A Study on Natural Frequencies of Timoshenko Beam with Rapidly Varying Stiffness

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Abstract

Vibrations of Timoshenko beams with properties periodically varying along the axis are under consideration. The tolerance method of averaging differential operators with highly oscillating coefficients is applied to obtain the governing equations with constant coefficients. The dynamics of Timoshenko beam with the effect of the cell length is described. A asymptotic model is then constructed, which is further studied in analysis of the low order natural frequencies. The proposed model is able to describe dynamics of beams made of non-slender cells.

Keywords: beam vibrations, periodic beams, tolerance modelling

1. Introduction

The analysis will be restricted to the linear free vibrations of elastic shear-deformable beam with rotational inertia. Considered structure consists of many small, identical and ordered pieces of length l, called periodicity cells. The geometric and material properties are varying periodically along longitudinal axis of the beam. A fragment of such beam is shown in Figure 1.

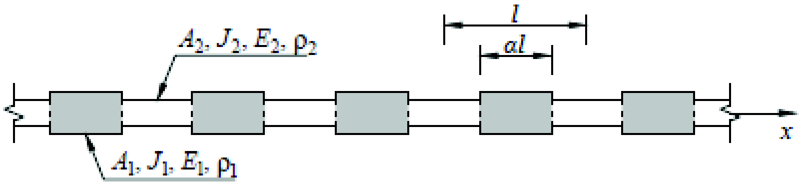

Figure 1. A fragment of periodically inhomogeneous Timoshenko beam

The direct analytical formulation of considered Timoshenko beam model leads to equations of motion which usually do have non-continuous, highly oscillating, periodic coefficients. Many methods have been developed in analysis of periodically inhomogeneous solids and structures. The most advanced are the analytical methods based on asymptotic homogenization of differential operators [1-2].

Here, the tolerance averaging technique [9-10] is applied in order to replace the differential equations with highly oscillating coefficients by equations with constant coefficients. The presented method enables continuous analysis of an equivalent homogeneous medium with effective properties. The model reduces the computational cost and disposes of numerical difficulties. The approach used here has been applied in analysis of many thermo-mechanical problems of periodic and almost-periodic solids and structures. To name only few, tolerance models of beams with periodically variable parameters are considered in [3, 8]. In [6] some aspects of modelling of dynamic problems of thin functionally graded plates with a special tolerance–periodic microstructure in planes parallel to the plate midplane are considered.

Detailed analytical solution of homogeneous Timoshenko beam is considered in [6]. A numerical example is shown for a non-slender beam to signify the differences among the Timoshenko, Bernoulli, shear and Rayleigh beam models.

2. Formulation of the problem

The strain-displacement relations in Timoshenko beam theory are assumed as

$$\kappa = \partial\theta, \quad \gamma = \partial w - \theta, \tag{1}$$

where w, θ, κ and γ represent the deflection, the cross-section rotation, the bending curvature, and the shear strain, respectively. The strain energy U and the kinetic energy K for a Timoshenko beam can be written as

$$U = \frac{1}{2}\int_0^L \left(EJ\kappa^2 + kGA\gamma^2\right)dx, \quad K = \frac{1}{2}\int_0^L \rho A\dot{w}^2 dx + \frac{1}{2}\int_0^L \rho J\dot{\theta}^2 dx, \tag{2}$$

where ρ, A, J, E, G and k are the mass density per unit volume, cross-section area of the beam, the area moment of the inertia, Young's modulus and shear modification coefficient, respectively. The equations of motion may be derived from Hamiltons principle (3).

