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ON REFLECTIONLESS EQUI-TRANSMITTING MATRICES

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Abstract. Reflectionless equi-transmitting unitary matrices are studied in connection to matching conditions in quantum graphs. All possible such matrices of size 6 are described explicitly. It is shown that such matrices form 30 six-parameter families intersected along 12 five-parameter families closely connected to conference matrices.

Keywords: quantum graphs, vertex scattering matrix, equi-transmitting matrices.

Mathematics Subject Classification: 34L25, 81U40, 35P25, 81V99.

1. INTRODUCTION

Unitary and Hermitian matrices are used to describe different problems in physics and are well-studied in mathematics. It might appear that all relevant questions have beed studied and there is completely nothing to add, but modern fields of research pose new questions and may shed light on old ones. Our interest in such matrices is related to quantum graphs – differential operators on metric graphs – an area of modern mathematical physics expanding in recent years, see for example [1, 9]. In these models transmission of waves through the vertices is described by finite unitary matrices, in general energy dependent. The size of these matrices, to be denoted by v, coincides with the number of edges connected at each vertex. The vertex scattering matrices are energy independent if they are not only unitary but are Hermitian as well. This explains our interest in unitary Hermitian matrices. It is natural to ask the following question: *Which scattering matrices describe edge couplings where the edges are indistinguishable?* In quantum mechanics transition probabilities are given by the modulus squares of matrix elements. Therefore we shall be interested in matrices S possessing modular permutation symmetry

$$\begin{cases} |s_{ii}| = |s_{jj}|, & i,j = 1,\dots,v, \\ |s_{ij}| = |s_{lm}|, \quad i \neq j, l \neq m, & i,j,l,m = 1,\dots,v, \end{cases} \tag{1.1}$$

where v is the valency (degree) of the corresponding vertex, i.e. the size of S. In what follows matrices satisfying (1.1) will be called *equi-transmitting*. Interest in such matrices in connection to quantum graphs is explained by the fact that so called standard matching conditions (the function is continuous and the sum of normal derivatives is zero, also known as Kirchhoff or Neumann conditions), widely used in the area, exhibit a non-physical behavior, since the corresponding vertex scattering matrix has a dominated diagonal

$$S^{\mathrm{st}} = \begin{pmatrix} -1+2/v & 2/v & 2/v & \dots \\ 2/v & -1+2/v & 2/v & \dots \\ 2/v & 2/v & -1+2/v & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \underset{v\to\infty}{\longrightarrow} -I. \tag{1.2}$$

This fact was first pointed out in [6], where it was suggested to study reflectionless equi-transmitting matrices – equi-transmitting matrices with zero diagonal. Examples of such symmetric matrices were constructed. It was noticed that the problem to obtain such matrices is related to the classical (not yet completely solved) problem of describing all Hadamard matrices. Constructed examples determine vertex scattering matrices, which are reflectionless and equi-transmitting at a certain energy, but not for all values of the energy parameter. In order to obtain vertex scattering matrices which are reflectionless and equi-transmitting for all energies one should require that such matrices are also Hermitian (see Theorem 3.1). Therefore it appears natural to include Hermiticity requirement into the definition of reflectionless equi-transmitting matrices (see Definition 3.2).

General equi-transmitting matrices (without necessarily requiring zero diagonal) have been studied in a series of papers by Turek and Cheon [10, 11]. It was noticed that real reflectionless equi-transmitting matrices have already been studied under the name of *conference matrices*. These are matrices with zero diagonal and ± 1 outside the diagonal, such that all rows and columns are mutually orthogonal. The main focus of [11] is on the relation between the absolute values of the diagonal and non-diagonal elements in an equi-transmitting matrix.

The goal of our paper is to study reflectionless equi-transmitting matrices systematically. Trivially such matrices exists only in even dimensions (Theorem 3.1). The dimensions 2 and 4 can easily be studied leading to a one-parameter and two three-parameter families respectively. The first interesting dimension is 6. It appears that all reflectionless equi-transmitting matrices form 30 six-parameter families intersecting along 12 five-parameter families. The five parameters can be interpreted as magnetic potentials between different edges. Removing these parameters one gets 12 conference matrices of size 6. The structure of this family provides a beautiful example, which may be used to study such matrices in higher dimensions.

2. VERTEX SCATTERING MATRICES

It is sufficient when discussing the matching conditions for a quantum graph, to consider a star graph. By a star graph Γ, we mean the union of v semi-infinite intervals emenating from a vertex V, i.e. $\Gamma = \cup_{j=1}^{v} [0, \infty)$. An exemplary star graph is illustrated in Figure 1.

