

Lingfei YOU  
Jianguo ZHANG  
Qiao LI  
Nan YE

## STRUCTURAL RELIABILITY ANALYSIS BASED ON FUZZY RANDOM UNCERTAINTY

### ANALIZA NIEZAWODNOŚCI STRUKTURALNEJ W OPARCIU O ROZMYTĄ NIEPEWNOŚĆ LOSOWĄ

*To address the fuzzy random uncertainty in structural reliability analysis, a novel method for obtaining the membership function of fuzzy reliability is proposed on the two orders four central moments (TOFM) method based on envelope distribution. At each cut level, the envelope distribution is first constructed, which is a new expression to describe the bound of the fuzzy random variable distribution. The central moments of the bound distribution are determined by generating samples from the envelope distribution, and they are used to calculate the central moments of the limit state function based on the first two orders of the Taylor expansion. Thereafter, the modern approximation method is used to approximate the polynomial expression for the limit state function probability density function (PDF) by considering the central moments as constraint conditions. Thus, the reliability boundaries can be calculated under the considered cut level, and the membership function of the fuzzy reliability is subsequently obtained. Three examples are evaluated to demonstrate the efficiency and accuracy of the proposed method. Moreover, a comparison is made between the proposed method, Monte Carlo simulation (MCS) method, and fuzzy first-order reliability method (FFORM). The results show the superiority of the proposed method, which is feasible for the analysis of structural reliability with fuzzy randomness.*

**Keywords:** *fuzzy random uncertainty, approximation method, envelope distribution, structure, cut level.*

*W pracy przedstawiono metodę, która pozwala na uwzględnienie rozmytej niepewności losowej w strukturalnej analizie niezawodności. Zaproponowana metoda określania funkcji przynależności niezawodności rozmytej wykorzystuje cztery momenty centralne dwóch rzędów czy czwarte momenty centralne drugiego rzędu obliczane w oparciu o rozkład obwiedni. Dla każdego poziomu cięcia, najpierw konstruuje się rozkład prawdopodobieństwa obwiedni, za pomocą którego opisuje się granice rozkładu rozmytych zmiennych losowych, a momenty centralne rozkładu ograniczonego wyznacza się poprzez generowanie prób z rozkładu obwiedni. Następnie, stosując nowoczesną metodę optymalnej aproksymacji, otrzymuje się aproksymowane wyrażenie wielomianowe funkcji gęstości prawdopodobieństwa rozkładu obwiedni, gdzie momenty centralne stanowią warunki ograniczające, które pozwalają aproksymować niezawodność za pomocą rozwinięcia Taylora drugiego rzędu funkcji stanu granicznego. W ten sposób granice niezawodności oblicza się na rozważanym poziomie cięcia, a następnie otrzymuje się funkcję przynależności niezawodności rozmytej. W artykule przeanalizowano trzy przykłady, na podstawie których wykazano skuteczność i trafność proponowanej metody. Przeprowadzono także porównanie z metodą symulacji Monte Carlo oraz metodą analizy rozmytej niezawodności pierwszego rzędu. Wyniki wskazują na wyższość omawianej metody, która pozwala analizować niezawodność strukturalną w warunkach losowości rozmytej.*

**Słowa kluczowe:** *rozmyta niepewność losowa, metoda aproksymacji, rozkład obwiedni, struktura, poziom cięcia.*

#### 1. Introduction

There are several uncertainties with respect to the analysis of structural reliability, and the fluctuations due to the uncertainty have a significant influence on the performance of structure products, which increases the requirements of the uncertainty analysis method for achieving reliable structures.

Traditionally, uncertainty is classified into two major categories, namely, aleatory or epistemic. Aleatory uncertainties in reliability analysis have been successfully addressed using the probability theory, which requires completely statistical information based on probability distributions to describe the aleatory uncertainties [2,5,30]. The probabilistic reliability analysis methods with random variables include the moments method [3, 10, 40], response surface method [16, 17], Monte Carlo method [36], and direct integration method [38]. Although the probabilistic methods have been successfully applied, the quality of the input information should be statistically guaranteed

by a sufficiently large set of sample elements to verify the used distributions.

In contrast to aleatory uncertainties, epistemic uncertainties are knowledge-based and arise from imprecise modelling, simplification, and limited data availability [11]. There are several approaches for modelling epistemic uncertainties, such as the convex model method [9], possibility theory method [18], interval modelling [15, 27], evidence theory [1, 37], and uncertainty theory [20]. As their representative, the fuzzy sets theory is widely used for reliability analysis [7, 32, 39]. By the membership functions [28, 31], fuzzy reliability analysis can account for inaccuracies and uncertainty in data, which typically occurs when insufficient data is available to provide a useful statistical description.

However, with significant research on physical modeling and reliability analysis, it is found that aleatory and epistemic uncertainties do not exist alone, i.e., certain information, precise values, and completely obscure information do not exist. Thus, the concept of the

fuzzy random variable was proposed [14], where uncertain structural parameters governed by probability distributions with fuzzy parameters were introduced. Moreover, the fuzzy random variable reconciles aleatory and epistemic uncertainties, allowing an uncertain expression with random distribution and incomplete information to be constructed.

Willner [34] proposed an engineering concept to address fuzzy randomness. Möller et al. [25,26] presented a method for describing and predicting fuzzy time-series based on fuzzy random uncertainties. Liu et al. [21] used fuzzy random variables as basic variables to establish a relationship between fuzzy random variables, in addition to fuzzy random events. Körner [13] evaluated the properties of the variations in fuzzy random variables, and then applied to linear regression and limit theorems. Möller et al. [24] introduced a method for estimating the membership function of the safety index under the consideration of fuzzy randomness. A fuzzy first-order reliability method (FFORM) was developed using fuzzy random variables. Terán [33] presented probabilistic results toward a framework for modelling measurements based on fuzzy random variables. Wang et al. [35] solved the time dependent reliability problem for systems with fuzzy random uncertainties using saddle point approximation simulations. Koç et al. [12] used the theory of fuzzy random variables with fuzzy Monte Carlo simulations for reliability-based risk analysis of a rubble-mound breakwater. Shapiro [29] modelled the future lifetime as a fuzzy random variable, where the essential feature of the model was combined the stochastic component of mortality with the fuzzy component. In the study conducted by Jahani et al. [8], uncertain variables were modeled as fuzzy random variables. In addition, an interval Monte Carlo simulation (IMC) and the interval finite element method were used to evaluate the failure probability. Hryniewicz [6] presented a Bayesian approach to analysis the reliability under fuzzy random data. Li et al. [19] proposed a fuzzy reliability calculation method based on the error synthesis principle for fuzzy random uncertainty inputs.

The abovementioned methods can be divided into three categories: namely, iteration algorithms, sampling algorithms, and approximation algorithms. However, the application of fuzzy random uncertainties in addressing the reliability presents several problems when the abovementioned methods are used. With the combination of an iteration algorithm and traditional reliability algorithm, the calculation efficiency is not satisfied, and the accuracy is insufficient for high nonlinearity limit state functions. Moreover, sampling algorithms requires significant operations in the membership interval, for which the efficiency is insufficiently low for complex structures. For the application of approximation algorithms, it requires cumbersome transformations, which has a tremendous possibility of improvement.