$$\delta\int_{t_0}^{t_1} (U - K)dt = 0. \tag{3}$$

3. Introductory concepts, fundamental assumptions

The domain occupied by the beam is given by one-dimensional $\Lambda = (0,L)$, where L is the beam length. It is assumed that the cell length is much smaller than the beam length, $l \ll L$. Following the book edited by Woźniak cf. [9], some introductory concepts of the tolerance modelling are used, i.e. the averaging operator, tolerance system, slowly-varying function $SV^{\alpha}_{\xi}(\Lambda,\Delta)$, tolerance-periodic function $TP^{\alpha}_{\xi}(\Lambda,\Delta)$, highly oscillating function $HO^{\alpha}_{\xi}(\Lambda,\Delta)$, fluctuation shape function $FS^{\alpha}_{\xi}(\Lambda,\Delta)$, where ξ is the tolerance parameter and α is a positive constant determining kind of the function. The basic concept of the modelling technique is the averaging operator, for an integrable function f defined by:

$$\langle f\rangle(x) = \frac{1}{l}\int_{\Delta(x)} f(x)dx. \tag{4}$$

The unknown deflection w, and rotation θ are decomposed into their averaged and fluctuating part:

$$
\begin{aligned}
& w(x,t) = W(x,t) + h^A(x)V^A(x,t), \quad A = 1,\dots,N, \\
& \theta(x,t) = \Theta(x,t) + p^R(x)Z^R(x,t), \quad R = 1,\dots,M, \\
& W(\cdot), V^A(\cdot), \Theta(\cdot), Z^R(\cdot) \in SV_d^1(\Lambda,\Delta), \quad h^A(\cdot), p^R(\cdot) \in FS_d^1(\Lambda,\Delta),
\end{aligned}
\tag{5}
$$

The new basic kinematic unknowns $W(x,t)$ and $\Theta(x,t)$ are called the transverse macro-displacement and the macro-rotation; $V^A(x,t)$, $Z^R(x,t)$ are additional kinematic unknowns, called the fluctuation amplitudes. The unknown functions are assumed to be slowly-varying (*SV*) together with their first derivatives. The highly oscillating fluctuation shape functions (*FSF*s) h^A and p^R are assumed a priori in every problem under consideration in order to describe the unknown fields oscillations caused by the structure inhomogeneity. Apart from the restriction of l-periodicity, the *FSF*s have to satisfy the following conditions:

$$
\langle \rho A h^A \rangle = 0, \quad \langle \rho J p^K \rangle = 0. \tag{6}
$$

4. The tolerance model of a Timoshenko beam

The Lagrange function for considered problem is given as follows:

$$
L = U - K = \frac{1}{2}\left[EJ\partial\theta\partial\theta + kGA(\partial w\partial w - 2\partial w\theta + \theta\theta) - \rho A\dot{w}\dot{w} - \rho J\dot{\theta}\dot{\theta} \right]. \tag{7}
$$

As the basic modelling assumption micro-macro decompositions (5) of the unknown deflection w, longitudinal displacement u_0 and shear angle θ are introduced into Lagrangian. Applying averaging operator (4) and the tolerance averaging approximations, the tolerance averaged form $\langle \mathcal{L} \rangle$ of Lagrangian (7) is obtained in the form:

$$
\begin{aligned}
\langle \mathcal{L} \rangle = \frac{1}{2}\Big[& \langle EJ \rangle \partial\Theta\partial\Theta + 2\langle EJ\partial p^R \rangle \partial\Theta Z^R + \langle EJ\partial p^R \partial p^S \rangle Z^R Z^S + \langle kGA \rangle \partial W \partial W + \\
& + 2\langle kGA\partial h^A \rangle \partial W V^A + \langle kGA\partial h^A \partial h^B \rangle V^A V^B - 2\langle kGA \rangle \partial W \Theta - 2\langle kGAp^R \rangle \partial W Z^R + \\
& - 2\langle kGA\partial h^A \rangle \Theta V^A + -2\langle kGA\partial h^A p^R \rangle V^A Z^R + \langle kGA \rangle \Theta\Theta + 2\langle kGAp^R \rangle \Theta Z^R + \\
& + \langle kGAp^R p^S \rangle Z^R Z^S - \langle \rho A \rangle \dot{W}\dot{W} + -2\langle \rho A h^A \rangle \dot{W}\dot{V}^A - \langle \rho A h^A h^A \rangle \dot{V}^A \dot{V}^B + \\
& - \langle \rho J \rangle \dot{\Theta}\dot{\Theta} - 2\langle \rho J p^R \rangle \dot{\Theta}\dot{Z}^R - \langle \rho J p^R p^S \rangle \dot{Z}^R \dot{Z}^S
\end{aligned}
\tag{8}
$$

