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ON THE DIMENSION OF ARCHIMEDEAN SOLIDS

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Abstract. We study the dimension of graphs of the Archimedean solids. For most of these graphs we find the exact value of their dimension by finding unit-distance embeddings in the euclidean plane or by proving that such an embedding is not possible.

Keywords: Archimedean solid, unit-distance graph, dimension of a graph.

Mathematics Subject Classification: 05C10.

1. INTRODUCTION

Throughout this paper, we consider simple connected graphs and their geometric representation in an Euclidean space: given a graph G and an integer $n \geq 2$, let $D(G)$ be a drawing of G in $\mathbb{R}^n$ in such a way that each edge is a linear segment. In particular, we are interested in the case when all edges of $D(G)$ are of unit length (a unit-distance representation); note that there always exists the unit-distance representation of G for $n = |V(G)| - 1$. The smallest n such that there exists unit-distance representation $D(G)$ of G in $\mathbb{R}^n$ is the *graph dimension* $\dim(G)$ of G. If $\dim(G) = 2$, G is called *unit-distance graph*.

Note that, in most cases, we assume that the geometric representation $D(G)$ of a graph G is non-degenerate, that is, two distinct vertices u, v of G correspond to distinct points U, V of $D(G)$. However, sometimes we will also consider drawings in which a vertex (a geometric point) corresponds to different vertices of a graph; such drawings are called *degenerate*.

The graph dimension was first defined by Erdős, Harary and Tutte in [3]. In the same paper, they have determined the exact values and upper bounds for dimensions of graphs of several classes (including complete bipartite graphs, wheels, hypercubes and graphs with fixed chromatic number or large girth). Generally, the problem of determining the graph dimension seems to be hard (see the note [2] and the paper [6] on possible algorithms for producing a unit-distance drawing of a graph) and the unit-distance representations are difficult to find even for small graphs, see [4] for unit-distance drawings of the Heawood graph (which has been conjectured to have graph dimension at least 3).

On the other hand, certain graphs are linked, in a natural way, with geometric objects which allow us to determine their graph dimension easily. Important examples are the graphs of Platonic and Archimedean solids. Based on their definition (see [5] for a discussion on differences between existing notions of semiregularity of polyhedra), it follows that all edges of these solids have the same length, hence the dimension of their graphs is at most 3. It is easy to check that the tetrahedron, octahedron and icosahedron graphs are not unit-distance; the unit-distance drawing of a cube and a dodecahedron graph are found in [3] and [7], respectively. The aim of this paper is to find out which Archimedean solids possess graphs which are unit-distance embeddable in the plane: we show that this is the case for prisms, several truncated solids (truncated tetrahedron, cube, octahedron, dodecahedron, icosahedron and cuboctahedron), rhombicuboctahedron and icosidodecahedron, whereas for antiprisms, snub solids (snub cube and dodecahedron) and cuboctahedron, no unit-distance drawing in the plane exists. For two remaining Archimedean solids – the rhombicosidodecahedron and truncated icosidodecahedron – the existence of an unit-distance drawing is open, although, for truncated icosidodecahedron, we were able to find a degenerate unit-distance drawing in the plane which might be possibly transformed, using the methods described in Section 3, into a non-degenerate one.

The rest of the paper is devoted to presenting a detailed explanation of approaches used to construct, for graphs of particular Archimedean solids, their unit-distance drawings or to prove their nonexistence. According to this, we will divide all Archimedean solids into four groups:

1. cube-like solids: truncated cube, truncated cuboctahedron, rhombicuboctahedron and prisms,
2. solids involving a kind of "rotation symmetry": the icosidodecahedron, truncated icosahedron and truncated dodecahedron,
3. solids with "bad triangles": antiprisms, snub cube, snub dodecahedron, cuboctahedron,
4. the rest.

2. CUBE-LIKE SOLIDS

The common idea for constructing unit-distance drawings of truncated cube, truncated cuboctahedron, rhombicuboctahedron and prisms in the plane is inspired by the unit-distance drawing of the cube as presented in [3]. Each of these solids admits a plane symmetry with respect to a plane that passes through midpoints of selected edges (see Fig. 1). From this it follows that we can decompose the edge set of each corresponding graph into three subsets such that two of them induce disjoint subgraphs H_1, H_2 which are isomorphic and the third one induces a matching. First, we construct a unit-distance drawing $D(H_1)$ of H_1; then we obtain a unit-distance drawing of H_2 by translating all points of $D(H_1)$ by a suitable unit vector (note that it may be chosen in such a way that the vertices of $D(H_1)$ and $D(H_2)$ do not overlap). Finally, to obtain the unit-distance drawing of the considered solid graph, we join the pairs of equivalent vertices under this translation by new unit edges (see Fig. 2 for final unit-distance graphs).

