colorings, it is not hard to find a rainbow Steiner S-tree for every set S of three vertices of $Amal(W_n, v, t)$.

Case 3. $n \geq 6$.

For odd n, suppose that $srx_3\left(Amal(W_n, v, t)\right) \leq \left(\left\lceil \frac{n-5}{2} \right\rceil + 1\right) t + 1$. Let c be a strong 3-rainbow coloring of $Amal(W_n, v, t)$ using $\left(\left\lceil \frac{n-5}{2} \right\rceil + 1\right) t + 1$ colors. For each $i \in [1, t]$, we need at least $\left\lceil \frac{n-5}{2} \right\rceil + 1$ distinct colors to color edges vv_i^1 and $v_i^1v_i^p$ for all $p \in [4, n-2]$ by Observation 3.11(iii)-(iv). Since we only have $\left(\left\lceil \frac{n-5}{2} \right\rceil + 1\right) t + 1$ available colors, we consider the following two subcases.

Subcase 3.1. There exists $i \in [1, t]$ which uses exactly $\left\lceil \frac{n-5}{2} \right\rceil + 2$ colors to color edges vv_i^1 and $v_i^1v_i^p$ for all $p \in [4, n-2]$.

Without loss of generality, let $i = 1$. Note that for each $j \in [2, t]$, we have exactly $\lceil \frac{n-5}{2} \rceil + 1$ distinct colors to color edges vv_j^1 and $v_j^1 v_j^p$ for all $p \in [4, n-2]$, where each color is assigned to exactly two spokes. Now, by symmetry, consider spoke $v_1^1 v_1^3$. By Lemma 2.4 and considering $\{v_1^1, v_1^3, v_j^p\}$ for all $j \in [2, t]$ and $p \in [4, n-2]$, we have $c(v_1^1 v_1^3) \notin \{c(vv_1^1), c(v_1^1 v_1^q), c(vv_j^1), c(v_j^1 v_j^p)\}$ for all $q \in [5, n-2]$. This forces $c(v_1^1 v_1^3) = c(v_1^1 v_1^4)$. Next, consider spoke $v_2^1 v_2^3$. By Lemma 2.4, $c(v_2^1 v_2^3) \notin \{c(vv_2^1), c(v_2^1 v_2^p)\}$ for all $p \in [4, n-2]$. This forces $c(v_2^1 v_2^3) \in \{c(vv_j^1), c(v_j^1 v_j^p)\}$ for some $j \in [1, t]$ with $j \neq 2$ and $p \in [4, n-2]$. However, there is no rainbow Steiner $\{v_2^1, v_2^3, v_j^p\}$-tree since the tree must contain spokes $v_2^1 v_2^3$, vv_j^1, and $v_j^1 v_j^p$, a contradiction.

The subcase above implies the following subcase.

Subcase 3.2. For each $i \in [1, t]$ and $p \in [4, n-2]$, we use exactly $\lceil \frac{n-5}{2} \rceil + 1$ distinct colors to color edges vv_i^1 and $v_i^1 v_i^p$.

Note that we have exactly one color left, say color a. Now, consider spoke $v_1^1 v_1^3$. Since n is odd, it follows by Lemma 2.4 that $c(v_1^1 v_1^3) \notin \{c(vv_1^1), c(v_1^1 v_1^p)\}$ for all $p \in [4, n-2]$. This forces $c(v_1^1 v_1^3) \in \{c(vv_j^1), c(v_j^1 v_j^p), a\}$ for some $j \in [2, t]$ and $p \in [4, n-2]$. If $c(v_1^1 v_1^3) \in \{c(vv_j^1), c(v_j^1 v_j^p)\}$, then there is no rainbow Steiner $\{v_1^1, v_1^3, v_j^p\}$-tree, since the tree must contain spokes $v_1^1 v_1^3$, vv_j^1, and $v_j^1 v_j^p$. Hence, $c(v_1^1 v_1^3) = a$. Similarly, $c(v_1^1 v_1^{n-1}) = a$. Therefore, $c(v_1^1 v_1^3) = c(v_1^1 v_1^{n-1}) = a$, contradicts Lemma 2.4.

Thus, $srx_3(Amal(W_n, v, t)) \geq \left(\lceil \frac{n-5}{2} \rceil + 1\right) t + 2$ for odd n. An argument similar to that used in the proof of odd n will verify the lower bound for even n.

Next, we prove the upper bound. Let $x = \lceil \frac{n-5}{2} \rceil + 1$. For odd n, we define a strong 3-rainbow coloring $c : E(Amal(W_n, v, t)) \to \left[\left(\lceil \frac{n-5}{2} \rceil + 1\right) t + 2\right]$ as follows.