Therefore, a novel structure reliability analysis method on TOFM based on envelope distribution is developed by combining the modern approximation algorithm, which considers the basic input variables as fuzzy random variables, and reliability analysis is expressed with respect to fuzzy numbers using the  $\alpha$  cut level approach. In this study, modern approximation algorithms such as the maximum entropy model [1] and optimal square approximation method [22,41] were used to approximate the fuzzy probability density function (FPDF) with fuzzy random variable inputs. Only the central moments are used in the approximation without considering the actual distribution. At each cut level, a new measure distribution named envelope distribution is used to establish an accurate description for the envelope of the fuzzy random distribution, which is the boundary of the distribution family of fuzzy random variables. In addition, the first four central moments of the envelope distribution are obtained using a statistical method and then the moments of limit state function are approximated based on envelope distribution moments according to the first two orders of magnitude of the Taylor expansion on limit state function. Thereafter, by considering the central moments as the

constrained conditions, the undetermined polynomial coefficients are fitted by employing the modern approximation method. Hence, the approximated polynomial expression of the limit state function PDF boundary is obtained. Thereafter, the boundary of the reliability membership function is calculated, and the fuzzy reliability is obtained by the application of the abovementioned operation at each cut level.

Compared with traditional methods, the proposed method can solve the drawback of high computation loads, poor accuracy, and instability due to fuzzy random uncertainties. It facilitates reliability analysis without iterative algorithms at each cut level, whereas the classical reliability analysis method requires computationally complex searches or optimization procedures. Furthermore, the proposed method only uses moments obtained from the statistical analysis of basic data, which is convenient for practical operations.

This article is structured as follows. Section 2 presents a brief introduction to the fuzzy random variable. In Section 3, the concept of moment generation based on the sampling of the envelope distribution is presented. Section 4 presents a discussion on fuzzy reliability, in addition to modern approximation algorithms using the central moments of envelope distribution. Finally, in Section 5, three examples are provided to illustrate the method.

## 2. Fuzzy Random Variable and Reliability

A fuzzy random variable  $\tilde{x}$  is a random variable for which its distribution parameters are fuzzy numbers.  $\tilde{x}$  can be defined on a fuzzy probability space  $\Omega, F, P$ , wherein  $\Omega$  is the space of the fuzzy random elementary events, and  $F$  and  $P$  are the subsets and fuzzy probability measure, respectively. A fuzzy random variable  $\tilde{x}$  defines a mapping relationship from  $(\Omega, \mu(\Omega))$  to  $(R^n, \mu(R^n))$ , i.e.,  $(\Omega, \mu(\Omega)) \rightarrow (R^n, \mu(R^n))$  [35][4], where  $\mu(\cdot)$  is the membership degree. Each fuzzy random variable  $\tilde{x}$  contains a basic realization random variable  $x$  as the initial of  $\tilde{x}$ . The  $\alpha$  cut level approach is used to conduct fuzzy arithmetic operations. Hence, the fuzzy probability cumulative distribution function (FPCDF) of a fuzzy random variable can be expressed as follows:

$$\tilde{F}(x) = \left\{ (F_\alpha(x), \mu(F_\alpha(x))) \mid F_\alpha(x) = [F_\alpha^L(x), F_\alpha^U(x)], \mu(F_\alpha(x)) = \alpha \forall \alpha \in (0,1] \right\} \quad (1)$$

where  $F_\alpha(x)$  is the FPCDF under the  $\alpha$  cut level, and  $F_\alpha^L(x)$  and  $F_\alpha^U(x)$  are the lower and upper bounds of  $F_\alpha(x)$ , respectively. There is a set of distributions under different membership degrees. Fig. 1 presents a fuzzy random variable  $\tilde{x}$  with FPCDF  $\tilde{F}(x)$  and FPDF  $\tilde{f}(x)$ . The dashed and solid lines indicate probability functions with fuzzy parameters that correspond to membership degrees of value 0 and 1, respectively.

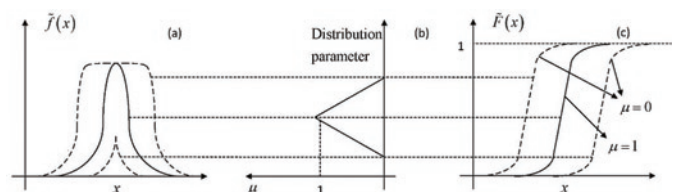


Fig. 1. (a) Fuzzy probability density function; (b) fuzzy parameter; (c) fuzzy probability distribution functions.

Fuzzy random reliability is based on the use of fuzzy random variables as the basic variables for the reliability problem, which is meas-

ured using the membership degree [34,23]. The limit state function  $y$  of the reliability model is defined as:

$$y = g(\tilde{\mathbf{X}}) = g(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n) \quad (2)$$

where  $\tilde{\mathbf{X}} = \{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n\}$  are  $n$  dimensional fuzzy random variables, which have FPDFs of  $f_\alpha(x_i)$  ( $i=1,2,\dots,n$ ) at the  $\alpha$  cut level. The fuzzy theory-based reliability defined as  $\tilde{R} = \left\{ \left\{ (P_r)_\alpha, \mu((P_r)_\alpha) \right\} \mid (P_r)_\alpha = \Pr\{g(\mathbf{X}) > 0\}, \mathbf{X} \sim f_\alpha(x_i) (i=1,2,\dots,n) \right\}$ . It represents the influence of fuzziness on reliability based on different membership levels. Once the basic variables are defined using fuzzy membership functions at various membership levels, the reliability interval at the  $\alpha$  cut level  $(P_r)_\alpha = \left[ (P_r)_\alpha^L, (P_r)_\alpha^U \right]$  can be obtained with respect to  $F_\alpha^L(x)$  and  $F_\alpha^U(x)$ , respectively;  $(P_r)_\alpha^L$  and  $(P_r)_\alpha^U$  are the lower and upper bounds of  $P_r$  at the  $\alpha$  cut level, respectively.

### 3. Moment Generation Based on Envelope Distribution

Moment generation based on sampling from the envelope distribution is presented in this section. The envelope distribution is an envelope line that consists of the upper and lower boundaries of the FPCDFs of fuzzy random variables. The objective of envelope distribution is to comprehensively describe the boundary distribution of  $\tilde{F}(x)$ , and the central moments can be obtained using a statistical method. This is used for reliability analysis, which will be discussed in the next section.

For convenience, the process of generating an envelope distribution is illustrated by assuming the basic realization of a fuzzy random variable as normal distribution, as well as the other distribution.  $\tilde{\mu}$  and  $\tilde{\sigma}$  are the fuzzy mean value and fuzzy standard deviation of a fuzzy random variable  $\tilde{x} = N(\tilde{\mu}, \tilde{\sigma})$ , respectively. All the membership functions are assumed to be fuzzy triangular number. Hence, the fuzzy mean and standard deviation can be expressed as  $\tilde{\mu} = \mu_{Low}; \mu_{Mid}; \mu_{Up}$  and  $\tilde{\sigma} = \sigma_{Low}; \sigma_{Mid}; \sigma_{Up}$ , respectively, where the subscripts Low, Mid, and Up are the lower bound, median bound, and upper bound, respectively (in the following, these labels will be written as superscripts once the cut level expression is introduced). A fuzzy random variable with  $\tilde{\mu} = \langle -0.5, 0, 0.5 \rangle$  and  $\tilde{\sigma} = \langle 0.9, 1, 1.1 \rangle$  is generated in MATLAB as an example, which is shown in Fig. 2, where the black line in Fig. 2(a) and the middle black line in Fig. 2(b) correspond to a membership degree of 1. According to the curves in Fig. 2(b), the boundary of the FPCDF is found to be an envelope of a set of curves. The upper and lower black lines in Fig. 2(c) are the envelope curves can be obtained by following operation: at each  $\alpha$  cut level, after the bound of the interval numbers,  $\left[ \mu_\alpha^L, \mu_\alpha^U \right]$  and  $\left[ \sigma_\alpha^L, \sigma_\alpha^U \right]$  are obtained, the upper bound PDF ( $F_\alpha^U(x)$ ) of the fuzzy random variables is constructed by sampling from  $x \sim N(\mu_\alpha^U, \sigma_\alpha^L)$  on the left side of  $\mu_\alpha^U$  and from  $x \sim N(\mu_\alpha^U, \sigma_\alpha^U)$  on the right side of  $\mu_\alpha^U$ . The set of sampling points is defined as  $\tilde{\mathbf{X}}_\alpha = \{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n\}$ . In contrast, the lower bound PDF ( $F_\alpha^L(x)$ ) is constructed by sampling from  $x \sim N(\mu_\alpha^L, \sigma_\alpha^U)$  on the left side of  $\mu_\alpha^L$ , and from  $x \sim N(\mu_\alpha^L, \sigma_\alpha^L)$  on the right side of  $\mu_\alpha^L$ . These sampling points are defined as  $\tilde{\mathbf{X}}_\alpha = \{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n\}$ . It should note that the envelope curve can be directly computed from CDFs if it could be expressed expediently, but in some cases the proposed generation method is really needed: 1. The expression of CDF is very complex, such as the marginal distribution under the joint distribution of polar diameter and polar angle in two-dimensional irregular walking issue. 2. Those truncated distributions that are hard to express CDF, which is applied widely in engineering