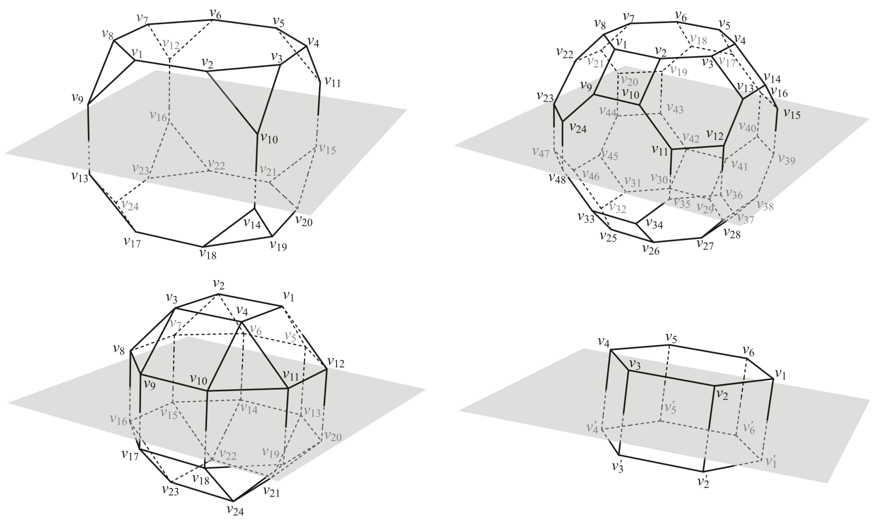

Fig. 1

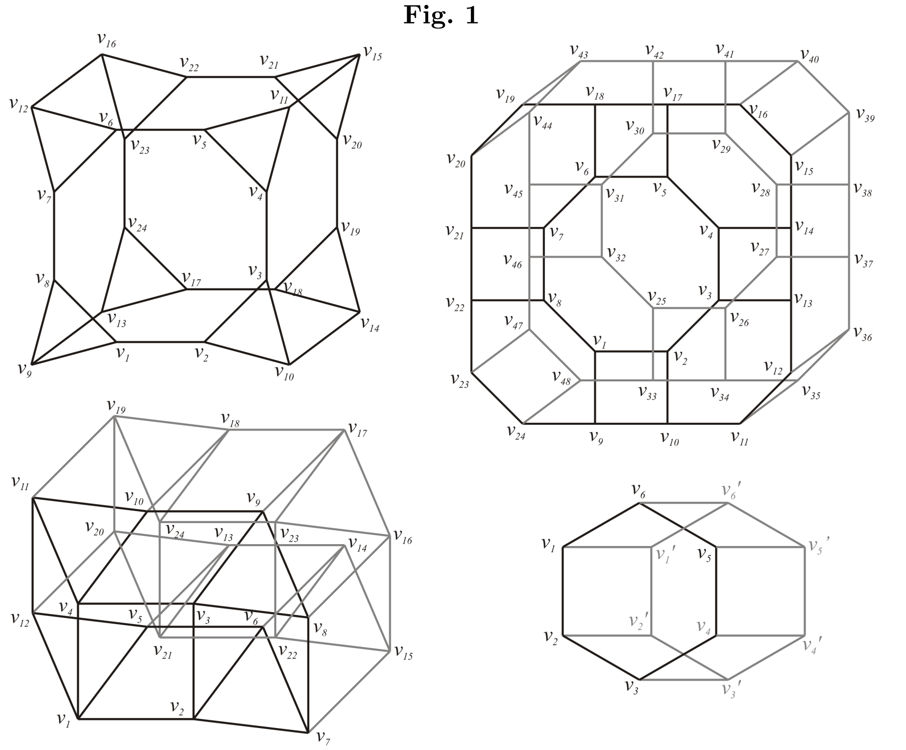

Fig. 2

3. SOLIDS WITH ROTATION-SYMMETRIC UD-EMBEDDINGS

This group includes the icosidodecahedron, truncated icosahedron and truncated dodecahedron. The method of construction of their unit-distance drawings combines the approach used for the solid graphs in the first group: we find isomorphic subgraphs on half of the vertices, but their unit-distance drawings are now equivalent under a certain rotation.

Consider first the truncated icosahedron (see Fig. 3):

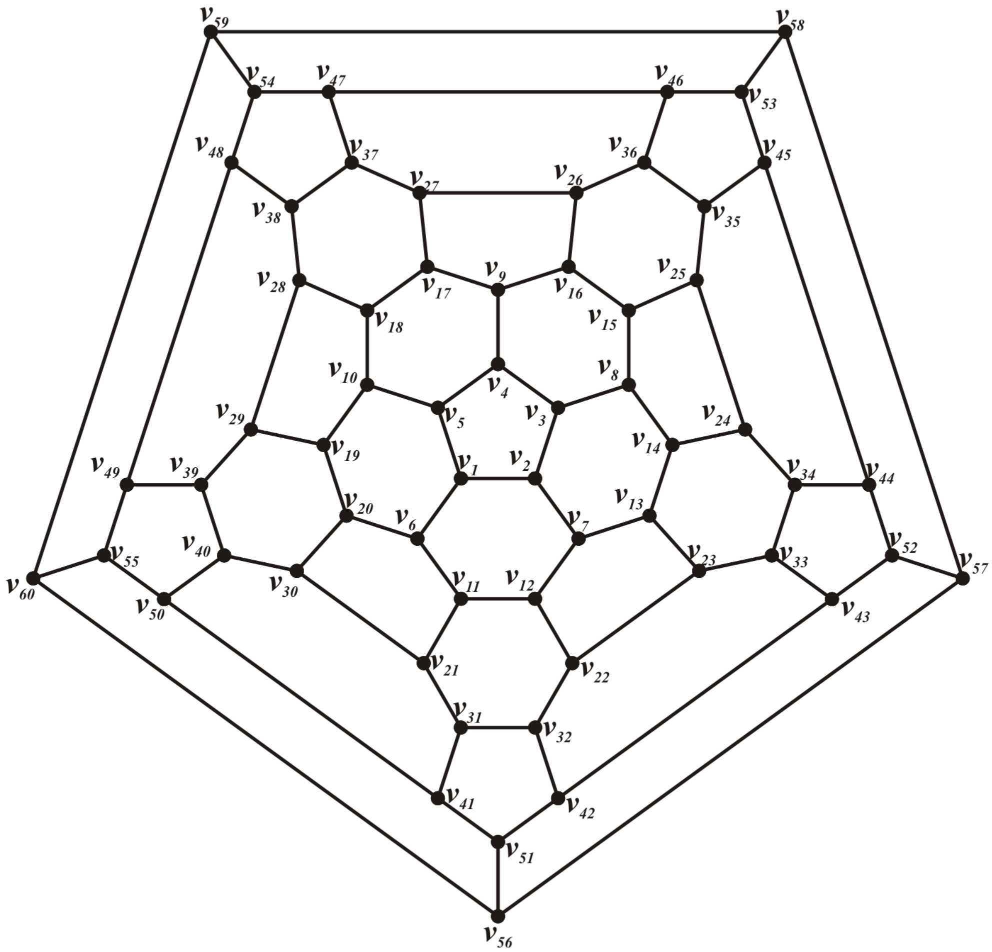

Fig. 3

We decompose the edge set of this graph into three subsets, two of them inducing disjoint isomorphic subgraphs R_1, R_2 on 30 vertices (R_1 has vertex set $\{v_i : i \in [1, 30]\}$, R_2 the vertex set $\{v_i : i \in [31, 60]\}$) and the third one being a matching. First, we find a plane unit-distance drawing of R_1. The central 5-face containing vertices v_1 to

v_5 is drawn as a regular pentagon. We continue with the 6-face $v_1v_2v_7v_{12}v_{11}v_6$: the vertex v_6 is placed on the axis of the outer angle to $\angle v_5v_1v_2$ in unit-distance from v_1; similarly, v_7 is placed on the axis of the outer angle to $\angle v_1v_2v_3$ in unit distance from v_2. The image of v_1 in axial symmetry with respect to axis v_6v_7 will be the location of vertex v_{11}, and, similarly, v_{12} is the image of v_2 under the same symmetry. An analogous approach is used for the remaining 6-faces around the central 5-face. Finally, we need to perform the construction for the "peripheral" 5-faces. We explain this for 5-face $v_6v_{11}v_{21}v_{30}v_{20}$. We only need to place vertices v_{21} and v_{30}: the vertex v_{21} is placed in unit-distance from vertex v_{11} (in the following, we denote the measure of angle $\angle v_6v_{11}v_{21}$ as α). We place vertex v_{30} in a similar way in unit-distance from v_{20} (the measure of angle $\angle v_6v_{20}v_{30}$ is denoted β). Now the distance of vertices v_{21} and v_{30} can be expressed as a function depending on α and β. We apply an analogous approach to the remaining "peripheral" 5-faces. While doing this we make sure not to break the rotational symmetry with respect to the angle of measure $\frac{2\pi}{5}$ with the center of symmetry located at the center of the regular pentagon $v_1v_2v_3v_4v_5$. Now, having a unit-distance drawing of R_1, we create the unit-distance drawing of R_2 as the image of unit-distance drawing of R_1 in a rotation through angle of measure π about the center of the regular pentagon. Note that, at this point, the assignment of indices to vertices is important (see black and red subgraph in Fig. 4).