(i) For each $i \in [1, t]$, assign colors $1 + x(i-1)$ to the spoke vv_i^1, colors $\lfloor \frac{p}{2} \rfloor + x(i-1)$ to the spokes $v_i^1 v_i^p$ for all $p \in [4, n-2]$, color $xt + 1$ to the spokes $v_1^1 v_1^2$ and $v_1^1 v_1^3$, and color $xt + 2$ to the spokes $v_1^1 v_1^{n-1}$ and $v_1^1 v_1^n$.

(ii) For each $i \in [1, t]$, define $c(vv_1^2) = c(v_i^1 v_i^4)$, $c(v_i^2 v_i^3) = c(vv_i^n) = c(vv_i^1)$, $c(v_i^p v_i^{p+1}) = c(v_i^1 v_i^{p+1})$ for odd $p \in [3, n-2]$, and $c(v_i^p v_i^{p+1}) = c(v_i^1 v_i^{p-1})$ for even $p \in [4, n-1]$.

For even n, we define an edge-coloring $c : E(Amal(W_n, v, t)) \to \left[\left(\lceil \frac{n-5}{2} \rceil + 1\right) t + 1\right]$ as follows.

(i) For each $i \in [1, t]$, assign colors $1 + x(i-1)$ to the spoke vv_i^1, colors $\lceil \frac{p}{2} \rceil + x(i-1)$ to the spokes $v_i^1 v_i^p$ for all $p \in [2, n-2]$, and color $xt + 1$ to the spokes $v_1^1 v_1^{n-1}$ and $v_1^1 v_1^n$.

(ii) For $n = 6$ and each $i \in [1, t]$, define $c(vv_i^2) = c(v_i^5 v_i^6) = c(v_i^1 v_i^3)$, $c(v_i^2 v_i^3) = c(v_i^4 v_i^5) = c(vv_i^6) = c(vv_i^1)$, and $c(v_i^3 v_i^4) = xt + 1$.

(iii) For $n \geq 8$ and each $i \in [1, t]$, define $c(vv_i^2) = c(v_i^1 v_i^3)$, $c(v_i^p v_i^{p+1}) = c(v_i^1 v_i^p)$ for even $p \in [2, n-2]$, $c(v_i^p v_i^{p+1}) = c(v_i^1 v_i^{p+2})$ for odd $p \in [3, n-3]$, $c(v_i^{n-1} v_i^n) = c(vv_i^1)$, and $c(vv_i^n) = c(v_i^{n-2} v_i^{n-1})$.

By the colorings above, it is not hard to find a rainbow Steiner S-tree for every set S of three vertices of $Amal(W_n, v, t)$. $\square$

Figure 5 gives examples of strong 3-rainbow colorings of $Amal(W_6, v, 4)$ and $Amal(W_7, v, 4)$.

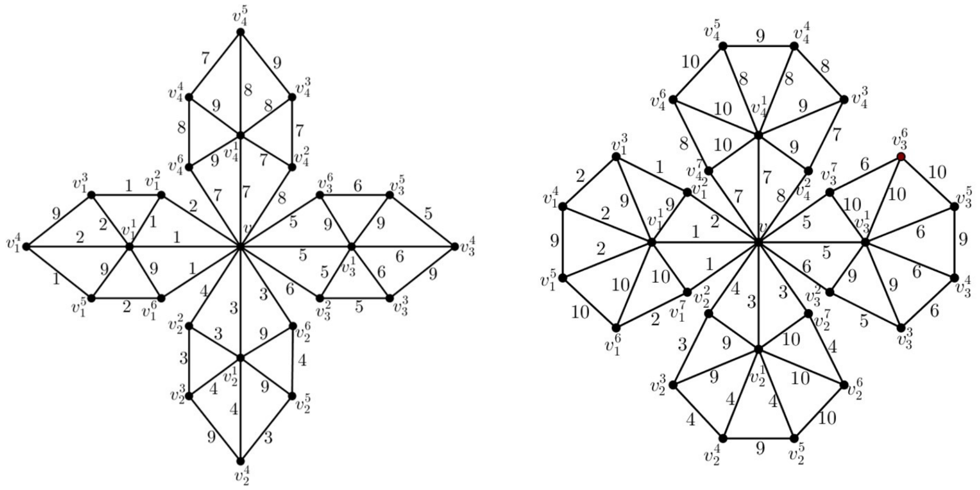

Fig. 5. Strong 3-rainbow colorings of $Amal(W_6, v, 4)$ and $Amal(W_7, v, 4)$ where v is not the center vertex of W_n

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