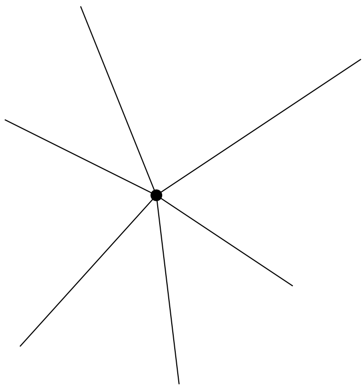

Fig. 1. Star graph with 6 semi-infinite edges

On the star graph Γ, consider the operator $L = -\frac{d^2}{dx^2}$ defined on the vector function $\vec{u}$ from the Sobolev space $W_2^2\left([0,\infty), \mathbb{C}^v\right)$ sastisfying the matching conditions

$$i\,(S - I)\,\vec{u} = (S + I)\,\partial_n \vec{u}, \tag{2.1}$$

where S is a unitary $v \times v$ matrix. Here the value of a function u and its normal derivative at the vertex V are denoted by $\vec{u} = \vec{u}\,(0)$ and $\partial_n \vec{u} = \vec{u}'\,(0)$ respectively. Let us calculate the corresponding vertex scattering matrix $S_V\,(k)$. Suppose the vectors representing the amplitudes of the incoming and outgoing waves are denoted by $\vec{b}$ and $\vec{a}$ respectively. The solution of the differential equation $-u'' = \lambda u$ can be written as $u\,(x) = \vec{b}e^{-ikx} + \vec{a}e^{ikx}$, $\vec{a}, \vec{b} \in \mathbb{C}^v$, $\lambda = k^2$. Then the values of the vectors $\vec{u}$ and $\partial_n \vec{u}$ at V are given by

$$\vec{u} = \vec{b} + \vec{a} \qquad \text{and} \qquad \partial_n \vec{u} = -ik\vec{b} + ik\vec{a}. \tag{2.2}$$

The vertex scattering matrix can be given in terms of the amplitutes of the incoming and outgoing waves and so take the form $\vec{a} = S_V\,(k)\,\vec{b}$. Substituting this into (2.2) we obtain

$$\vec{u} = \vec{b} + S_V\,(k)\,\vec{b} \qquad \text{and} \qquad \partial \vec{u} = -ik\vec{b} + ikS_V\,(k)\,\vec{b}.$$

With these values of $\vec{u}$ and $\partial_n \vec{u}$, (2.1) becomes

$$i\,(S - I)\,(I + S_V\,(k))\,\vec{b} = ik\,(S + I)\,(-I + S_V\,(k))\,\vec{b}.$$

From this we obtain an expression for the vertex scattering matrix which in general is energy dependent:

$$S_V(k) = \frac{(k+1)S + (k-1)I}{(k-1)S + (k+1)I}. \tag{2.3}$$

It is given in terms of the unitary matrix S used to parameterize the matching conditions. When $k = 1$, then $S_V(k)$ coincides with S.

Since S is unitary, it has the spectral representation given by

$$S = \sum_{n=1}^{v} e^{i\theta_n} \langle \vec{e}_n, \cdot \rangle \vec{e}_n,$$

where $\theta_n \in [-\pi, \pi]$, and $\vec{e}_n \in \mathbb{C}^v$ are the eigenvectors of S which are chosen orthonormal. Substituting this spectral representation into (2.3), we obtain

$$S_V(k) = \sum_{n=1}^{v} \frac{k\left(e^{i\theta_n} + 1\right) + \left(e^{i\theta_n} - 1\right)}{k\left(e^{i\theta_n} + 1\right) - \left(e^{i\theta_n} - 1\right)} \langle \vec{e}_n, \cdot \rangle \vec{e}_n. \tag{2.4}$$

The unitary matrix $S_V(k)$ has the same eigenvectors as the matrix S, but the eigenvalues in general depend on the energy. However eigenvalues ± 1 which coincide with the eigenvalues ± 1 of S do not depend on k. All other eigenvalues tend to 1 as $k \to \infty$. The high energy limit of $S_V(k)$ exists [4, 5, 8] and is given by

$$S_V(\infty) = \lim_{k\to\infty} S_V(k) = -P_{-1} + (I - P_{-1}) = I - 2P_{-1},$$

where P_{-1} is the spectral projection into the eigenspace of the eigenvlaue -1.

3. DEFINITION OF REFLECTIONLESS EQUI-TRANSMITTING MATRICES

Our aim is to study matrices S leading to so-called reflectionless equi-transmitting vertex scattering matrices. Such matrices have zero diagonal. The following theorem proves that all such matrices have to be not only unitary but also Hermitian.

Theorem 3.1. *Suppose S is a $v \times v$ unitary matrix and let $(S_V(k))_{jj} = 0$, $j = 1, \ldots, v$. Then v is even and S is Hermitian.*

Proof. Let the eigenvalues of S be denoted by λ_n and the corresponding eigenvectors by $\vec{e}_n = (z_1^n, z_2^n, \ldots, z_v^n)$. Then for $j = 1, \ldots, v$, we have

$$(S_V(k))_{jj} = \sum_{n=1}^{v} \frac{k(\lambda_n + 1) + (\lambda_n - 1)}{k(\lambda_n + 1) - (\lambda_n - 1)} \left|z_j^n\right|^2 = 0. \tag{3.1}$$

This gives us a $v \times v$ system of equations. For brevity let us introduce the notation $a_n = \lambda_n + 1$ and $b_n = \lambda_n - 1$. Then for $j = 1, \ldots, v$ the system of equations becomes

$$\sum_{n=1}^{v} \frac{ka_n + b_n}{ka_n - b_n} \left|z_j^n\right|^2 = 0.$$

Since the eigenvectors $\{\vec{e}_n\}$ form an orthonormal basis, we have $|z_1^n|^2+\ldots+|z_v^n|^2=1$, $n=1,\ldots,v$. So summing all the equations in the system we end up with the single equation

$$\sum_{n=1}^{v}\frac{ka_n+b_n}{ka_n-b_n}=0. \tag{3.2}$$

It follows that

$$\sum_{n=1}^{v}\left\{(ka_n+b_n)\prod_{\substack{m=1\\ m\neq n}}^{v}(ka_m-b_m)\right\}=0. \tag{3.3}$$