Obviously, there exists a rotational symmetry of the obtained partial embedding about the center of the regular pentagon through the angle of measure $\frac{2\pi}{5}$. This yields that all the edges in the set $\{v_{22}v_{23}, v_{24}v_{25}, v_{26}v_{27}, v_{28}v_{29}, v_{21}v_{30}, v_{31}v_{32}, v_{33}v_{34}, v_{35}v_{36}, v_{37}v_{38}, v_{39}v_{40}\}$ will be of equal length. For the same reason, also the edges in set $\{v_{21}v_{31}, v_{22}v_{32}, v_{23}v_{33}, v_{24}v_{34}, v_{25}v_{35}, v_{26}v_{36}, v_{27}v_{37}, v_{28}v_{38}, v_{29}v_{39}, v_{30}v_{40}\}$ are, in this partial embedding, of equal length. The lengths of the edges in both sets can be expressed as functions of α and β in the following way (we assume both equal 1):

$$
\begin{aligned}
|v_{26}v_{27}|^2 = {} & \left[2\cos\left(\frac{\pi}{10}\right) - \cos\left(\alpha - \frac{\pi}{10}\right) - \cos\left(\beta - \frac{\pi}{10}\right)\right]^2 + \\
& + \left[\sin\left(\alpha - \frac{\pi}{10}\right) - \sin\left(\beta - \frac{\pi}{10}\right)\right]^2 = 1, \\
|v_{27}v_{37}|^2 = {} & \left[1 - 2\cos\frac{\pi}{5} + \sin\frac{\pi}{10} + \sin\left(\beta - \frac{\pi}{10}\right) - \sin\left(\alpha - \frac{3\pi}{10}\right) + \frac{1}{2\sin\frac{\pi}{5}} - \frac{1}{2\tan\frac{\pi}{5}}\right]^2 + \\
& + \left[\cos\left(\beta - \frac{\pi}{10}\right) - \cos\frac{\pi}{10} + \frac{1}{2} + \cos\left(\alpha - \frac{3\pi}{10}\right)\right]^2 = 1.
\end{aligned}
$$

We need to prove that there exists a solution $[\alpha_0, \beta_0]$ of the above nonlinear system. To show this, we use the implicit function theorem to express α as a continuous function f of $\beta \in I_0 = (1.11, 1.12)$ from the first equation and β as a continuous function g of $\alpha \in I_1 = (1.66, 1.67)$ from the second equation. As $f(I_0) \subset I_1$ and $g(I_1) \subset I_0$ and both f and g are continuous, there exists a point $[\alpha_0, \beta_0]$ where $\beta_0 = g(\alpha_0)$ and $\alpha_0 = f(\beta_0)$. For these values, both edges $v_{26}v_{27}$ and $v_{27}v_{37}$ are of unit length, thus yielding a unit-distance drawing of the whole graph. We used the computer algebra system Maple for determining a numerical approximation of the solution, obtaining $\alpha_0 \doteq 1.630\,067\,2$ and $\beta_0 \doteq 1.140\,945\,5$.

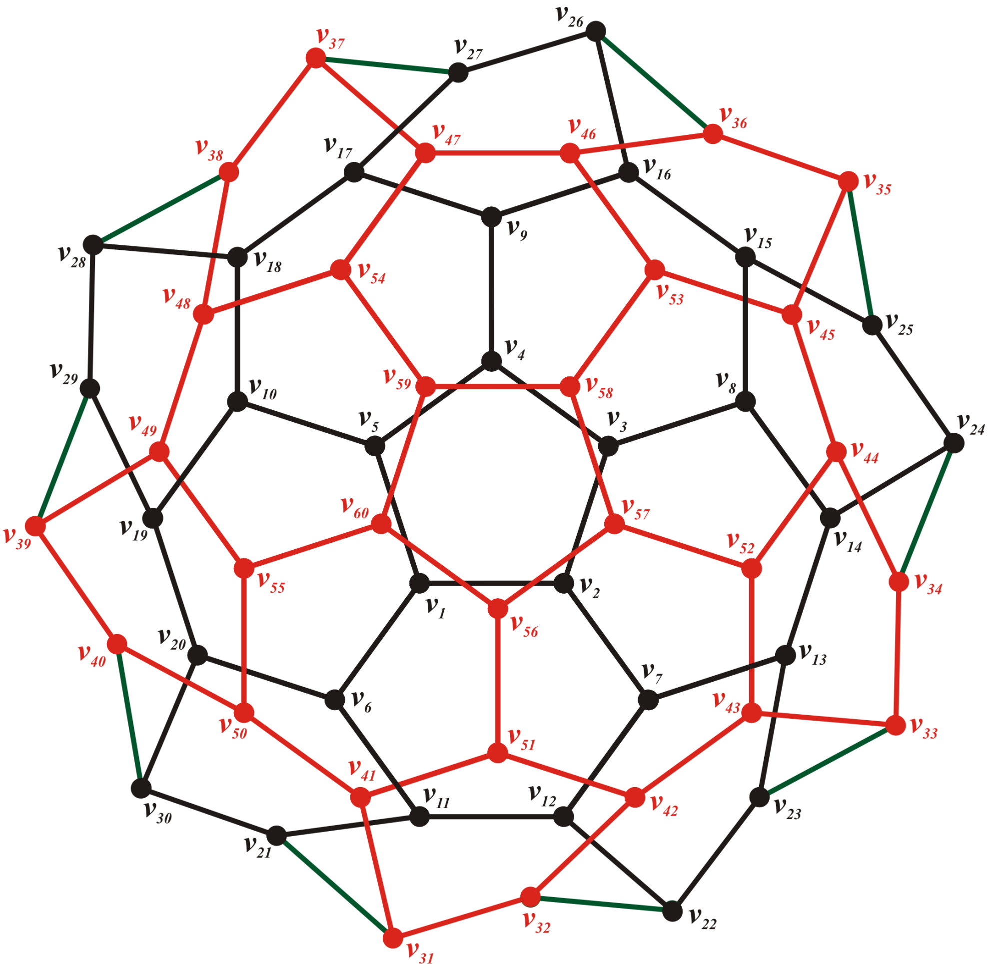

Fig. 4

The other constructions for the solids in this group are similar – all of them are based on rotational symmetry. Sometimes it may be necessary to alter some steps, as the result may be degenerate if unaltered. We usually have to find solutions for two implicit functions, in other cases it may be sufficient to deal with a single function.

Let us continue with the graph of the icosidodecahedron (see Fig. 5).