3. The distribution which central moments can not be expressed, e.g., a Cauchy distribution.

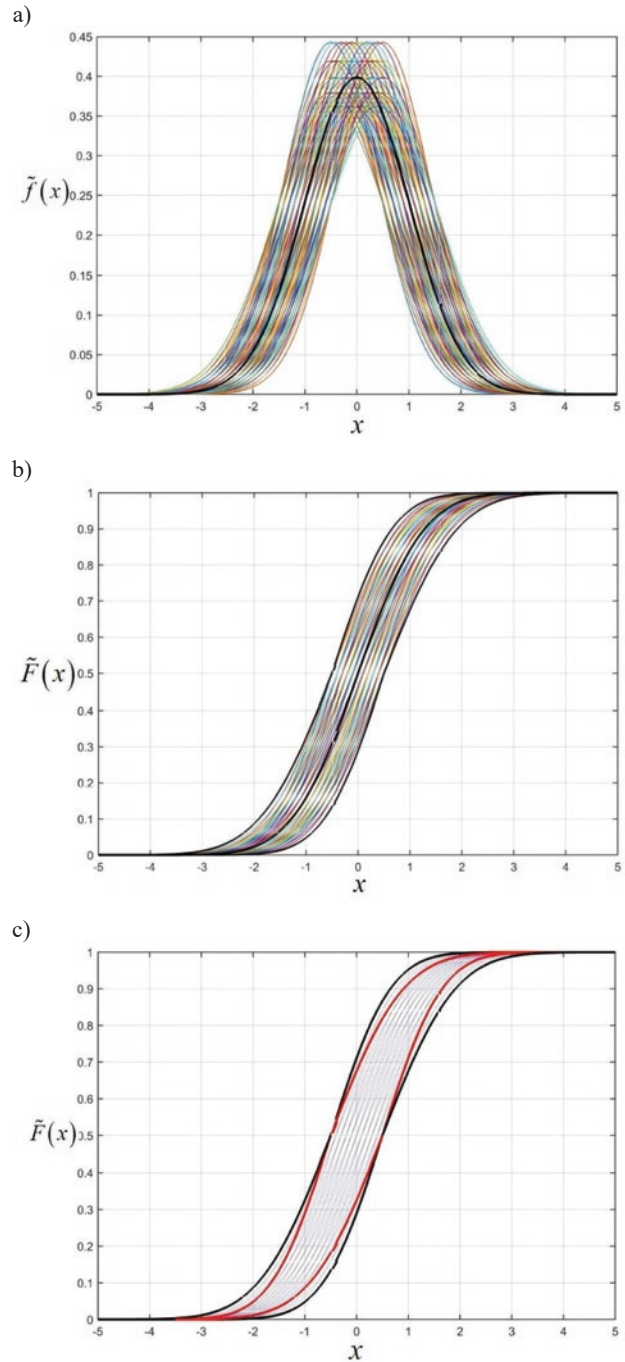


Fig. 2. (a) FPDF of fuzzy random variable; (b) FPCDF fuzzy random variable (c) envelope distribution

The envelope curve can encapsulate the boundaries of the distribution family. If only the upper and lower bound of the mean value are considered as sampling centers in the entire region, instead of separate on both sides of  $\mu_\alpha^U$  and  $\mu_\alpha^L$ , e.g.,  $x \sim N(\mu_\alpha^U, \sigma_\alpha^L)$  or  $x \sim N(\mu_\alpha^U, \sigma_\alpha^U)$ . This will produce inaccurate results, as indicated with the red line in Fig. 2(c).

The  $i$ th central moments of  $F_\alpha^U(x)$  and  $F_\alpha^L(x)$  are expressed as  $v_{x_\alpha^U}^i$  and  $v_{x_\alpha^L}^i$ , respectively, which used in next section. Based on  $\tilde{\mathbf{X}}_\alpha$  and  $\tilde{\mathbf{X}}_\alpha$ , they can be calculated using a statistical method or a

simple method for generating central moments, e.g., a universal generating function. Thereafter, the modern approximation method is employed to calculate the reliability interval at a given cut level.

**4. Modern approximation method based on central moments of the envelope distribution**

In this section, modern approximation methods that considers the central moments of the envelope distribution as constraint conditions is presented, which are used to approximate the fuzzy reliability  $(P_r)_\alpha = [(P_r)_\alpha^L, (P_r)_\alpha^U]$  composed of different  $\alpha$  cut levels. TOFM based on the envelope distribution can prevent large amount of iterations and complex transformations. As typical modern approximation algorithms, the maximum entropy model and optimal square approximation methods are employed in TOFM in this study. These methods are extensively used due to their satisfactory fitting effect and easy implementation. The limit state function is defined as  $Z = g(\tilde{\mathbf{X}})$ , where  $\tilde{\mathbf{X}}$  is the set of fuzzy random variables  $\tilde{\mathbf{X}} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ . Based on the FCDF bounds  $F_\alpha^U(\mathbf{X})$  and  $F_\alpha^L(\mathbf{X})$  of  $\tilde{\mathbf{X}}$ , the upper and lower bounds of  $Z_\alpha^U$  and  $Z_\alpha^L$  are approximated by the first two orders of magnitude of the Taylor expansion at the MPP (most probable point, i.e. the point of greatest contribution to failure probability)  $\mathbf{x}_\alpha^{*U}(\mathbf{x}_\alpha^{*L})$  as follows:

$$\begin{cases} Z_\alpha^U \approx g_{x_\alpha^U}(\mathbf{x}_\alpha^{*U}) + (\mathbf{X}_\alpha^U - \mathbf{x}_\alpha^{*U})^T \nabla g_{x_\alpha^U}(\mathbf{x}_\alpha^{*U}) + \frac{1}{2}(\mathbf{X}_\alpha^U - \mathbf{x}_\alpha^{*U})^T \nabla^2 g_{x_\alpha^U}(\mathbf{x}_\alpha^{*U})(\mathbf{X}_\alpha^U - \mathbf{x}_\alpha^{*U}) \\ Z_\alpha^L \approx g_{x_\alpha^L}(\mathbf{x}_\alpha^{*L}) + (\mathbf{X}_\alpha^L - \mathbf{x}_\alpha^{*L})^T \nabla g_{x_\alpha^L}(\mathbf{x}_\alpha^{*L}) + \frac{1}{2}(\mathbf{X}_\alpha^L - \mathbf{x}_\alpha^{*L})^T \nabla^2 g_{x_\alpha^L}(\mathbf{x}_\alpha^{*L})(\mathbf{X}_\alpha^L - \mathbf{x}_\alpha^{*L}) \end{cases} \quad (3)$$