Assume first that v is *odd*. The function in the above equation can be seen as a polynomial in k and it is identically zero only if all coefficients at different powers are zero. In particular, the coefficient of k^v gives us

$$v\prod_{n=1}^{v}a_n=0. \tag{3.4}$$

Without loss of generality we take $a_v=0$ which yields $\lambda_v=-1$. Substituting this into equation (3.2), the last term in that sum equals -1. Hence we obtain

$$\sum_{n=1}^{v-1}\frac{ka_n+b_n}{ka_n-b_n}=1. \tag{3.5}$$

Simplifying this equation we have

$$\sum_{n=1}^{v-1}\left\{(ka_n+b_n)\prod_{\substack{m=1\\ m\neq n}}^{v-1}(ka_m-b_m)\right\}=\prod_{n=1}^{v-1}(ka_n-b_n). \tag{3.6}$$

Proceeding as before and comparing the coefficients of the highest power of k we see that

$$(v-1)\prod_{n=1}^{v-1}a_n=\prod_{n=1}^{v-1}a_n \qquad\Rightarrow\qquad (v-2)\prod_{n=1}^{v-1}a_n=0. \tag{3.7}$$

Since v is odd, $v-2\neq 0$ and so without loss of generality we take $a_{v-1}=0$ so that $\lambda_{v-1}=-1$. Continuing in a similar manner, we observe that at every step, the coefficient $(v-l)$ in the second equation in (3.7) is such that $l\in 2\mathbb{N}$. Therefore $v-l\neq 0$ for all applicable l. This has the consequence that $a_n=0$ for all $n=1,\ldots,v$ so that $\lambda_n=-1$ for all $n=1,\ldots,v$. But that is impossible since the trace of S is zero. Hence v is *even*.

For convenience of manipulation, we set $v=2N$, $N\in\mathbb{N}$. By the precceding calculations we see that $a_{2N}=a_{2N-1}=\ldots=a_{N+1}=0$ which implies that $\lambda_n=-1$, $n=N+1,\ldots,2N$. Taking into account that the trace of S is zero we get:

$$\mathrm{Tr} S = \sum_{n=1}^{2N} \lambda_n = \sum_{n=1}^{N} \lambda_n + (-1)\, N = 0 \quad \Rightarrow \quad \sum_{n=1}^{N} \lambda_n = N.$$

Since $|\lambda_n| = 1$ for all $n = 1, \dots, 2N$ we obtain that $\lambda_n = 1$, $n = 1, \dots, N$. Therefore all the eigenvalues of S are not only on the unit circle but are also real and precisely are -1 and 1. Hence S is also Hermitian. □

In [6] Harrison, Smilansky and Winn studied reflectionless equi-transmitting matrices. The definition used there does not require that the matrix be Hermitian. In view of Theorem 3.1 it is natural to require that reflectionless equi-transmitting matrices are not only unitary, but Hermitian as well. Such matrices lead to energy independent vertex scattering matrices as can be deduced from (2.4). We therefore give the following definition

Definition 3.2. An $n \times n$ matrix S is called *reflectionless equi-transmitting* (RET-matrix) if it is unitary Hermitian and it has zero diagonal while the off-diagonal elements have the same absolute value, i.e.

$$S = S^{-1} = S^*,$$

$$s_{jj} = 0, \quad j = 1, \dots, n,$$

$$|s_{ij}| = |s_{lm}|, \quad i \neq j, l \neq m, \quad i, j, l, m = 1, \dots, n.$$

4. REFLECTIONLESS EQUI-TRANSMITTING MATRICES OF SIZES TWO AND FOUR

For the sake of completeness we describe here all RET-matrices for $n = 2, 4$. It is clear that RET-matrices in dimension two have the form

$$S = \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix} = \mathrm{diag}\left\{1, e^{-i\theta}\right\} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathrm{diag}\left\{1, e^{i\theta}\right\}, \tag{4.1}$$

where $\theta \in [-\pi, \pi]$. We have just one one-parameter family.

Every RET-matrix in dimension four possesses the representation

$$S = \mathrm{diag}\left\{1, e^{-i\theta_1}, e^{-i\theta_2}, e^{-i\theta_3}\right\} \frac{1}{\sqrt{3}}\, C\, \mathrm{diag}\left\{1, e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}\right\}, \tag{4.2}$$

where $\theta_n \in [-\pi, \pi]$ and

$$C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a & b \\ 1 & \bar{a} & 0 & c \\ 1 & \bar{b} & \bar{c} & 0 \end{pmatrix}. \tag{4.3}$$

The numbers $a, b, c \in \mathbb{C}$ have absolute value one and are chosen such that the rows (and columns) are orthogonal. It should be observed that the rows (and columns) of C are already normalized. Using the orthogonality conditions of the rows of C and solving for a, b and c we obtain $a = \pm i$, $b = -a$ and $c = -\overline{a}$. From these values of the parameters we obtain the following two matrices, which are complex conjugate or transpose of each other:

$$C_1 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & i & -i \\ 1 & -i & 0 & i \\ 1 & i & -i & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -i & i \\ 1 & i & 0 & -i \\ 1 & -i & i & 0 \end{pmatrix}. \tag{4.4}$$

Hence the set of all RET-matrices in dimension 4 consists of two 3-parameter non-intersecting families.