We construct the drawing in such a way that the whole embedding possesses a rotational symmetry through an angle of measure $\frac{2\pi}{5}$ about the point $(0,0)$ in a fixed Cartesian coordinate system. We start with embedding the 5-face $v_1v_2v_3v_4v_5$ as a regular pentagon centered at $(0,0)$. There are two possible points we could place vertex v_6 (as it forms a 3-face with vertices v_1 and v_2), but we choose its location outside the central pentagon. Analogously, we place vertices $v_7, \ldots, v_{10}$. To place vertices $v_{11}, \ldots, v_{20}$, denote the measure of $\angle v_1v_6v_{11}$ as α and place vertices v_{13},

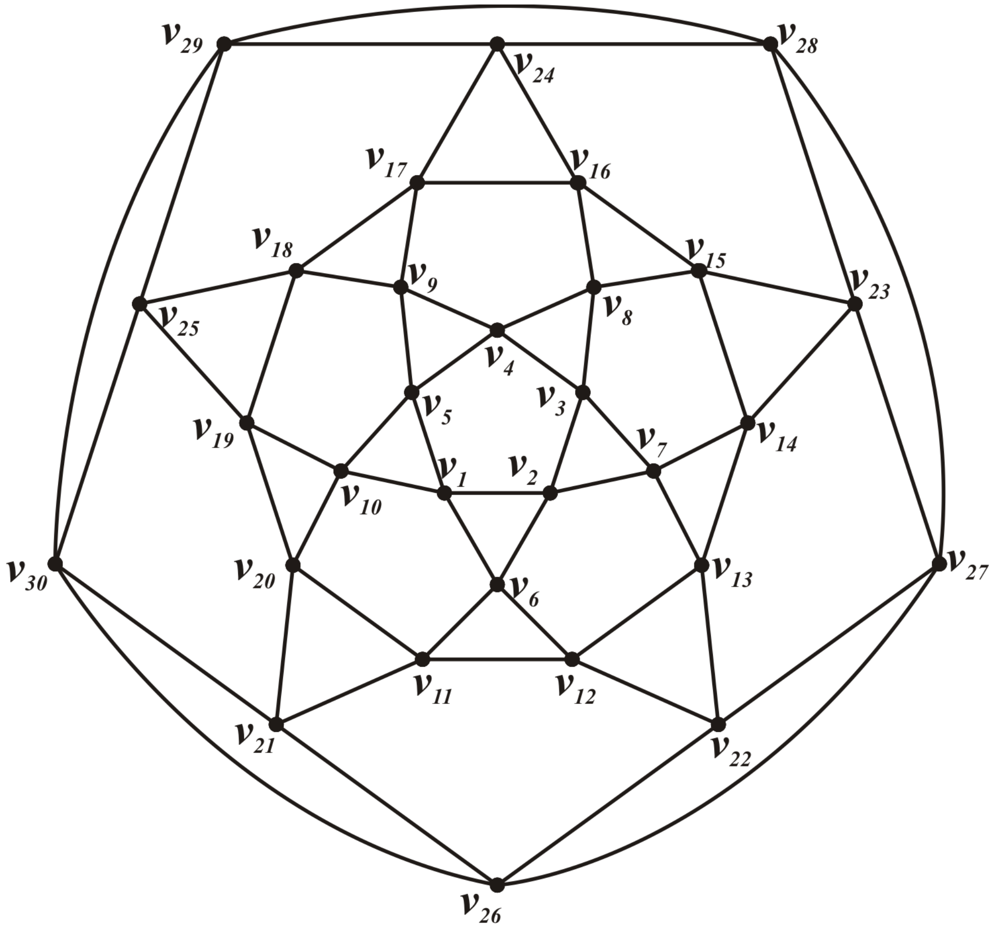

Fig. 5

v_{15}, v_{17} and v_{19} so that $|\angle v_2v_7v_{13}| = |\angle v_3v_8v_{15}| = |\angle v_4v_9v_{17}| = |\angle v_5v_{10}v_{19}| = \alpha$. Now there are two possible locations for the vertex v_{12}; both are equally admissible, it is just important to choose the location of vertices v_{14}, v_{16}, v_{18} and v_{20} accordingly (with respect to vertices v_{13}, v_{15}, v_{17} and v_{19} respectively). Now we can express the positions of vertices $v_{11}, \ldots, v_{20}$ as functions of α. Except for edges $v_{12}v_{13}$, $v_{14}v_{15}$, $v_{16}v_{17}$, $v_{18}v_{19}$ and $v_{11}v_{20}$, all edges constructed so far are of unit length. Because of the rotational symmetry preserved by this construction, edges $v_{12}v_{13}$, ..., $v_{11}v_{20}$ are of equal length, so it suffices to find an embedding where one of these edges is of unit length. Depending on how we choose the locations of vertices v_1 to v_{20}, the length of such an edge can be expressed as

$$h(\alpha) = \left[2\cos\frac{2\pi}{15} - \cos\left(\alpha+\frac{2\pi}{15}\right) - \sin\left(\alpha+\frac{11\pi}{30}\right)\right]^2 + \left[\sin\left(\alpha+\frac{2\pi}{15}\right) - \cos\left(\alpha+\frac{11\pi}{30}\right)\right]^2.$$

It is easy to show that there exists a real number α_0 such that $h(\alpha_0) = 1$ (using Maple software, we found $\alpha_0 \doteq 0.554\,390\,551\,5$).

The last thing we need to do is to place vertices v_{21} to v_{30}. Place vertex v_{21} so that it is the image of vertex v_{10} in axial symmetry with the axis passing through vertex v_{20} and point $(0,0)$. Place vertices v_{22} to v_{25} analogously (v_{22} being the image of v_6 with axis $(0,0)$ and v_{12} and so on). The position of vertices v_{26} to v_{30} is determined by the position of vertices v_{21} to v_{25} (we place them on the cycle that contains vertices v_1 to v_5). We illustrate the unit-distance embedding of this graph by Figure 6.