where  $\mathbf{X}_\alpha^U = (x_{1\alpha}^U, x_{2\alpha}^U, \dots, x_{n\alpha}^U)$  and  $\mathbf{X}_\alpha^L = (x_{1\alpha}^L, x_{2\alpha}^L, \dots, x_{n\alpha}^L)$  are the variables that with distributions  $F_\alpha^U(\mathbf{X})$  and  $F_\alpha^L(\mathbf{X})$ , respectively.  $\nabla g_X(\cdot)$  is the partial derivative vector,  $\nabla^2 g_X(\cdot)$  is the Hessian matrix. For convenience, the proposed method will be illustrated with the upper bound PDF approximation in the following. A standard normally distributed random variable  $Z_\alpha^U$  can be normalized as  $Y_\alpha^U = (Z_\alpha^U - \mu_{Z_\alpha^U}) / \sigma_{Z_\alpha^U}$ , where  $\mu_{Z_\alpha^U}$  and  $\sigma_{Z_\alpha^U}$  are the mean and standard deviation of  $Z_\alpha^U$ , respectively. Hence, the  $i$ th central moments of the upper bound at the  $\alpha$  cut level can be calculated using  $v_{Y_\alpha^U}^i = E[(Y_\alpha^U)^i] = \int_{-\infty}^{+\infty} (y)^i f_{Y_\alpha^U}(y) dy$ , which is a function of  $v_{x_{j\alpha}^U}, i=0,1,2,3,4, j=1,2,\dots,n$  according to Eq. (3). In addition,  $v_{x_{j\alpha}^L}, i=0,1,2,3,4, j=1,2,\dots,n$  are generated from  $\bar{\mathbf{X}}_{j,\alpha}, j=1,2,\dots,n$  using the method discussed in Section 3. Thereafter, TOFM based on the envelope distribution can be implemented using modern approximation algorithms. To clearly demonstrate the proposed algorithm, the basic theories of the maximum entropy model and optimal square approximation method are briefly reviewed and combination with central moments are investigated in the following subsection.

**4.1. Maximum entropy model based on central moments**

Shannon entropy is a measure of the degree of uncertainty of an event prior to its occurrence. Moreover, it is a measurement of the amount of information obtained from the event after the event (information content). Under given conditions, there is a distribution of all possible probability distributions, which maximizes the information entropy. This is referred to as the Jaynes maximum entropy principle.

Under the constraint of known information, the information entropy is greatest, and the probability distribution is the least biased. The entropy of the continuous random variable  $x$  with PDF  $f(x)$  is defined as [16]:

$$H = -c \int_{-\infty}^{+\infty} f_X(x) \ln f_X(x) dx \quad (4)$$

where  $H$  is referred to as the Shannon entropy and  $c$  is Boltzmann's constant, which is greater than 0. Considering the central moments  $v_{Z_\alpha^U}^i (i=0,1,2,3,4)$  of the limit state function  $Z_\alpha^U$  as the constraint condition after normalization, the maximum entropy model of the upper bound of  $Z$  at the  $\alpha$  cut level can be expressed as follows:

$$\begin{cases} \max H = -c \int_{-\infty}^{+\infty} f_\alpha^U(z) \ln f_\alpha^U(z) dz \\ \text{s.t.} \quad E[(Y_\alpha^U)^i] = v_{Y_\alpha^U}^i, i=0,1,2,3,4 \end{cases} \quad (5)$$

The Lagrange multiplier method is therefore employed to solve the maximum entropy model, i.e.,  $L = H + \sum_{i=0}^4 \lambda_i [E[(Y_\alpha^U)^i] - v_{Y_\alpha^U}^i]$ . The undetermined constant is defined as  $a_0 = 1 - \frac{\lambda_0}{c}$ , where  $a_i = -\frac{\lambda_i}{c} (i=1,2,3,4)$ , and the approximate expression of the probability density function of the limit state function is:

$$f_{Y_\alpha^U}(y) = \exp\left(-\sum_{j=0}^4 a_j y^j\right) \quad (6)$$

On the other hand, the first four moments of the upper bound at the  $\alpha$  cut level are calculated from  $v_{x_{j\alpha}^U}, i=0,1,2,3,4, j=1,2,\dots,n$ , which are the moments of the envelope distribution as mentioned above. Substituting Eq. (6) into Eq. (5) yields Eq. (7):

$$\int_{-\infty}^{+\infty} y^i \exp\left(-\sum_{j=0}^4 a_j y^j\right) dy = v_{Y_\alpha^U}^i, i=0,1,2,3,4 \quad (7)$$

The polynomial fitting coefficients  $a_0, a_1, \dots, a_m$  of  $f_{Y_\alpha^U}(y)$  could then be determined.

**4.2. Optimal square approximation model based on central moments**

The theoretical basis of the optimal square approximation method is as follows. If the central moments of two random variables are equal at each order, they have the same probability distribution characteristics and eigenvalues. The undetermined coefficients of the PDF polynomials can be obtained by considering the central moments of each order as constraints in a given inner product space, thus determining the probability distribution [22, 41].

According to the above analysis, the FPDF bound  $(f_\alpha^U(z))$  must be approximated at the given cut level. The optimal square approximation model involved in fuzzy random variables can be expressed as follows:

$$\begin{cases} \min I = \int_a^b [p_\alpha^U(z) - f_\alpha^U(z)]^2 \rho(z) dz \\ \text{s.t.} \quad p_\alpha^U(z) = \sum_{i=0}^m \lambda_i p_{\alpha,i}^U(z) \end{cases} \quad (8)$$

where  $f_\alpha^U(z)$  is the upper bound PDF of the limit state function at the  $\alpha$  cut level,  $p_\alpha^U(z)$  is the approximate polynomial expression of  $f_\alpha^U(z)$ ,  $p_{\alpha,i}^U(z) (i=0,1,2,\dots,m)$  is a continuous function of  $m+1$  linearly independent functions based on the limit state function interval  $[a,b]$ ,  $\lambda_i (i=0,1,\dots,m)$  is the respective coefficient, and  $\rho(x)$  is the weight function of the power on interval  $[a,b]$ .

The necessary condition  $\frac{\partial I}{\partial \lambda_i} = 0$  for the extremum of a multivariate function can be used to determine the system of linear equations with coefficients  $\lambda_0, \lambda_1, \dots, \lambda_m$ , as follows:

$$\mathbf{A}\boldsymbol{\lambda} = \mathbf{B} \quad (9)$$

where the respective components of the matrix elements and vectors are:

$$A_{ij} = \int_a^b p_{\alpha,i}^U(z) p_{\alpha,j}^U(z) \rho(z) dz = 0, i, j = 0, 1, \dots, m \quad (10)$$

$$B_i = \int_a^b f_\alpha^U(z) p_{\alpha,i}^U(z) \rho(z) dz = 0, i = 0, 1, \dots, m \quad (11)$$

Given that  $p_{\alpha,0}^U(z), p_{\alpha,1}^U(z), \dots, p_{\alpha,m}^U(z)$  are linearly independent, and  $\mathbf{A}$  is an  $m+1$  order non-singular matrix, Eq. (9) has unique solutions. Let  $p_{\alpha,i}^U(z) = z^i, i=1,2,\dots,m$  and  $\rho(z)=1$ ;  $\mathbf{A}$  and  $\mathbf{B}$  can then be calculated using Eqs. (12) and (13), respectively:

$$A_{ij} = \frac{b^{i+j+1} - a^{i+j+1}}{i+j+1}, i, j = 0, 1, 2, \dots, m \quad (12)$$

$$B_i = v_{Y_\alpha^U}, i = 0, 1, 2, \dots, m \quad (13)$$

Based on the general case of the optimal square approximation method, the estimates of  $a$  and  $b$  are related to the skewness coefficients of  $Z_\alpha^U$ :

$$a = \begin{cases} -3.5, v_{Y_\alpha^U} \geq 0 \\ -4.0, v_{Y_\alpha^U} < 0 \end{cases}, b = \begin{cases} 3.5, v_{Y_\alpha^U} \leq 0 \\ 4.0, v_{Y_\alpha^U} > 0 \end{cases} \quad (14)$$

Thereafter,  $\lambda_i (i=0,1,\dots,m)$  can be solved using Eq. (9), yielding the approximate polynomial expression  $p_\alpha^U(z) = \sum_{i=0}^m \lambda_i z^i$ . TOFM can be used when  $m = 4$ .