Subsequently, variation of above Lagrangian leads to four equations of motion with constant coefficients.

$$
\begin{aligned}
& \langle kGA \rangle (\partial^2 W - \partial\Theta) + \langle kGA\partial h^A \rangle \partial V^A - \underline{\langle kGAp^R \rangle} \partial Z^R - \langle \rho A \rangle \ddot{W} - \underline{\langle \rho A h^A \rangle} \ddot{V}^A = 0, \\
& \langle kGA\partial h^A \rangle (\partial W - \Theta) + \langle kGA\partial h^A \partial h^B \rangle V^B - \underline{\langle kGA\partial h^A p^R \rangle} Z^R + \underline{\langle \rho A h^A \rangle} \ddot{W} + \\
& + \underline{\langle \rho A h^A h^B \rangle} \ddot{V}^B = 0,
\end{aligned}
\tag{9}
$$

$$
\begin{gathered}
\langle EJ\rangle\partial^2\Theta+\langle EJ\partial p^R\rangle\partial Z^R+\langle kGA\rangle(\partial W-\Theta)+\langle kGA\partial h^A\rangle V^A-\underline{\langle kGAp^R\rangle}Z^R+\\
-\langle\rho J\rangle\ddot{\Theta}-\underline{\langle\rho Jp^R\rangle}\ddot{Z}^R=0,\\
\langle EJ\partial p^R\rangle\partial\Theta+\langle EJ\partial p^R\partial p^S\rangle Z^S-\underline{\langle kGAp^R\rangle}(\partial W-\Theta)-\underline{\langle kGA\partial h^A p^R\rangle}V^A+\\
+\underline{\langle kGAp^R p^S\rangle}Z^S+\underline{\langle\rho Jp^R\rangle}\ddot{\Theta}+\underline{\langle\rho Jp^R p^S\rangle}\ddot{Z}^S=0.
\end{gathered}
\tag{9}
$$

The underlined terms depend on the microstructure size.

5. Asymptotic model equations

Neglecting the terms dependent on the cell length l, we obtain the system of equations of the asymptotic model. It describes the behaviour of Timoshenko beam only in the macroscale:

$$
\begin{gathered}
\langle kGA\rangle(\partial\partial W-\partial\Theta)+\langle kGA\partial h^A\rangle\partial V^A-\langle\rho A\rangle\ddot{W}=0,\\
\langle kGA\partial h^A\rangle(\partial W-\Theta)+\langle kGA\partial h^A\partial h^B\rangle V^B=0,\\
\langle EJ\rangle\partial\partial\Theta+\langle EJ\partial p^R\rangle\partial Z^R+\langle kGA\rangle(\partial W-\Theta)+\langle kGA\partial h^A\rangle V^A-\langle\rho J\rangle\ddot{\Theta}=0,\\
\langle EJ\partial p^R\rangle\partial\Theta+\langle EJ\partial p^R\partial p^S\rangle Z^S=0.
\end{gathered}
\tag{10}
$$

Equations $(10)_2$ and $(10)_4$ can be rewritten as

$$
V^A=-\frac{\langle kGA\partial h^A\rangle}{\langle kGA\partial h^A\partial h^B\rangle}(\partial W-\Theta),\quad Z^R=-\frac{\langle EJ\partial p^R\rangle}{\langle EJ\partial p^R\partial p^S\rangle}\partial\Theta.
\tag{11}
$$

We can further define the effective shear stiffness H^{eff} and effective bending stiffness D^{eff} which are constant:

$$
\langle kGA\rangle-\frac{\langle kGA\partial h^A\rangle\langle kGA\partial h^B\rangle}{\langle kGA\partial h^A\partial h^B\rangle}=H^{eff},\quad \langle EJ\rangle-\frac{\langle EJ\partial p^R\rangle\langle EJ\partial p^S\rangle}{\langle EJ\partial p^R\partial p^S\rangle}=D^{eff}.
\tag{12}
$$

Combining equations (10-12), we obtain the following system of differential equations which represents the asymptotic model of considered Timoshenko beam:

$$
\begin{gathered}
H^{eff}(\partial\partial W-\partial\Theta)-\langle\rho A\rangle\ddot{W}=0,\\
D^{eff}\partial\partial\Theta+H^{eff}(\partial W-\Theta)-\langle\rho J\rangle\ddot{\Theta}=0.
\end{gathered}
\tag{13}
$$

It can be noted that the above equations have the same form as the equations for a homogeneous beam, cf. [4].