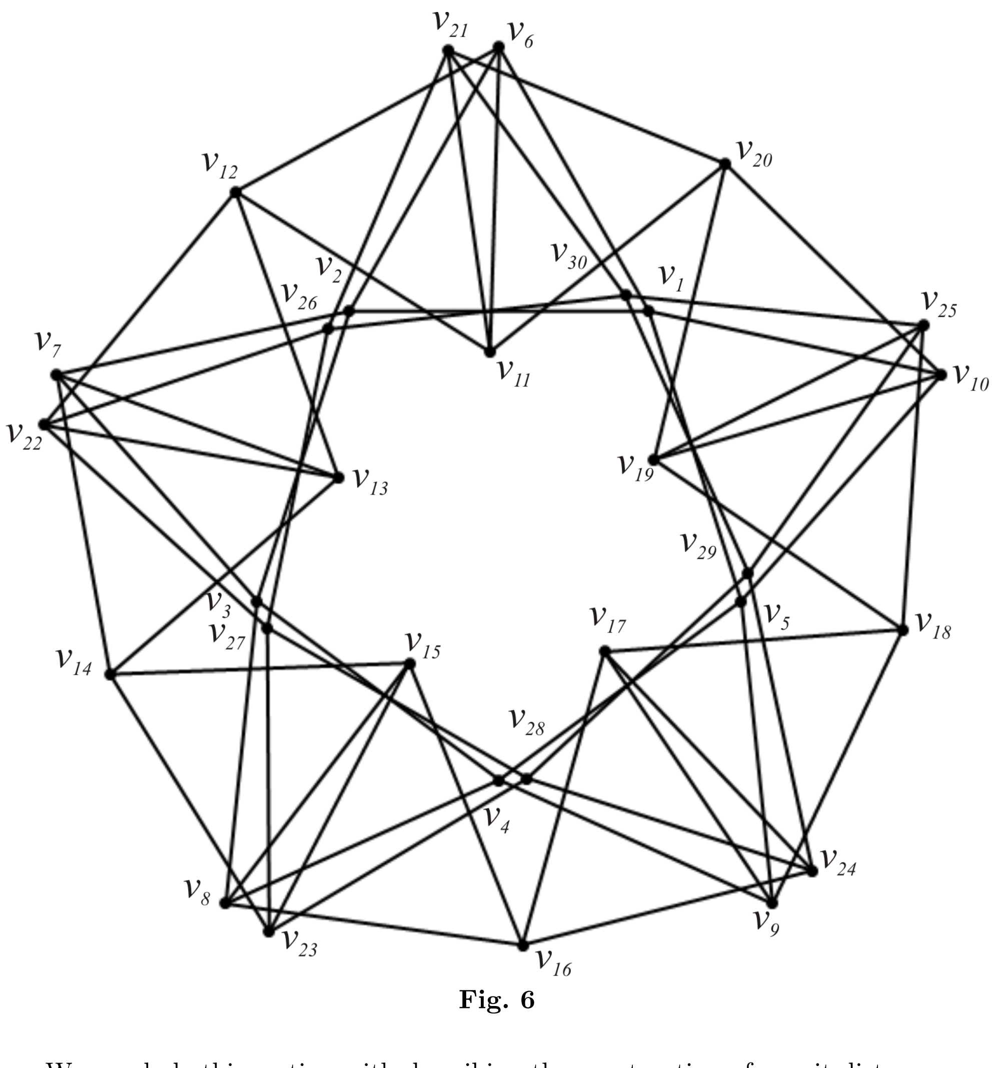

Fig. 6

We conclude this section with describing the construction of a unit-distance embedding of the graph of the truncated dodecahedron (see Fig. 7).

We start by embedding the subgraph induced on vertices v_1 to v_{30}; denote this

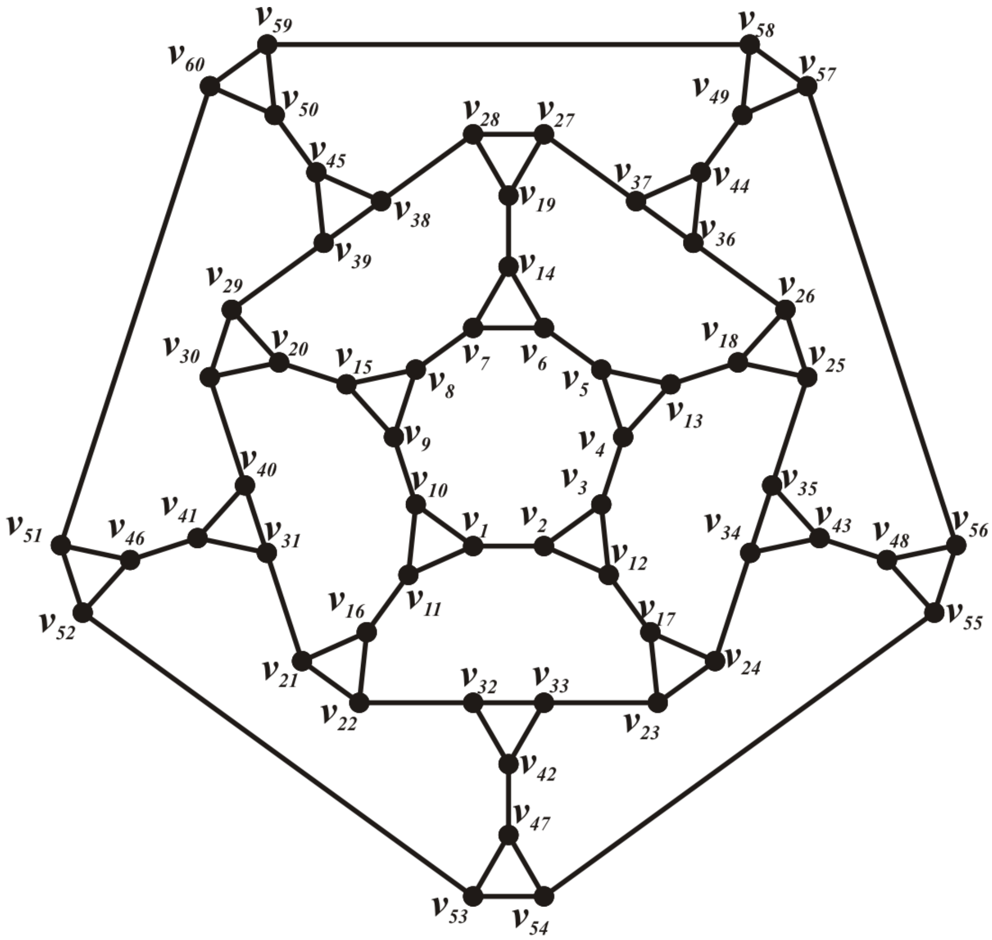

Fig. 7

subgraph H_1. Again, we preserve a certain rotational symmetry (the rotation through angle of measure $\frac{2\pi}{5}$). First we draw the central 10-face $v_1 \dots v_{10}$ as a regular decahedron (denote its center as S). There are two possible positions where to place vertex v_{11} (they are determined by the position of vertices v_1 and v_{10}); we choose the position outside the decahedron. Analogously, we place vertices v_{12} to v_{15}. We place vertex v_{16} on the line passing through S and v_{11}, in unit distance from point v_{11} in such a way that $|S, v_{16}| > |S, v_{11}|$. Repeat an analogous placement for vertices v_{17} to v_{20}. Denote the measure of angle $\angle v_{11}v_{16}v_{21}$ as α and place vertices v_{23}, v_{25}, v_{27} and v_{29} so that $|\angle v_{12}v_{17}v_{23}| = |\angle v_{13}v_{18}v_{25}| = |\angle v_{14}v_{19}v_{27}| = |\angle v_{15}v_{20}v_{29}| = \alpha$, in such a way, that the image of vertices v_{21}, v_{23}, v_{25}, v_{27} and v_{29} under the rotation through an angle of measure $\frac{2\pi}{5}$ lies in the set $\{v_{21}, v_{23}, v_{25}, v_{27}, v_{29}\}$. There are two possible positions for vertex v_{22}, both equally admissible; it is important just to pick the location of vertices v_{24}, v_{26}, v_{28} and v_{30} analogously (with respect to their neighbours and rotational symmetry). The embedding constructed so far has a rotational symmetry through an angle of measure $\frac{2\pi}{5}$ about S (see Fig. 8).