### 4.3. TOFM based on envelope distribution using modern approximation method

As discussed above, based on the envelope distribution, which is a conservative description of fuzzy random variables, the fuzzy randomness problem is transformed into an approximate fitting problem

on the interval of cut level. By employing the modern approximation method in TOFM, the approximate polynomial expression of the FPDF of limit state function can be calculated.

It should be noted that due to the extension of the expansion in Eq. (3) to the second moment, only the first four moments of  $v_{Y_\alpha^U}, i=0,1,2,3,4$  can be expressed. If higher central moments required, Pearson family curves could be used to develop the relationship between each central moment of the family curves, as follows:

$$v_{Y_\alpha^U}^{i+1} = -\frac{k}{1+(k+2)c_2} \left[ c_0 v_{Y_\alpha^U}^{i-1} + c_1 v_{Y_\alpha^U}^i \right], i = 4, 5, \dots \quad (15)$$

where  $c_i, i=1,2,3$  are the Pearson family curve parameters, which can be expressed in terms of the first four moments. It should be noted that the used constraint conditions order number is dependent on the specific case, and the order increases lead to an increase in the calculation time consumption. After obtaining the polynomial expression of  $f_\alpha^U(z)$ , the upper bound of the fuzzy reliability probability at the  $\alpha$  cut level under different modern approximation method is:

$$(P_r)_\alpha^U \approx \Pr(Z_\alpha^U \geq 0) = \Pr\left(Y \geq -\frac{\mu_{Z_\alpha^U}}{\sigma_{Z_\alpha^U}}\right) = \begin{cases} \int_{-\infty}^{-\frac{\mu_{Z_\alpha^U}}{\sigma_{Z_\alpha^U}}} \sum_{i=0}^m \lambda_i y^i dy, \text{ optimal square approximation} \\ \int_{-\infty}^{-\frac{\mu_{Z_\alpha^U}}{\sigma_{Z_\alpha^U}}} \sum_{i=0}^m \exp\left(-\sum_{j=0}^m a_j y^j\right) dy, \text{ maximum entropy} \end{cases} \quad (16)$$

The lower bound of the reliability probability at the  $\alpha$  cut level  $R_\alpha^L$  is:

$$(P_r)_\alpha^L \approx \Pr(Z_\alpha^L \geq 0) = \Pr\left(Y \geq -\frac{\mu_{Z_\alpha^L}}{\sigma_{Z_\alpha^L}}\right) = \begin{cases} \int_{-\infty}^{-\frac{\mu_{Z_\alpha^L}}{\sigma_{Z_\alpha^L}}} \sum_{i=0}^m \lambda_i y^i dy, \text{ optimal square approximation} \\ \int_{-\infty}^{-\frac{\mu_{Z_\alpha^L}}{\sigma_{Z_\alpha^L}}} \sum_{i=0}^m \exp\left(-\sum_{j=0}^m a_j y^j\right) dy, \text{ maximum entropy} \end{cases} \quad (17)$$

Thus, the membership degree of reliability is obtained by performing the abovementioned process at each cut level. The procedure involved in the TOFM based on the envelope distribution method can be summarized as follows:

*Step 1.* The family distribution of the fuzzy random variables under the given cut level can be obtained according to the membership interval of the fuzzy random variables.

*Step 2.* The envelope distribution is constructed for each fuzzy random variable at each cut level using the method presented in Section 3.

*Step 3.* Based on the envelope distribution, the respective bound central moments  $v_{x_\alpha^U}, i=0,1,2,3,4$  and  $v_{x_\alpha^L}, i=0,1,2,3,4$  of  $f_\alpha^U(x)$  and  $f_\alpha^L(x)$  are obtained using a statistical method.

*Step 4.* The bounds of the limit state function  $Z = g(\tilde{\mathbf{X}})$  at the  $\alpha$  cut level  $Z_\alpha^U$  and  $Z_\alpha^L$  are approximated by the first two orders of the Taylor series expansion Each are normalized to  $Y_\alpha^U$  and  $Y_\alpha^L$ .

Step 5. The bound central moments  $v_{Y_\alpha^U, i}, i=0,1,2,3,4$  and  $v_{Y_\alpha^L, i}, i=0,1,2,3,4$  are calculated using Eq. (3), and the higher order moments are calculated using Eq. (15), if required.

Step 6. Employing the modern approximation method by considering the bound central moments as the constraint conditions. In particular, for the maximum entropy model, as mentioned in section 4.1, the approximate polynomial expression is obtained using Eq. (7). For the optimal square approximation method, as mentioned in Section 4.2, **A** and **B** are calculated using Eqs. (12) and (13), respectively. The fitting polynomial coefficients  $\lambda_j, j=0,1,2,\dots,m$  are then obtained using Eq. (9).

Step 7. The bounds of the reliability at the  $\alpha$  cut levels  $(P_r)_\alpha^U$  and  $(P_r)_\alpha^L$  are calculated using Eqs. (16) and (17), respectively. The steps above can be repeated at each  $\alpha$  cut level, and yielding the fuzzy reliability. Fig. 3 shows a flowchart of the proposed method.

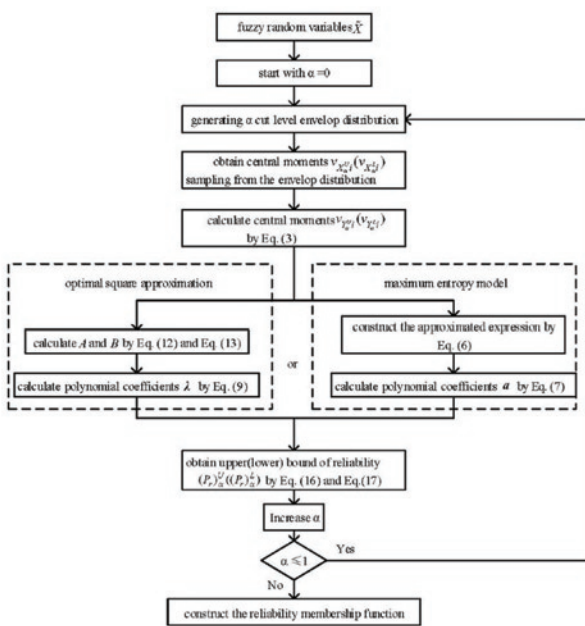


Fig. 3. Flowchart of TOFM based on the envelope distribution using the modern approximation method.

5. Examples

Three examples are presented to illustrate the proposed method. The first is a pure mathematical example. The second and third examples demonstrate the applicability of the proposed method in engineering, i.e., loads or materials that are considered with respect to fuzzy randomness uncertainty. Results from the MCS and FFORM methods are compared with those from the proposed method as these are classical approaches to fuzzy random uncertainties. The numerical results illustrate the superiority of the present approach in terms of efficiency and accuracy. The results contain sharp enclosures for all values of the reliability probability based on the proposed method compared with those obtained by MCS and FFORM approaches.