6. Asymptotic model solution

We assume that functions W, Θ share the same time solution $T(t)$:

$$
\begin{bmatrix}W(x,t)\\ \Theta(x,t)\end{bmatrix}=\begin{bmatrix}W(x)\\ \Theta(x)\end{bmatrix}T(t).
\tag{14}
$$

After substitution of (14), equations (13) can be rewritten in matrix form:

$$0=\begin{bmatrix} H^{eff} & 0 \\ 0 & D^{eff} \end{bmatrix}\begin{bmatrix} \partial\partial W \\ \partial\partial\Theta \end{bmatrix}+\begin{bmatrix} 0 & -H^{eff} \\ H^{eff} & 0 \end{bmatrix}\begin{bmatrix} \partial W \\ \partial\Theta \end{bmatrix}+ \\ +\begin{bmatrix} \langle\rho A\rangle\omega^2 & 0 \\ 0 & -H^{eff}+\langle\rho J\rangle\omega^2 \end{bmatrix}\begin{bmatrix} W \\ \Theta \end{bmatrix}. \tag{15}$$

These equations can be decoupled to yield

$$\partial\partial\partial\partial\Theta+\left(\frac{\langle\rho A\rangle}{H^{eff}}+\frac{\langle\rho J\rangle}{D^{eff}}\right)\omega^2\partial\partial\Theta-\frac{\langle\rho A\rangle}{D^{eff}}\left(1-\frac{\langle\rho J\rangle}{H^{eff}}\omega^2\right)\omega^2\Theta=0, \\ \partial\partial\partial\partial W+\left(\frac{\langle\rho A\rangle}{H^{eff}}+\frac{\langle\rho J\rangle}{D^{eff}}\right)\omega^2\partial\partial W-\frac{\langle\rho A\rangle}{D^{eff}}\left(1+\frac{\langle\rho J\rangle}{H^{eff}}\omega^2\right)\omega^2 W=0. \tag{16}$$

The differential equations for $W(x)$ and $\Theta(x)$ have the same form, so that it is assumed that he solutions also have the same form and differ by a constant as

$$\begin{bmatrix} W(x) \\ \Theta(x) \end{bmatrix}=d\mathbf{u}e^{rx}. \tag{17}$$

The characteristic equation is given by

$$r^4+\left(\frac{\langle\rho A\rangle}{H^{eff}}+\frac{\langle\rho J\rangle}{D^{eff}}\right)\omega^2 r^2-\frac{\langle\rho A\rangle}{D^{eff}}\left(1-\frac{\langle\rho J\rangle}{H^{eff}}\omega^2\right)\omega^2=0, \tag{18}$$

therefore the eigenfrequencies can be expressed as:

$$r_i=\pm\sqrt{-\left(\frac{\langle\rho A\rangle}{H^{eff}}+\frac{\langle\rho J\rangle}{D^{eff}}\right)\omega^2\pm\sqrt{\left(\frac{\langle\rho A\rangle}{H^{eff}}+\frac{\langle\rho J\rangle}{D^{eff}}\right)^2\omega^4+\frac{\langle\rho A\rangle}{D^{eff}}\left(1-\frac{\langle\rho J\rangle}{H^{eff}}\omega^2\right)\omega^2}}, \tag{19}$$

i = 1, 2, 3, 4, and from the following equation:

$$\begin{bmatrix} H^{eff}r^2+\langle\rho A\rangle\omega^2 & -H^{eff}r \\ H^{eff}r & D^{eff}r^2-H^{eff}r+\langle\rho J\rangle\omega^2 \end{bmatrix}\mathbf{u}=0, \tag{20}$$

the corresponding eigenvectors **u** are obtained:

$$\mathbf{u}_i=\begin{bmatrix} H^{eff}r \\ H^{eff}r^2+\langle\rho A\rangle\omega^2 \end{bmatrix}\ \text{lub}\ \begin{bmatrix} D^{eff}r^2-H^{eff}r+\langle\rho J\rangle\omega^2 \\ -H^{eff}r \end{bmatrix}. \tag{21}$$