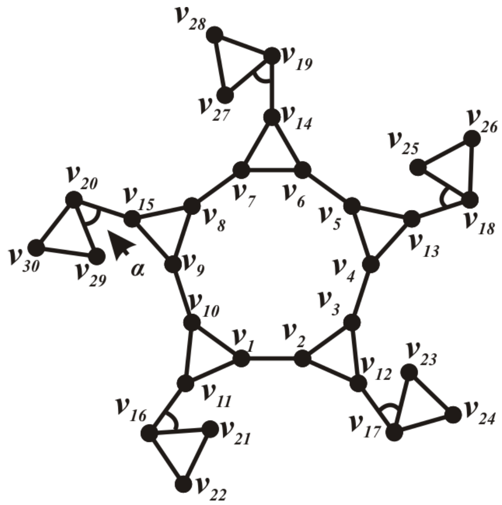

Fig. 8

We construct subgraph H_2 induced by vertices v_{31} to v_{60}, which is isomorphic to H_1, using the same construction, but we replace unknown α by a new unknown β, which can differ from α. The next step is to join the two embeddings into one, usually rotating one through an angle of measure π about the center of symmetry (in this case point S). Note, that if subgraph H_2 is rotated through the angle π about S, the obtained embedding would be degenerate, with the vertices of the central regular decahedra of both subgraphs being identified. So instead we rotate the partial embedding of H_2 by an angle of measure $\pi+\omega$, where $\omega = \frac{k\pi}{180}$ for some $k \in [1, 18]$. The position of all vertices can now be expressed as a function of α, β and the parameter k. The length of the missing edges can be expressed as:

$$\begin{aligned} l_1 = &\left[c_2 \cos\left(\omega + \frac{3\pi}{10}\right) - \sin\left(\frac{16\pi}{30} + \beta - \omega\right) + \sin\alpha\right]^2 + \\ &+ \left[c_2 \sin\left(\omega + \frac{3\pi}{10}\right) - \cos\left(\frac{16\pi}{30} + \beta - \omega\right) - c_2 + \cos\alpha\right]^2 = 1, \\ l_2 = &\left[c_2 \cos\left(\omega + \frac{7\pi}{10}\right) + \cos\left(\frac{3\pi}{10} + \beta - \omega\right) + \sin\left(\alpha + \frac{\pi}{3}\right)\right]^2 + \\ &+ \left[c_2 \sin\left(\omega + \frac{7\pi}{10}\right) - \sin\left(\frac{3\pi}{10} + \beta - \omega\right) - c_2 + \cos\left(\alpha + \frac{\pi}{3}\right)\right]^2 = 1, \end{aligned}$$

where $c_2 = 1 + \frac{\sqrt{3}}{2} + \frac{\cos\left(\frac{\pi}{10}\right)}{2\sin\left(\frac{\pi}{10}\right)}$ (because of the rotational symmetry, we only need to express the length of two edges, all other missing edges will be of the same length).

Analogously as for the truncated icosahedron, it can be shown, using the implicit function theorem, that, for a fixed k, this system of equations will have a real-valued solution. For $k = 9$ we found an approximation of a solution using Maple; $\alpha_0 \doteq 5.852\,028\,177$ and $\beta_0 \doteq -7.571\,555\,037$. The unit-distance embedding is illustrated by Figure 9.

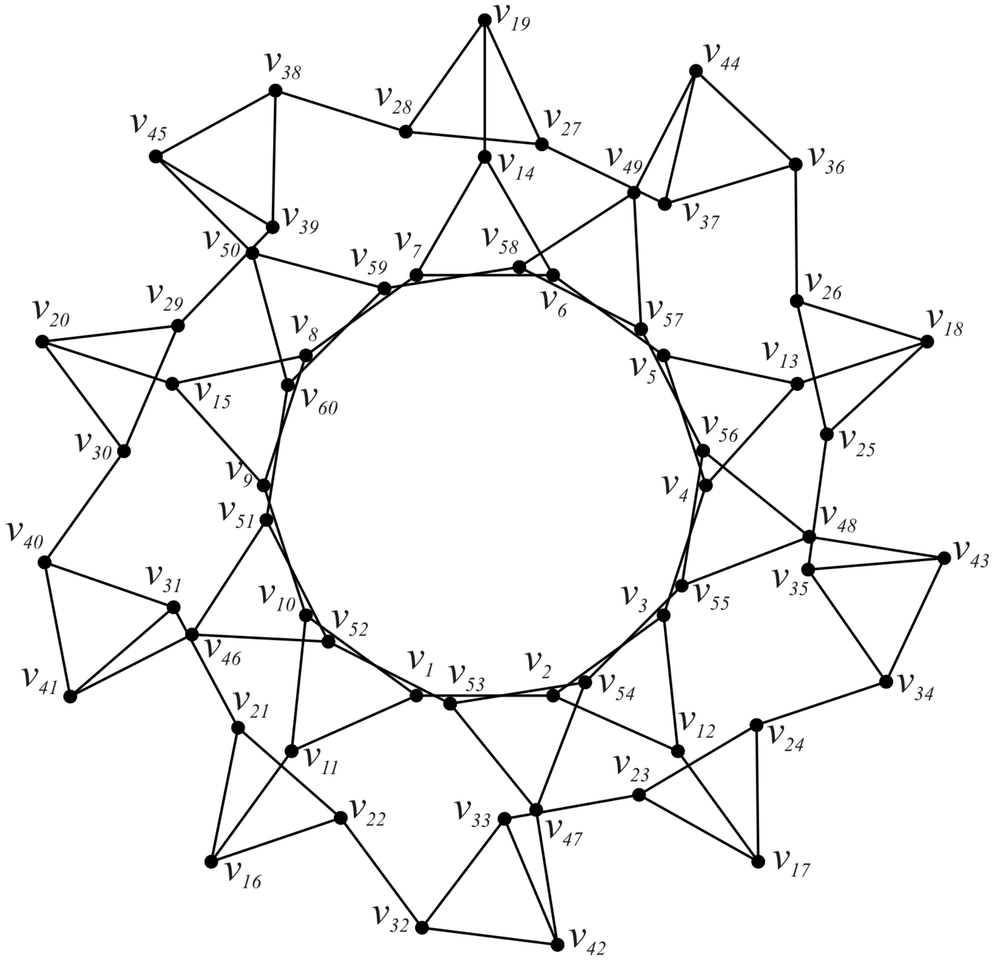

Fig. 9

4. SOLIDS WITH "BAD TRIANGLES"

The proofs of non-existence of a unit-distance drawing of graphs of snub solids and antiprisms are based on a simple counting argument which is illustrated for the snub cube graph: assuming that there exists its unit-distance drawing D, take a 4-cycle C

corresponding to a 4-face of the snub cube. Note that C is a rhombus and each 3-cycle of D is an equilateral triangle. Since every vertex x of C belongs to four 3-cycles whose edges do not cross, the angle of two sides of C incident with x is $\frac{2\pi}{3}$. Hence, the sum of inner angles of C is $\frac{8\pi}{3}$, a contradiction.

The proof for the snub dodecahedron is analogous. The core idea of the proof for the antiprisms is that, in their unit-distance drawings, all 3-cycles necessarily form a straight belt.