5. REFLECTIONLESS EQUI-TRANSMITTING MATRICES OF SIZE SIX

To construct RET-matrices in dimension six we use the following lemma.

Lemma 5.1. *The sum of four complex numbers z_1, z_2, z_3, z_4 of equal magnitude equals zero if and only if at least one of the following cases occur:*

$$z_2 = -z_1, \; z_4 = -z_3,$$
$$or$$
$$z_3 = -z_1, \; z_4 = -z_2,$$
$$or$$
$$z_4 = -z_1, \; z_3 = -z_2.$$

To see that the lemma holds one only needs to observe that the sum of the four complex numbers $z_j, j = 1, 2, 3, 4$ which have the same absolute value equals zero if and only if they form the sides of a rhombus.

Similar to dimension four, equi-transmitting matrices in dimension six take the form

$$S = D^{-1} \frac{1}{\sqrt{5}} C \, D, \tag{5.1}$$

where $D = \text{diag}\left\{1, e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}, e^{i\theta_5}\right\}$, $\theta_n \in [0, 2\pi)$ and

$$C = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & b & c & d \\ 1 & \overline{a} & 0 & e & f & g \\ 1 & \overline{b} & \overline{e} & 0 & h & l \\ 1 & \overline{c} & \overline{f} & \overline{h} & 0 & m \\ 1 & \overline{d} & \overline{g} & \overline{l} & \overline{m} & 0 \end{pmatrix}. \tag{5.2}$$

The parameters $a, b, c, d, e, f, g, h, l, m \in \mathbb{C}$ have absolute value one and are chosen so that the rows (columns) of S are orthogonal. The orthogonality conditions yield the following 15 equations in 10 unknowns:

$$\begin{cases} a+b+c+d=0, & (1)\\ \overline{a}+e+f+g=0, & (2)\\ \overline{b}+\overline{e}+h+l=0, & (3)\\ \overline{c}+\overline{f}+\overline{h}+m=0, & (4)\\ \overline{d}+\overline{g}+\overline{l}+\overline{m}=0, & (5)\end{cases} \qquad \begin{cases} 1+b\overline{e}+c\overline{f}+d\overline{g}=0, & (6)\\ 1+ae+c\overline{h}+d\overline{l}=0, & (7)\\ 1+af+bh+d\overline{m}=0, & (8)\\ 1+ag+bl+cm=0, & (9)\end{cases} \tag{5.3}$$

$$\begin{cases} 1+\overline{a}b+f\overline{h}+g\overline{l}=0, & (10)\\ 1+\overline{a}c+eh+g\overline{m}=0, & (11)\\ 1+\overline{a}d+el+fm=0, & (12)\end{cases} \qquad \begin{cases} 1+\overline{b}c+\overline{e}f+l\overline{m}=0, & (13)\\ 1+\overline{b}d+\overline{e}g+hm=0, & (14)\end{cases} \quad \left\{ 1+\overline{c}d+\overline{f}g+\overline{h}l=0. \quad (15)\right.$$

Below we give just two examples illustrating how all RET-matrices of size 6 can be obtained.

Example 5.2 (Case 1.1). We recall that the parameters in the matrix C are complex numbers of absolute value one. Therefore for any of these parameters x we have that $x\bar{x} = |x|^2 = 1$. Applying Lemma 5.1 to equation (1) in the system (5.3) we end up with the following three cases:

$$\begin{aligned} &\text{Case 1}: \quad b=-a, \quad d=-c,\\ &\text{Case 2}: \quad c=-a, \quad d=-b,\\ &\text{Case 3}: \quad d=-a, \quad c=-b. \end{aligned} \tag{5.4}$$

Let us consider the first possibility (cases 2 and 3 can be treated in a similar way). Substituting the values of b and d into (5.3), we obtain the following system:

$$\begin{cases} \overline{a}+e+f+g=0, & (2)\\ -\overline{a}+\overline{e}+h+l=0, & (3)\\ \overline{c}+\overline{f}+\overline{h}+m=0, & (4)\\ -\overline{c}+\overline{g}+\overline{l}+\overline{m}=0, & (5)\end{cases} \qquad \begin{cases} 1-a\overline{e}+c\overline{f}-c\overline{g}=0, & (6)\\ 1+ae+c\overline{h}-c\overline{l}=0, & (7)\\ 1+af-ah-c\overline{m}=0, & (8)\\ 1+ag-al+cm=0, & (9)\end{cases} \tag{5.5}$$

$$\begin{cases} f\overline{h}+g\overline{l}=0, & (10)\\ 1+\overline{a}c+eh+g\overline{m}=0, & (11)\\ 1-\overline{a}c+el+fm=0, & (12)\end{cases} \qquad \begin{cases} 1-\overline{a}c+\overline{e}f+l\overline{m}=0, & (13)\\ 1+\overline{a}c+\overline{e}g+hm=0, & (14)\end{cases} \quad \left\{ \overline{f}g+\overline{h}l=0. \quad (15)\right.$$