5.1. Investigation 1 (numerical)

It is assumed that the limit state function of the structure is  $Z=x_1 * x_2 - x_3 - 1200$ , where  $x_1 \sim N(\tilde{\mu}_{x_1}, \tilde{\sigma}_{x_1})$ ,  $x_2 \sim N(\tilde{\mu}_{x_2}, \tilde{\sigma}_{x_2})$ ,

and  $x_3 \sim N(\tilde{\mu}_{x_3}, \tilde{\sigma}_{x_3})$ . The basic realization of  $x_i, i=1,2,3$  are assumed as normal distribution. The mean  $\mu$  and standard deviation  $\sigma$  of the basic variables are considered as triangular fuzzy numbers:

$$\mu_{x_1} = 37.5;38;40 ; \sigma_{x_1} = 1.6;2;2.4$$

$$\mu_{x_2} = 53.5;54;56 ; \sigma_{x_2} = 3.6;4;4.4$$

$$\mu_{x_3} = 19.7;20;21 ; \sigma_{x_3} = 1;1.5;2$$

$\alpha$  discretization is used for mapping the fuzzy space to the interval random space. Moreover,  $\alpha$  is varied from 0 to 1, and the fuzzy numbers are evaluated at the following  $\alpha$ : 0.0, 0.2, 0.4, 0.6, 0.8, and 1.0. At each level, an interval is obtained for each distribution parameter, and the entire support domain of the problem is obtained for  $\alpha = 0$ . Permissible domains for the distribution parameters could be easily calculated for different values of  $\alpha$ . The results from the MCS and FFORM methods are compared in this example, as shown in Fig. 4.

For the MCS method, 64 combinations are used at each  $\alpha$  cut level ( $N(\bar{\mu}, \bar{\sigma}), \bar{\mu} = [\mu_{x_i}^{low}, \mu_{x_i}^{up}], \bar{\sigma} = [\sigma_{x_i}^{low}, \sigma_{x_i}^{up}], i=1,2,3$ ), and 1,000,000 analyses are required for each combination, thus the MCS method required  $64 \times 1,000,000 \times 6(384,000,000)$  runs. For the FFORM method, four iterations are performed for constructing the bound fuzzy reliability index, which indicates that the FFORM method required  $64 \times 4 \times 6 (1536)$  runs. In comparison, the proposed method required two repetitions of the process at the upper and lower boundaries. Moreover, the maximum error at a given  $\alpha$  cut level is found to be  $1.54 \times 10^{-2}$  at  $\alpha = 0$  of the lower distribution, as shown in Table 1. This error level is acceptable compared to the entire reliability membership function. The proposed method provides a clear improvement in the calculation efficiency, and the results obtained by the three methods are similar. In addition, 1000 samples are used to construct the envelope distribution, thus the proposed method required  $6 \times 1,000 \times 6 (36,000)$  sampling operations. A comparison of the computation time is shown in Table 1, which illustrate the great advantage of the conventional methods. The result of the proposed method is included in the MCS and FFORM methods as shown in Fig. 4, that's because the boundary extremum occurs when  $N(\mu_{x_i}^{low}, \mu_{x_i}^{up})$  and  $N(\mu_{x_i}^{up}, \mu_{x_i}^{low})$  are operated. This indicates that the proposed method has the effect of correcting and amplifying reliability when the extremum is conservative. This example demonstrates the superiority of TOFM based on the envelope distribution approach with respect to other approaches in the reliability assessment of structures. In the following two examples, the efficiency of the proposed method is illustrated based on evaluation.

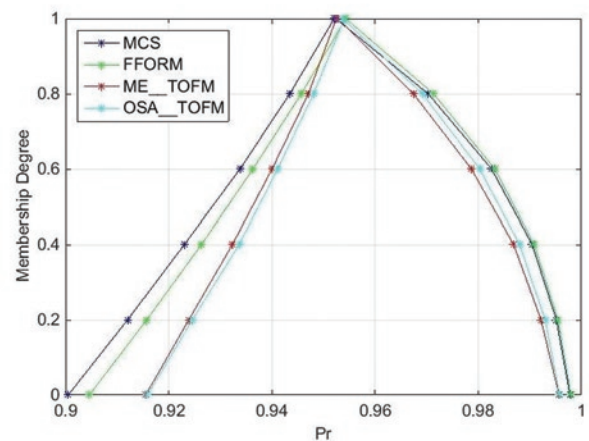


Fig. 4. Reliability membership function in Example 1

Table 1. Fuzzy reliability probability for the example 1

$\alpha$	ME_TOFM Reliability probability		OSA_TOFM Reliability probability		FFORM Reliability probability		MCS Reliability probability	
	$R_L$	$R_U$	$R_L$	$R_U$				
1.0	0.95247	0.95279	0.95405	0.95372	0.95433	0.95433	0.95208	0.95241
0.8	0.94707	0.96744	0.94807	0.96920	0.94570	0.97134	0.94347	0.97030
0.6	0.94004	0.97868	0.94111	0.98028	0.93634	0.98320	0.93382	0.98258
0.4	0.93233	0.98687	0.93363	0.98802	0.92633	0.99088	0.92301	0.99049
0.2	0.92389	0.99218	0.92455	0.99297	0.91573	0.99547	0.91201	0.99517
0.0	0.91553	0.99569	0.91591	0.99578	0.90460	0.99796	0.90051	0.99780
Computation time	123.98 s		115.55 s		151.41 s		188.73 s	

5.2. Investigation 2 (model)

A roof truss is presented in Fig. 5, for which the top boom and compression bars are reinforced with concrete, and the bottom boom and the tension bars are made of steel. This evaluation is conducted under the assumption that a uniformly distributed load  $q$  is applied to the roof truss, and that a uniformly distributed load can be transformed into the nodal load  $P = ql / 4$ . The perpendicular deflection  $\Delta_c$  of node  $C$  can be obtained through mechanical analysis, and it is a function of the basic variables, i.e.,  $\Delta_c = \frac{ql^2}{2} \left( \frac{3.81}{A_c E_c} + \frac{1.13}{A_s E_s} \right)$ , where  $A_c$ ,  $A_s$ ,  $E_c$ ,  $E_s$ , and  $l$  are the cross-sectional area, elastic modulus, length of concrete, and length of the steel bars. With respect to safety and applicability, the limit condition is that  $\Delta_c$  of node  $C$  could not exceed 3.1 cm, and the limit state function could be constructed using  $g = 0.031 - \Delta_c$ . The values of  $l$  and  $q$  are shown in Table 2.

Table 2. Variables in Example 2

Variable	Value
$l(m)$	12
$q(10^4 N)$	2

In this example, the basic realization of the fuzzy random variables  $A_c$ ,  $A_s$ ,  $E_c$ , and  $E_s$  are assumed as the normal distribution. The mean and standard deviation of the variables are considered as triangular fuzzy numbers:

$$\mu_{A_c} = 3.85; 4.0; 4.1 \times 10^{-2} m^2 ; \sigma_{A_c} = 0.29; 0.32; 0.35 \times 10^{-2} m^2$$

$$\mu_{A_s} = 0.99; 1.0; 1.01 \times 10^{-3} m^2 ; \sigma_{A_s} = 0.055; 0.06; 0.065 \times 10^{-3} m^2$$

$$\mu_{E_c} = 1.64; 1.67; 1.7 \times 10^{10} Pa ; \sigma_{E_c} = 0.13; 0.14; 0.18 \times 10^{10} Pa$$

$$\mu_{E_s} = 0.90; 0.91; 0.92 \times 10^{11} Pa ; \sigma_{E_s} = 0.037; 0.04; 0.045 \times 10^{11} Pa$$

$\alpha$  is varied from 0 to 1, and the fuzzy numbers are evaluated at the following  $\alpha$ : 0.0, 0.2, 0.4, 0.6, 0.8, and 1.0. The results from the MCS and FFORM methods are compared in this example.