The spatial solutions are given by

$$\begin{bmatrix} W_m(x) \\ \Theta_m(x) \end{bmatrix}=\sum_{i=1}^{4}d_i\mathbf{u}_i e^{r_i x}=d_1\mathbf{u}_1e^{bx}+d_2\mathbf{u}_2e^{-bx}+d_3\mathbf{u}_3e^{\mathrm{i}ax}+d_4\mathbf{u}_4e^{-\mathrm{i}ax}, \tag{22}$$

where

$$a=\sqrt{\left(\frac{\langle\rho A\rangle}{H^{eff}}+\frac{\langle\rho J\rangle}{D^{eff}}\right)\omega^2+\sqrt{\left(\frac{\langle\rho A\rangle}{H^{eff}}+\frac{\langle\rho J\rangle}{D^{eff}}\right)^2\omega^4+\frac{\langle\rho A\rangle}{D^{eff}}\left(1-\frac{\langle\rho J\rangle}{H^{eff}}\omega^2\right)\omega^2}}, \tag{23}$$

$$b=\sqrt{-\left(\frac{\langle\rho A\rangle}{H^{eff}}+\frac{\langle\rho J\rangle}{D^{eff}}\right)\omega^2+\sqrt{\left(\frac{\langle\rho A\rangle}{H^{eff}}+\frac{\langle\rho J\rangle}{D^{eff}}\right)^2\omega^4+\frac{\langle\rho A\rangle}{D^{eff}}\left(1-\frac{\langle\rho J\rangle}{H^{eff}}\omega^2\right)\omega^2}}\,. \tag{23}$$

Spatial solution (22) can be also written in terms of the sinusoidal and hyperbolic functions with real arguments:

$$\begin{bmatrix}W(x)\\ \Theta(x)\end{bmatrix}=\begin{bmatrix}C_1\\ D_1\end{bmatrix}\sin ax+\begin{bmatrix}C_2\\ D_2\end{bmatrix}\cos ax+\begin{bmatrix}C_3\\ D_3\end{bmatrix}\sinh bx+\begin{bmatrix}C_4\\ D_4\end{bmatrix}\cosh bx, \tag{24}$$

and only four from the constants C_1-C_4 and D_1-D_4 are independent, cf. [4].

Substituting (24) into the boundary conditions for W and Θ, we obtain a system of linear homogeneous equations for the suitable constants C and D. Then, the frequency equation is derived from the condition that the determinant of coefficients matrix has to vanish.

7. Application

In this section, analysis of influence of geometrical parameters in a cell on the first natural frequency of hinged-hinged beam with periodically varying cross-section, cf. Figure 1, is performed. The boundary conditions for considered beam are:

$$\partial\Theta=0 \quad \text{and} \quad W=0 \quad \text{for} \quad x=0, L. \tag{25}$$

The frequencies were obtained in the framework of the proposed model and compared with the results from a finite element model (30 elements, 60 degrees of freedom) with Hermite polynomials as shape functions.

The fluctuation shape functions defining the fluctuating parts of unknown displacements were assumed in the form of trigonometric series:

$$h^A(y)=l\sin\frac{2A\pi y}{l},\quad p^R(y)=l\sin\frac{2R\pi y}{l}, \tag{26}$$

to ensure non-zero correctors in calculating the effective shear and bending stiffness (12). The conditions (6) are satisfied for considered symmetric unit cell. The number of functions (26) has been selected by the analysis of the effective stiffness convergence, and the satisfactory results were obtained for $N = M = 10$.