Equation (15) can be eliminated since it is a multiple of equation (10). Application of Lemma 5.1 to equation (2) yields the following cases:

$$\begin{aligned}
&\text{Case } 1.1: \quad e = -\bar{a}, \quad g = -f,\\
&\text{Case } 1.2: \quad f = -\bar{a}, \quad g = -e,\\
&\text{Case } 1.3: \quad g = -\bar{a}, \quad f = -e.
\end{aligned} \tag{5.6}$$

We pick the first case again, and substitute the corresponding values of e and g into (5.5) and obtain the following system:

$$\begin{cases} -\overline{a} - a + h + l = 0, & (3)\\ \overline{c} + \overline{f} + \overline{h} + m = 0, & (4)\\ -\overline{c} - \overline{f} + \overline{l} + \overline{m} = 0, & (5) \end{cases} \qquad \begin{cases} 1 + a^2 + c\overline{f} + c\overline{f} = 0, & (6)\\ c\overline{h} - c\overline{l} = 0, & (7)\\ 1 + af - ah - c\overline{m} = 0, & (8)\\ 1 - af - al + cm = 0, & (9) \end{cases} \tag{5.7}$$

$$\begin{cases} f\overline{h} - f\overline{l} = 0, & (10)\\ 1 + \overline{a}c - \overline{a}h - f\overline{m} = 0, & (11)\\ 1 - \overline{a}c - \overline{a}l + fm = 0, & (12) \end{cases} \qquad \begin{cases} 1 - \overline{a}c - af + l\overline{m} = 0, & (13)\\ 1 + \overline{a}c + af + hm = 0. & (14) \end{cases}$$

From equation (7) in the above system we have that $l = h$. Substituting this value of l into equation (3) of the same system, we have that $\Re(a) = h$. This means that $h \in \mathbb{R}$ and since it has absolute value one it follows that $h = \pm 1$. This in turn implies that $l = \pm 1$ and $a = \pm 1$. Adding equations (4) and (5) and then substituting the values of l and h, we obtain $m = \mp 1$. Substituting $a = \pm 1$ into equation (6) yields $c\overline{f} = -1 \Rightarrow f = -c$. The parameter c can be chosen arbitrary leading to the following matrices:

$$C_{1.1} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1\\ 1 & 0 & \pm 1 & \mp 1 & c & -c\\ 1 & \pm 1 & 0 & \mp 1 & -c & c\\ 1 & \mp 1 & \mp 1 & 0 & \pm 1 & \pm\\ 1 & \overline{c} & -\overline{c} & \pm 1 & 0 & \mp 1\\ 1 & -\overline{c} & \overline{c} & \pm 1 & \mp 1 & 0 \end{pmatrix}. \tag{5.8}$$

□

Remark 5.3. The case we have considered will be numbered **Case 1.1**. This means that we have applied Lemma 5.1 twice and that in each case we choose the first option. The first application is to equation (1) of the initial system which yields three possibilities. We choose the first option from the three possibilities with the corresponding substitution giving us the second system. This system now has at most fourteen equations, equation (1) having been eliminated by the above substitution. The second application of Lemma 5.1 is now to eqution (2) of the second system which also yields three possibilities. The second "1" in the notation means that we choose the first option again from the second three possibilities in order to determine the matrix.

Example 5.4 (Case 3.1.1). Suppose that now we consider the third case in (5.4). Substituting the values of $d = -a$ and $c = -b$ into the system (5.3) we obtain the following system:

$$\begin{cases} \overline{a} + e + f + g = 0, & (2) \\ \overline{b} + \overline{e} + h + l = 0, & (3) \\ -\overline{b} + \overline{f} + \overline{h} + m = 0, & (4) \\ -\overline{a} + \overline{g} + \overline{l} + \overline{m} = 0, & (5) \end{cases} \qquad \begin{cases} 1 + b\overline{e} - b\overline{f} - a\overline{g} = 0, & (6) \\ 1 + ae - b\overline{h} - a\overline{l} = 0, & (7) \\ 1 + af + bh - a\overline{m} = 0, & (8) \\ 1 + ag + bl - bm = 0, & (9) \end{cases}$$
$$\begin{cases} 1 + \overline{a}b + f\overline{h} + g\overline{l} = 0, & (10) \\ 1 - \overline{a}b + eh + g\overline{m} = 0, & (11) \\ el + fm = 0, & (12) \end{cases} \qquad \begin{cases} \overline{e}f + l\overline{m} = 0, & (13) \\ 1 - a\overline{b} + \overline{e}g + hm = 0, & (14) \end{cases} \quad \begin{cases} 1 + a\overline{b} + \overline{f}g + \overline{h}l = 0. & (15) \end{cases} \tag{5.9}$$

From this system, we see that equation (13) is a multiple of equation (12) and so can be discarded. Applying Lemma 5.1 to equation (2) of the system (5.9) we obtain the following three cases:

$$\begin{aligned} &\text{Case 3.1}: && e = -\bar{a}, && g = -f, \\ &\text{Case 3.2}: && f = -\bar{a}, && g = -e, \\ &\text{Case 3.3}: && g = -\bar{a}, && f = -e. \end{aligned}$$