The MCS method require 256 combinations at each cut level in this example with 1,000,000 runs for each combination, i.e., the MCS method required  $256 \times 1,000,000 \times 6$  (1,536,00,000) runs. For the FFORM method, six iterations are performed for each combination,

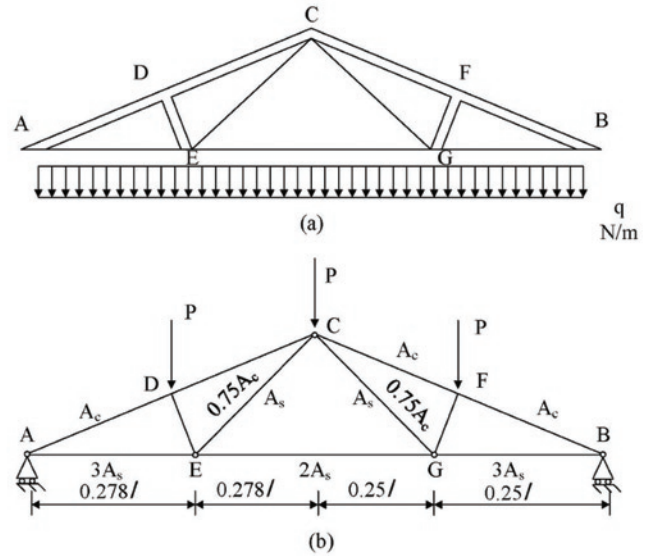


Fig. 5. Truss stress diagram

i.e., the FFORM method required  $256 \times 6 \times 6$  (9216) runs. In comparison, the proposed method requires two repetitions at the upper and lower boundaries. The maximum error for a given  $\alpha$  cut level is  $5.15 \times 10^{-3}$  at Level 0, as shown in Table 3, which illustrates the reliability probability at each level using the three methods. In addition, 1000 samples are used to construct the envelope distribution, thus the proposed method required  $8 \times 1,000 \times 6$  (48,000) sampling operations, illustrating its advantage over the MCS method. In this case, the computational efficiency of the TOFM based on the envelope distribution is evident in this example compared with the results from the MCS and FFORM methods. As the number of dimensions and nonlinearity increase, the advantage of this method is demonstrated, as showed by computation time in Table 3. In contrast with Example 1, the result of the MCS and FFORM methods are included in the proposed method, that's because the proposed method boundary extremum occurs when  $N(\mu_{x_i}^{low}, \mu_{x_i}^{low})$  and  $N(\mu_{x_i}^{up}, \mu_{x_i}^{up})$  are operated.

This indicates that the proposed method has the effect of correcting and diminishing reliability when the extremum occurs on the side combination instead of the cross combination of mean and standard deviation as shown in Example 1. The application of the proposed method in this paper to complex structures is presented below.

Table 3. Fuzzy reliability probability for Example 2

$\alpha$	ME_TOFM Reliability probability		OSA_TOFM Reliability probability		FFORM Reliability probability		MCS Reliability probability	
	$R_L$	$R_U$	$R_L$	$R_U$	$R_L$	$R_U$	$R_L$	$R_U$
1.0	0.99780	0.99785	0.99796	0.99789	0.99769	0.99769	0.99765	0.99784
0.8	0.99617	0.99887	0.99591	0.99927	0.99637	0.99859	0.99643	0.99866
0.6	0.99403	0.99941	0.99350	0.99992	0.99447	0.99918	0.99466	0.99919
0.4	0.99078	0.99971	0.99007	0.99999	0.99186	0.99954	0.99224	0.99954
0.2	0.98595	0.99986	0.98522	0.99970	0.98837	0.99976	0.98867	0.99975
0.0	0.97929	0.99994	0.97934	0.99969	0.98383	0.99988	0.98444	0.99987
Computation time	136.2 s		139.31 s		187.04 s		239.45 s	

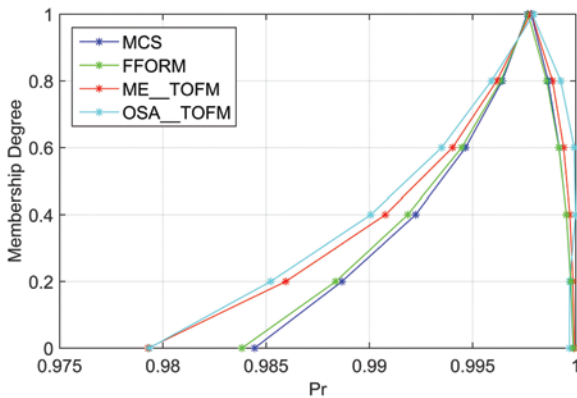


Fig. 6. Reliability membership function in Example 2.

5.3. Investigation 3 (model)

A truss member structural system is one of the most common structural forms in structural engineering. Fig. 7 shows the square grid structure of the square plate. The length of the upper chord plane is 5.0 m, the length of the lower chord plane is 4.0 m, the length of the string is 1.0 m, the height of the net frame is 0.7 m (the vertical distance between the upper and lower chords), the upper chord plane is hinged, and the lower chord are free. The bar is made of steel, and the mean values of the rod diameters are  $4.91 \times 10^{-4} m^2$  with density of  $7.8 \times 10^3 kg / m^3$ . In addition, the mean modulus of elasticity and Poisson's ratio are 207 GPa and 0.3, respectively. Three loads labelled P44, P49, and P54 (location node 44, 49 and 54) are considered as independent fuzzy random variables, as shown in Fig. 7. The loads are applied along the negative Z direction ( $\Delta_Z$ ). The serviceability limit state of the deflection is considered. The vertical deflection limit at any node is set as 4.57 cm. The limit state function could be constructed using  $g = 4.57 - (\Delta_Z)_{max}$ . The assumed diameter of the rods, modulus of elasticity, and Poisson's ratio are shown in Table 4.

The basic realization of the fuzzy random loads are assumed to be normal distribution. The mean and standard deviation are considered as triangular fuzzy numbers:

$$\mu_{P44} = 4800; 5000; 5200 N ; \sigma_{P44} = 386; 400; 404 N ;$$

Table 4. Variable value of Example 3

Variable	Value
Diameter of rods	$4.91 \times 10^{-4} m^2$
Modulus of elasticity	207 GPa
Poisson ratio	0.3

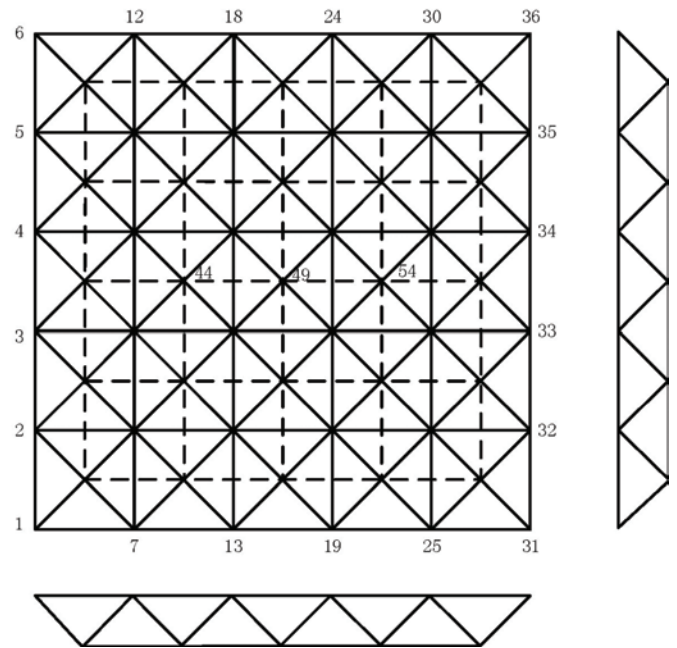


Fig. 7. Space-truss structure

$$\mu_{P49} = 9500; 10000; 10500 N ; \sigma_{P49} = 792; 800; 808 N ;$$

$$\mu_{P54} = 4800; 5000; 5200 N ; \sigma_{P54} = 386; 400; 404 N$$

$\alpha$  discretization is used to map this fuzzy space to the interval random space. Moreover,  $\alpha$  is varied from 0 to 1 in intervals of 0.2. The results from MCS and FFORM approach are compared in this example.