The nonexistence of a unit-distance drawing for the cuboctahedron graph (see Fig. 10) is proved by contradiction: consider a 4-cycle $v_1v_2v_3v_4$ of such a drawing (which forms a rhombus, see Fig. 11) and let μ be its inner angle at v_3; note that, for fixed v_3 and v_4, the coordinates of v_1 and v_2 can be expressed as a function of μ. The vertex v_5 lies on 3-cycle $v_1v_2v_5$, so there are just two possible positions for its location; denote them $v_{5,1}$ and $v_{5,2}$. The similar holds for vertices v_6, v_7 and v_8. As v_9 lies on the 4-cycle $v_1v_5v_9v_8$, its coordinates are uniquely determined by the positions of v_1, v_5, v_8. Each pair $(v_{5,i}, v_{8,j}), i, j \in [1, 2]$ determines the unique position for vertex v_9, so there are four different possibilities $v_{9,k}, k \in [1, 4]$. The same applies for vertices v_{10} (determined by v_5 and v_6), v_{11} (v_6 and v_7) and v_{12} (v_7 and v_8). The interesting thing is that the positions $v_{9,k}, k \in [1, 4]$ have the same coordinates (expressed as a function of μ) as the positions $v_{10,k}$, $v_{11,k}$ and $v_{12,k}$, $k \in [1, 4]$. So we have four positions for four vertices and we want to place all vertices into different places so we have to use all four positions. It is simple to calculate the distances between these positions and to check that four of the distances are equal to $\sqrt{3}$. But as there are four edges (namely, v_9v_{10}, $v_{10}v_{11}$, $v_{11}v_{12}$, v_9v_{12}) between the vertices in these positions, at most two of them can be of unit length. However, this is a contradiction with the assumption of the existence of a unit-distance cuboctahedron drawing in the plane.

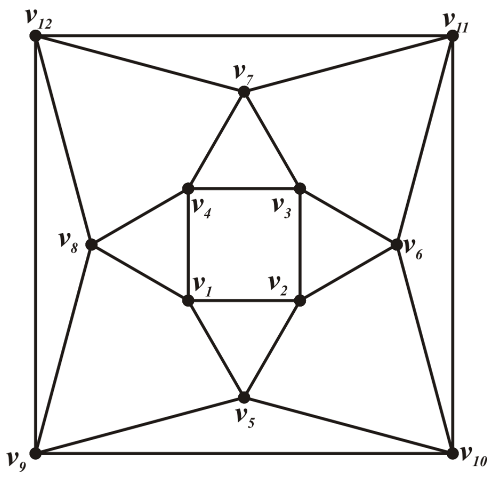

Fig. 10

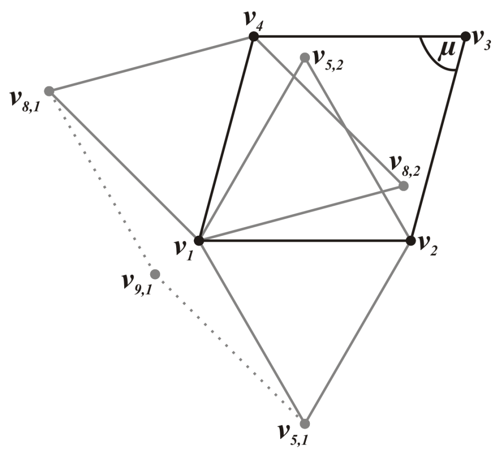

Fig. 11

5. THE REST

This group contains the truncated tetrahedron and the truncated octahedron (see Fig. 12 and 14). As there is no common approach, we deal separately with each of them.

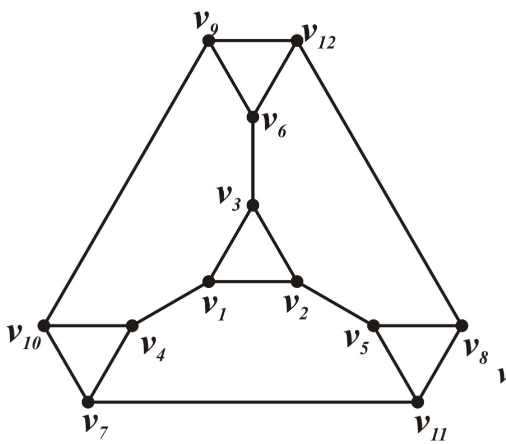

Fig. 12

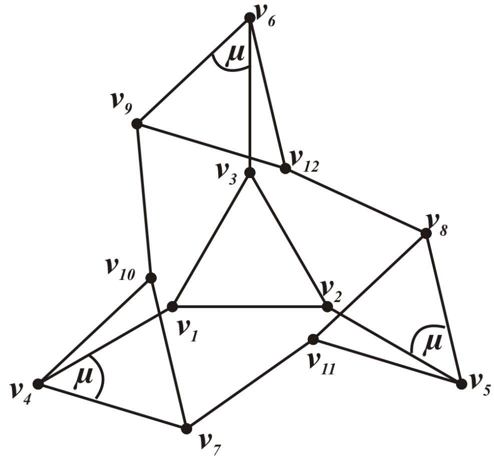

Fig. 13

The construction of a unit-distance drawing of truncated tetrahedron graph is illustrated on Figure 13. After fixing the equilateral triangle formed by v_1, v_2 and v_3, we choose the position of vertex v_4 in such a way that the line segment v_1v_4 lies on the axis of the outer angle at vertex v_1; the vertices v_5 and v_6 are placed analogously. Let μ be the measure of $\angle v_1v_4v_7$. Now, the position of vertex v_7 can be expressed as a function of μ. To keep a rotational symmetry in the construction, the measure of angles $\angle v_2v_5v_8$ and $\angle v_3v_6v_9$ is also μ. Now we just have to express the positions of vertices v_{10}, v_{11} and v_{12} as functions of μ; this is possible because each of them lies on an equilateral triangle. We have to note that for the simplicity of the construction, it is essential to keep the rotational symmetry at this point (we have two possible orientations for each of the triangles, so we choose one orientation for the first triangle and then choose the orientation of the remaining triangles accordingly). The symmetry ensures that the lengths of line segments v_7v_{11}, v_8v_{12} and v_9v_{10} are equal. The length of v_9v_{10} can be expressed as

$$|v_9v_{10}| = f(\mu) = \sqrt{\left(\frac{1+\sqrt{3}}{2} - 2\sin\mu\right)^2 + \left(\frac{3+\sqrt{3}}{2} - 2\cos\mu\right)^2}.$$

The function f is continuous and $f\left(\frac{\pi}{4}\right) < 1, f\left(\frac{\pi}{3}\right) > 1$; hence, by the intermediate value theorem, there exists $\mu^* \in \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$ such that $f(\mu^*) = 1$.

The unit-distance drawing for truncated octahedron graph is illustrated by Figure 15 (basically, it comes from certain plane projection of this solid); the coordinates of its vertices are listed in Table 1.