We pick the case 3.1 and substitute the corresponding values of e and g into the system (5.9) and thus obtain the following system:

$$\begin{cases} \overline{b} - a + h + l = 0, & (3) \\ -\overline{b} + \overline{f} + \overline{h} + m = 0, & (4) \\ -\overline{a} - \overline{f} + \overline{l} + \overline{m} = 0, & (5) \end{cases} \qquad \begin{cases} 1 - ab - b\overline{f} + a\overline{f} = 0, & (6) \\ -b\overline{h} - a\overline{l} = 0, & (7) \\ 1 + af + bh - a\overline{m} = 0, & (8) \\ 1 - af + bl - bm = 0, & (9) \end{cases}$$
$$\begin{cases} 1 + \overline{a}b + f\overline{h} - f\overline{l} = 0, & (10) \\ 1 - \overline{a}b - \overline{a}h - f\overline{m} = 0, & (11) \\ -\overline{a}l + fm = 0, & (12) \end{cases} \qquad \begin{cases} 1 - a\overline{b} + af + hm = 0, & (14) \\ a\overline{b} + \overline{h}l = 0. & (15) \end{cases} \tag{5.10}$$

We discard equation (4.1) because it is a multiple of equation (7). Here we need to apply Lemma 5.1 once more, this time to equation (2.2) of the system (5.10).

$$\begin{aligned} &\text{Case 3.1.1}: && b = \bar{a}, && l = -h, \\ &\text{Case 3.1.2}: && h = a, && l = -\bar{b}, \\ &\text{Case 3.1.3}: && l = a, && h = -\bar{b}. \end{aligned} \tag{5.11}$$

We then pick the first case. Substituting the values of b and l in the case 3.1.1 into the system (5.10) gives us the following system:

$$\begin{cases} -a+\overline{f}+\overline{h}+m=0, & (4)\\ -\overline{a}-\overline{f}-\overline{h}+\overline{m}=0, & (5)\end{cases} \qquad \begin{cases} -\overline{a}\overline{f}+a\overline{f}=0, & (6)\\ -\overline{a}\overline{h}+a\overline{h}=0, & (7)\\ 1+af+\overline{a}h-a\overline{m}=0, & (8)\\ 1-af-\overline{a}h-\overline{a}m=0, & (9)\end{cases} \tag{5.12}$$

$$\begin{cases} 1+\overline{a}^2+f\overline{h}+f\overline{h}=0, & (10)\\ 1-\overline{a}^2-\overline{a}h-f\overline{m}=0, & (11)\\ \overline{a}h+fm=0, & (12)\end{cases} \qquad \begin{cases} 1-a^2+af+hm=0. & (14)\end{cases}$$

From equation (7) we see that $a\in\mathbb{R}$, $\Rightarrow a=\pm1$. Adding equations (4) and (5) and then substituting the values of a we have that $\Re(m)=\pm1 \Rightarrow m=\pm1$. Substituting the values of a into equation (10) we obtain that $f\overline{h}=-1$, which implies that $h=-f$. From these we see that the corresponding matrix is

$$C_{3.1.1}=\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1\\ 1 & 0 & \pm1 & \pm & \mp1 & \mp1\\ 1 & \pm1 & 0 & \mp1 & f & -f\\ 1 & \pm1 & \mp1 & 0 & -f & f\\ 1 & \mp1 & \overline{f} & -\overline{f} & 0 & \pm1\\ 1 & \mp1 & -\overline{f} & \overline{f} & \pm1 & 0 \end{pmatrix}. \tag{5.13}$$

□

Continuing as in the above examples in all cases we obtain 30 (different) one-parameter families of matrices. A summary of all the cases can be seen in table 1. Below we list all the 30 different parameter dependent matrices where the subscript denotes the row (and column) which does not contain a parameter.

$$A_2=\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1\\ 1 & 0 & \pm1 & \pm1 & \mp1 & \mp1\\ 1 & \pm1 & 0 & \mp1 & \alpha & -\alpha\\ 1 & \pm1 & \mp1 & 0 & -\alpha & \alpha\\ 1 & \mp1 & \overline{\alpha} & -\overline{\alpha} & 0 & \pm1\\ 1 & \mp1 & -\overline{\alpha} & \overline{\alpha} & \pm1 & 0 \end{pmatrix}, B_2=\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1\\ 1 & 0 & \pm1 & \mp1 & \pm1 & \mp1\\ 1 & \pm1 & 0 & \beta & \mp1 & -\beta\\ 1 & \mp1 & \overline{\beta} & 0 & -\overline{\beta} & \pm1\\ 1 & \pm1 & \mp1 & -\beta & 0 & \beta\\ 1 & \mp1 & -\overline{\beta} & \pm1 & \overline{\beta} & 0 \end{pmatrix},$$