The beam length is $L = 1$ m, shear factor $k = 5/6$, the mass density of the material $\rho = 7860$ kg/m^3, Young modulus $E = 210$ GPa. The cross-section is rectangular, piecewise constant. The saturation parameter α changes in range 0.1-0.9, section width is $b_1 = b_2 = 20$ mm, section height is $h_1 = 20$ mm, $h_2 = a\ h_1 = \{0.95, 0.9, 0.85, 0.8, 0.75\}\ h_1$. The number of the cells is 10, $h_1 / l = 1/5$, hence the cell can be considered as moderately thick. Dependence of the first natural frequency ω for asymptotic model (lines) and finite element model (dots) is depicted in Figure 2, and the relative difference between these models, versus parameter α is shown in Figure 3. As it can be seen from the Figure 3, the results differ no more than 0.5% in the considered cases.

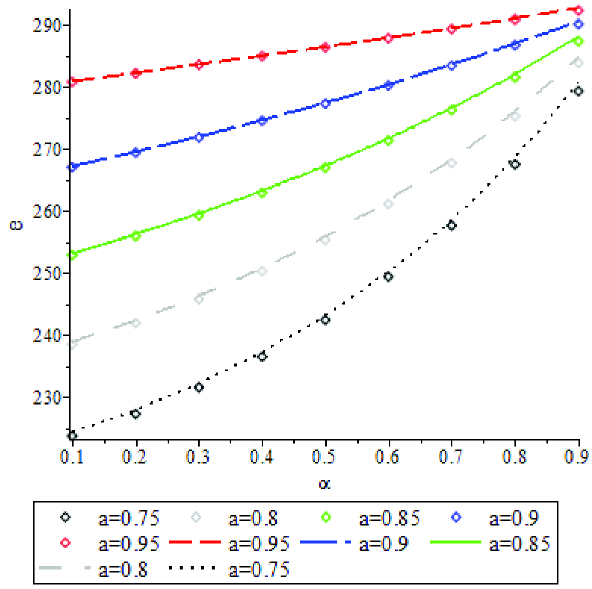

Figure 2. First natural frequency for various values of cross section height, dots - finite element model, lines – asymptotic model; $a=h_2/h_1$

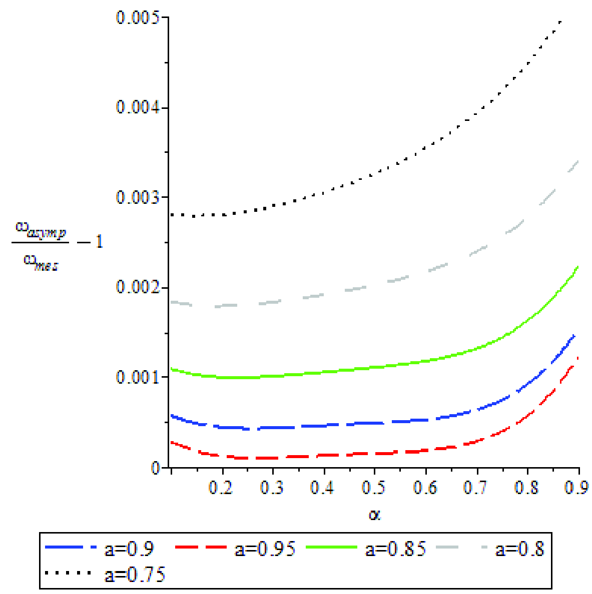

Figure 3. Relative difference between asymptotic and fem results

8. Conclusions

The natural vibration analysis of a periodic beam has been performed in the framework of tolerance modelling technique. The Timoshenko beam theory, including first order kinematic correction for shear strain, have been applied in order to analyse beams consisting of non-slender repetitive cells. The obtained system of differential equations with constant coefficients and additional degrees of freedom makes it possible to describe the dynamics of the beam in the macro-scale. The coefficients of these equations depend on so-called fluctuation shape functions which describe the vibrations of a periodicity cell.

A simplified version of the proposed model has been applied in analysis of first natural frequency of a variable cross-section beam. From the obtained results it can be concluded that application of approximate fluctuation shape functions leads to satisfactory results.

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