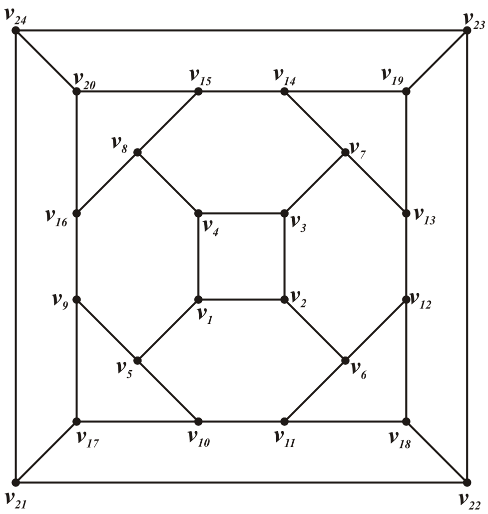

Fig. 14

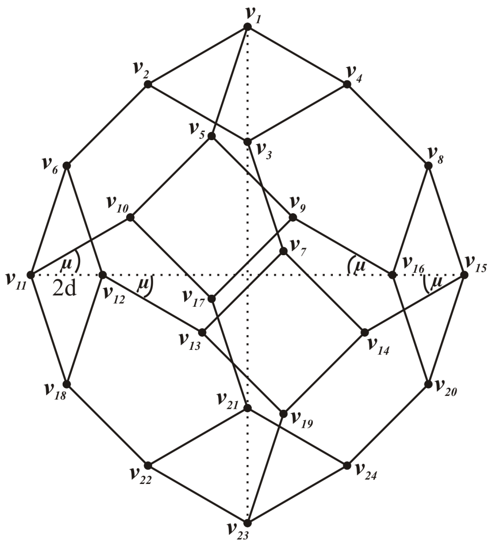

Fig. 15

Table 1

Vertex	$x-$coordinate	$y-$coordinate	Vertex	$x-$coordinate	$y-$coordinate
v_1	$\frac{\sqrt{3}+\sqrt{2}}{2}+\frac{5}{16}$	$\frac{\sqrt{2}+1}{2}+\frac{\sqrt{231}}{16}$	v_{13}	$\frac{5}{8}+\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
v_2	$\frac{\sqrt{2}}{2}+\frac{5}{16}$	$\frac{\sqrt{2}}{2}+\frac{\sqrt{231}}{16}$	v_{14}	$\frac{5}{8}+\frac{\sqrt{3}}{2}+\sqrt{2}$	$-\frac{1}{2}$
v_3	$\frac{\sqrt{3}+\sqrt{2}}{2}+\frac{5}{16}$	$\frac{\sqrt{2}-1}{2}+\frac{\sqrt{231}}{16}$	v_{15}	$\sqrt{2}+\sqrt{3}+\frac{5}{8}$	0
v_4	$\sqrt{3}+\frac{\sqrt{2}}{2}+\frac{5}{16}$	$\frac{\sqrt{2}}{2}+\frac{\sqrt{231}}{16}$	v_{16}	$\sqrt{2}+\sqrt{3}$	0
v_5	$\frac{\sqrt{3}+\sqrt{2}}{2}$	$\frac{\sqrt{2}+1}{2}$	v_{17}	$\frac{\sqrt{3}+\sqrt{2}}{2}$	$\frac{1-\sqrt{2}}{2}$
v_6	$\frac{5}{16}$	$\frac{\sqrt{231}}{16}$	v_{18}	$\frac{5}{16}$	$-\frac{\sqrt{231}}{16}$
v_7	$\frac{5}{8}+\frac{\sqrt{3}+\sqrt{2}}{2}$	$\frac{\sqrt{2}-1}{2}$	v_{19}	$\frac{5}{8}+\frac{\sqrt{3}+\sqrt{2}}{2}$	$-\frac{\sqrt{2}+1}{2}$
v_8	$\sqrt{3}+\sqrt{2}+\frac{5}{16}$	$\frac{\sqrt{231}}{16}$	v_{20}	$\sqrt{3}+\sqrt{2}+\frac{5}{16}$	$-\frac{\sqrt{231}}{16}$
v_9	$\frac{\sqrt{3}}{2}+\sqrt{2}$	$\frac{1}{2}$	v_{21}	$\frac{\sqrt{3}+\sqrt{2}}{2}+\frac{5}{16}$	$\frac{1-\sqrt{2}}{2}-\frac{\sqrt{231}}{16}$
v_{10}	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	v_{22}	$\frac{\sqrt{2}}{2}+\frac{5}{16}$	$-\frac{\sqrt{2}}{2}-\frac{\sqrt{231}}{16}$
v_{11}	0	0	v_{23}	$\frac{\sqrt{3}+\sqrt{2}}{2}+\frac{5}{16}$	$-\frac{1+\sqrt{2}}{2}-\frac{\sqrt{231}}{16}$
v_{12}	$\frac{5}{8}$	0	v_{24}	$\sqrt{3}+\frac{\sqrt{2}}{2}+\frac{5}{16}$	$-\frac{\sqrt{2}}{2}-\frac{\sqrt{231}}{16}$

6. OPEN PROBLEMS

As we stated at the beginning, our goal was to find the dimension of all the graphs of Archimedean solids. We were able to fulfill this to a good extend as we found the dimensions for all Archimedean solids with the exception of the rhombicosidodecahedron and the truncated icosidodecahedron. Nevertheless, we conjecture that their dimension is 2. This conjecture is supported partially by the fact that, for the truncated icosidodecahedron, we were able to find a degenerate unit-distance drawing (see Fig. 16); however, we have not managed to transform it to an non-degenerate unit-distance drawing using a kind of particular rotation symmetry as described in Section 3. If two vertices are identified, we use a double index on the vertex representing them, the lower left index being the index of one and the lower right index being the index of the other identified vertex (for example, vertex ${}_{62}v_{41}$ represents vertices v_{62} and v_{41}). Note also that, in this drawing, several edges of the original truncated icosidodecahedron graph are represented by the same line segment, for example, two edges which are incident with the vertex v_{61}.

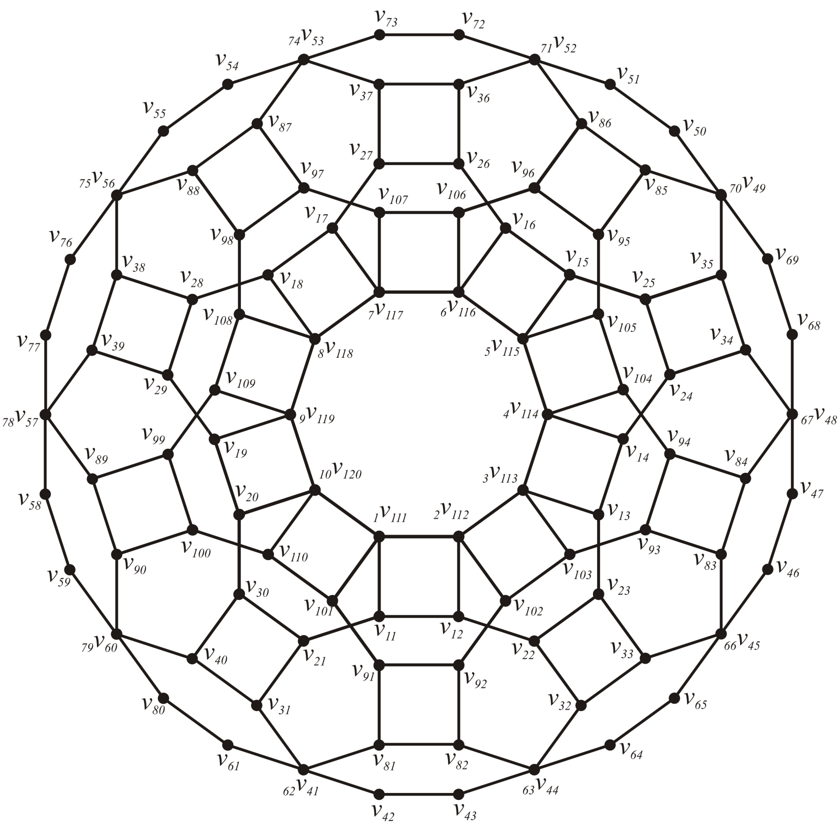

Fig. 16

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