$$C_2=\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1\\ 1 & 0 & \pm1 & \mp1 & \mp1 & \pm1\\ 1 & \pm1 & 0 & \gamma & -\gamma & \mp1\\ 1 & \mp1 & \overline{\gamma} & 0 & \pm1 & -\overline{\gamma}\\ 1 & \mp1 & -\overline{\gamma} & \pm1 & 0 & \overline{\gamma}\\ 1 & \pm1 & \mp1 & -\gamma & \gamma & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \mp 1 & \alpha & -\alpha \\ 1 & \pm 1 & 0 & \pm 1 & \mp 1 & \mp 1 \\ 1 & \mp 1 & \pm 1 & 0 & -\alpha & \alpha \\ 1 & \overline{\alpha} & \mp 1 & -\overline{\alpha} & 0 & \pm 1 \\ 1 & -\overline{\alpha} & \mp 1 & \overline{\alpha} & \pm 1 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \beta & \mp 1 & -\beta \\ 1 & \pm 1 & 0 & \mp 1 & \pm 1 & \mp 1 \\ 1 & \overline{\beta} & \mp 1 & 0 & -\overline{\beta} & \pm 1 \\ 1 & \mp 1 & \pm 1 & -\beta & 0 & \beta \\ 1 & -\overline{\beta} & \mp 1 & \pm 1 & \overline{\beta} & 0 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \gamma & -\gamma & \mp 1 \\ 1 & \pm 1 & 0 & \mp 1 & \mp 1 & \pm 1 \\ 1 & \overline{\gamma} & \mp 1 & 0 & \pm 1 & -\overline{\gamma} \\ 1 & -\overline{\gamma} & \mp 1 & \pm 1 & 0 & \overline{\gamma} \\ 1 & \mp & \pm 1 & -\gamma & \gamma & 0 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \mp 1 & \alpha & -\alpha \\ 1 & \pm 1 & 0 & \mp 1 & -\alpha & \alpha \\ 1 & \mp 1 & \mp 1 & 0 & \pm 1 & \pm 1 \\ 1 & \overline{\alpha} & -\overline{\alpha} & \mp 1 & 0 & \mp 1 \\ 1 & -\overline{\alpha} & \overline{\alpha} & \pm 1 & \mp 1 & 0 \end{pmatrix}, B_4 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \beta & \pm 1 & \mp 1 & -\beta \\ 1 & \overline{\beta} & 0 & \mp 1 & -\overline{\beta} & \pm 1 \\ 1 & \pm 1 & \mp 1 & 0 & \pm 1 & \mp 1 \\ 1 & \mp 1 & -\beta & \pm 1 & 0 & \beta \\ 1 & -\overline{\beta} & \pm 1 & \mp 1 & \overline{\beta} & 0 \end{pmatrix},$$

$$C_4 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \gamma & \pm 1 & -\gamma & \mp 1 \\ 1 & \overline{\gamma} & 0 & \mp 1 & \pm 1 & -\overline{\gamma} \\ 1 & \pm 1 & \mp 1 & 0 & \mp 1 & \pm 1 \\ 1 & -\overline{\gamma} & \pm 1 & \mp 1 & 0 & \overline{\gamma} \\ 1 & \mp 1 & -\gamma & \pm 1 & \gamma & 0 \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \alpha & -\alpha & \pm 1 & \mp 1 \\ 1 & \overline{\alpha} & 0 & \pm & \mp 1 & -\overline{\alpha} \\ 1 & -\overline{\alpha} & \pm 1 & 0 & \mp 1 & \overline{\alpha} \\ 1 & \pm 1 & \mp 1 & \mp 1 & 0 & \pm 1 \\ 1 & \mp 1 & \alpha & \alpha & \pm 1 & 0 \end{pmatrix}, B_5 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \beta & \mp 1 & -\beta \\ 1 & \pm 1 & 0 & -\beta & \mp 1 & \beta \\ 1 & \overline{\beta} & -\overline{\beta} & 0 & \pm 1 & \mp 1 \\ 1 & \mp 1 & \mp 1 & \pm 1 & 0 & \pm 1 \\ 1 & -\overline{\beta} & \overline{\beta} & \mp 1 & \pm 1 & 0 \end{pmatrix},$$

$$C_5 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \gamma & \pm 1 & \mp 1 & -\gamma \\ 1 & \overline{\gamma} & 0 & -\overline{\gamma} & \pm 1 & \mp 1 \\ 1 & \pm 1 & -\gamma & 0 & \mp 1 & \gamma \\ 1 & \mp 1 & \pm 1 & \mp 1 & 0 & \pm 1 \\ 1 & -\overline{\gamma} & \mp 1 & \overline{\gamma} & \pm 1 & 0 \end{pmatrix},$$

$$A_6 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \alpha & -\alpha & \pm 1 & \mp 1 \\ 1 & \overline{\alpha} & 0 & \mp 1 & -\overline{\alpha} & \pm 1 \\ 1 & -\overline{\alpha} & \mp 1 & 0 & \overline{\alpha} & \pm 1 \\ 1 & \pm 1 & -\alpha & \alpha & 0 & \mp 1 \\ 1 & \mp 1 & \pm 1 & \pm 1 & \mp 1 & 0 \end{pmatrix}, B_6 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \beta & \pm 1 & -\beta & \mp 1 \\ 1 & \overline{\beta} & 0 & -\overline{\beta} & \mp 1 & \pm 1 \\ 1 & \pm 1 & -\beta & 0 & \beta & \mp 1 \\ 1 & -\overline{\beta} & \mp & \overline{\beta} & 0 & \pm 1 \\ 1 & \mp 1 & \pm 1 & \mp 1 & \pm 1 & 0 \end{pmatrix},$$

$$C_6 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \gamma & -\gamma & \mp 1 \\ 1 & \pm 1 & 0 & -\gamma & \gamma & \mp 1 \\ 1 & \overline{\gamma} & -\overline{\gamma} & 0 & \mp 1 & \pm 1 \\ 1 & -\overline{\gamma} & \overline{\gamma} & \mp 1 & 0 & \pm 1 \\ 1 & \mp 1 & \mp 1 & \pm 1 & \pm 1 & 0 \end{pmatrix}.$$

The parameter dependent matrices are such that one row and one column with the same indexing are without the parameter. The parameter occurs twice in each of the remaining four rows and columns. Since the first row and column are fixed, there remains only four possibilities for taking up the two positions to be occupied by the parameter. This gives 6 matrices. Since there are five different ways in which the parameter free row (and column) can be taken up, we obtain the $6 \times 5 = 30$ matrices in agreement with the results obtained.