The negative direction of the Z axis  $\Delta_Z$  are calculated using finite element software ANSYS. Moreover, as a problem with implicit limit-state functions, multi-point approximations are constructed for the limit state, and the closed-form expressions could then be constructed to estimate the reliability bound. The Latin hypercube sampling technique is used to sample 35 design points in the abovementioned methods.

On this basis, the fuzzy reliability of the structure could be obtained using the proposed method, and the results for different cut levels are listed in Table 5. The maximum displacement along the Z axis is shown in Fig. 8. The results show that node 49 is the point where maximum displacement occurs. The reliability membership function is presented in Fig. 9.



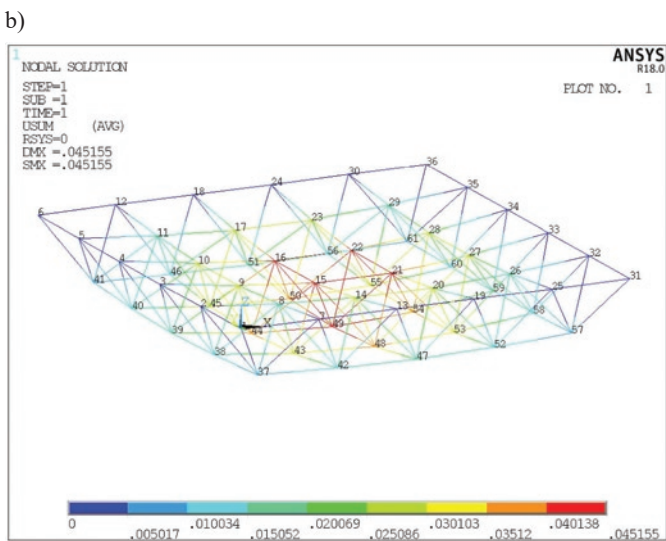
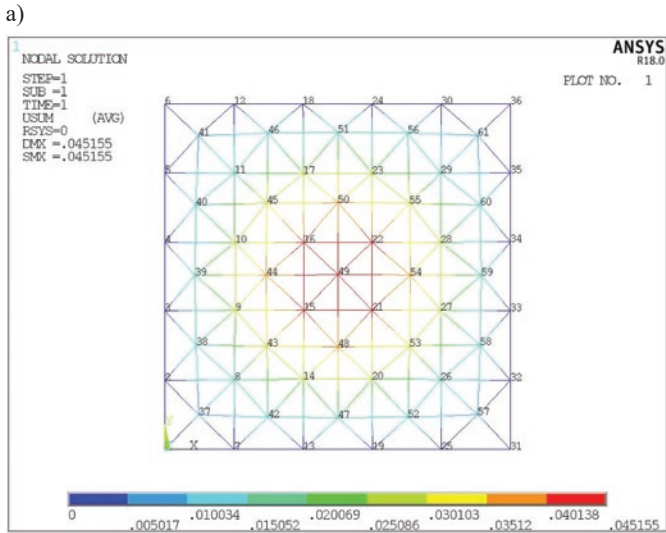


Fig. 8. Displacement diagram in ANSYS

Similarly, the MCS method had 64 combinations in this example, i.e., it required  $64 \times 1,000,000 \times 6$  (384,000,000) runs. For the FFORM method, seven iterations are used in the bound distribution, which required  $64 \times 7 \times 6$  (2688) runs. In comparison, the proposed method also requires two repetitions of the process at the upper and lower boundaries. Table 5 shows the reliability probability at each cut level. Moreover, 1000 samples are used to construct the envelope distribution. Therefore, the proposed method required  $6 \times 1,000 \times 6$  (36,000) samples. Table 5 shows a comparison of the computation

Table 5. Fuzzy reliability probability for the example 3

$\alpha$	ME_TOFM Reliability probability		OSA_TOFM Reliability probability		FFORM Reliability probability		MCS Reliability probability	
	$R_L$	$R_U$	$R_L$	$R_U$	$R_L$	$R_U$	$R_L$	$R_U$
	1.0	0.99630	0.99654	0.99647	0.99626	0.99788	0.99778	0.99772
0.8	0.99390	0.99837	0.99437	0.99814	0.99673	0.99873	0.9963	0.99877
0.6	0.99130	0.99921	0.99183	0.99904	0.99429	0.99929	0.99379	0.99927
0.4	0.98841	0.99959	0.98886	0.99964	0.99056	0.99962	0.99002	0.99961
0.2	0.98415	1.0	0.98468	1.0	0.98503	0.9998	0.98467	0.99981
0.0	0.97780	1.0	0.97854	1.0	0.97708	0.9999	0.97664	0.99990
Computation time	115.92 s		115.22 s		238.74 s		276.43 s	

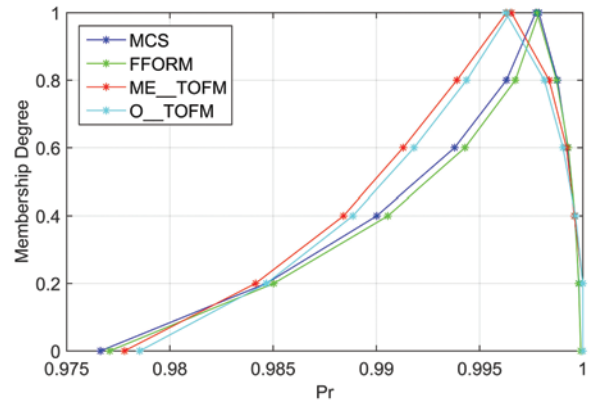


Fig. 9. Reliability membership function in Example 3

time with each method. Fig. 9 shows that the membership function of the proposed method exhibits conservative characteristics compared with the MCS method, which is more precise than the FFORM method due to the increased nonlinearity. There is a clear increase in efficiency that is significant considering the structural complexity.

### 6. Conclusion

In this study, a novel structural reliability analysis method with an uncertainty information model is applied to fuzzy random variables. The fuzzy reliability is calculated by using TOFM based on the envelope distribution. In the proposed method, based on the conservative characteristics of the bound distribution, the envelope distribution is used to describe the fuzzy random variables, which converts the fuzzy randomness into a probability problem. Hence, the bounds of the fuzzy reliability are calculated. Without the requirement of an iterative algorithm for calculating the reliability index  $\beta$ , the proposed method provides a significant advantage with respect to the simplification of the reliability calculation and the increased efficiency of the reliability analysis.

As illustrated in the examples, by combining with the modern approximation method, the proposed method only requires the central moments of each variable, which eliminates numerous iterative processes. Moreover, the calculation scale is considerably reduced compared with conventional reliability analysis methods, which significantly broadens its applicability. As the number of uncertainty variables increases, the efficiency of the proposed method is significant when the performance of the compared methods is unsatisfactory. The results show that the proposed method has the correction function. The fuzzy reliability can be appropriately increased or decreases according to the combination of mean and standard deviation when extreme value occurs.

The results of proposed method tend to be conservative, and they are suitable for engineering applications. In particular, the central moments of the envelope distribution appropriately describe the upper and lower bounds of the numerical characteristics of fuzzy random variables, which can be used as the input in reliability or other analyses. Furthermore, several aspects can also be evaluated, i.e., how to quickly select the most suitable value of  $m$  and accurately estimating  $a$  and  $b$ .

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**Lingfei YOU**

**Jianguo ZHANG**

**Qiao LI**

**Nan YE**

School of Reliability and Systems Engineering

Beihang University

XueYuan Road No.37,HaiDian District, BeiJing, China

E-mails: youlinfei@buaa.edu.cn, zjg@buaa.edu.cn,  
buaalqiao@gmail.com, yenan@buaa.edu.cn

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