It is natural to classify the one parameter matrices according to which row (and column) is parameter free. This gives us five families of one parameter matrices which we denote as follows:

$$\{A_i, B_i, C_i\}, \quad i = 2, 3, 4, 5, 6. \tag{5.14}$$

It is possible to obtain one matrix from another within a given family by permutations.

Assigning the values ± 1 to the parameters yields 60 parameter free matrices. We observe that there are only 12 distinct such matrices obtained from the 60. Each of these 12 is an intersection of certain five (out of 60) families of matrices. Equation (5.15) illustrates how the intersections are obtained. Below we also list the parameter free matrices.

$$D_1 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \pm 1 & \mp 1 & \mp 1 \\ 1 & \pm 1 & 0 & \mp 1 & \pm 1 & \mp 1 \\ 1 & \pm 1 & \mp 1 & 0 & \mp 1 & \pm 1 \\ 1 & \mp 1 & \pm 1 & \mp 1 & 0 & \pm 1 \\ 1 & \mp 1 & \mp 1 & \pm 1 & \pm 1 & 0 \end{pmatrix}, D_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \pm 1 & \mp 1 & \mp 1 \\ 1 & \pm 1 & 0 & \mp 1 & \mp 1 & \pm 1 \\ 1 & \pm 1 & \mp 1 & 0 & \pm 1 & \mp 1 \\ 1 & \mp 1 & \mp 1 & \pm 1 & 0 & \pm 1 \\ 1 & \mp 1 & \pm 1 & \mp 1 & \pm 1 & 0 \end{pmatrix},$$

$$D_3 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \mp 1 & \pm 1 & \mp 1 \\ 1 & \pm 1 & 0 & \pm 1 & \mp 1 & \mp 1 \\ 1 & \mp 1 & \pm 1 & 0 & \mp 1 & \pm 1 \\ 1 & \pm 1 & \mp 1 & \mp 1 & 0 & \pm 1 \\ 1 & \mp 1 & \mp 1 & \pm 1 & \pm 1 & 0 \end{pmatrix},$$

$$D_4 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \mp 1 & \mp 1 & \pm 1 \\ 1 & \pm 1 & 0 & \mp 1 & \pm 1 & \mp 1 \\ 1 & \mp 1 & \mp 1 & 0 & \pm 1 & \pm 1 \\ 1 & \mp 1 & \pm 1 & \pm 1 & 0 & \mp 1 \\ 1 & \pm 1 & \mp 1 & \pm 1 & \mp 1 & 0 \end{pmatrix}, D_5 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \mp 1 & \mp 1 & \pm 1 \\ 1 & \pm 1 & 0 & \pm 1 & \mp 1 & \mp 1 \\ 1 & \mp 1 & \pm 1 & 0 & \pm 1 & \mp 1 \\ 1 & \mp 1 & \mp 1 & \pm 1 & 0 & \pm 1 \\ 1 & \pm 1 & \mp 1 & \mp 1 & \pm 1 & 0 \end{pmatrix},$$

$$D_6 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \pm 1 & \mp 1 & \pm 1 & \mp 1 \\ 1 & \pm 1 & 0 & \mp 1 & \mp 1 & \pm 1 \\ 1 & \mp 1 & \mp 1 & 0 & \pm 1 & \pm 1 \\ 1 & \pm 1 & \mp 1 & \pm 1 & 0 & \mp 1 \\ 1 & \mp 1 & \pm 1 & \pm 1 & \mp 1 & 0 \end{pmatrix}.$$

Table 1. Summary of calculations: how the number of parameters is reduced

$\boldsymbol{n}_1$		$\boldsymbol{n}_2$		$\boldsymbol{n}_3$		$\boldsymbol{n}_4$		**Matrix**
1	$b=-a, d=-c$	1	$e=-\overline{a}, g=-f$					A_4
		2	$f=-\overline{a}, g=-e$	1	$e=a, l=-h$			A_3
				2	$h=\overline{a}, l=-\overline{e}$			A_6
				3	$l=\overline{a}, h=-\overline{e}$			B_2
		3	$g=-\overline{a}, f=-e$	1	$e=a, l=-h$			A_3
				2	$h=\overline{a}, l=-\overline{e}$			C_2
				3	$l=\overline{a}, h=-\overline{e}$			A_5
2	$c=-a, d=-b$	1	$e=-\overline{a}, g=-f$	1	$b=\overline{a}, l=-h$			A_2
				2	$h=a, l=-\overline{b}$			B_6
				3	$l=a, h=-\overline{b}$			B_3
		2	$f=-\overline{a}, g=-e$					B_5
		3	$g=-\overline{a}, f=-e$	1	$e=-b, l=-h$			C_4
				2	$h=-\overline{b}, l=-\overline{e}$			D_1
				3	$l=-\overline{b}, h=-\overline{e}$			D_4
3	$d=-a, c=-b$	1	$e=-\overline{a}, g=-f$	1	$b=\overline{a}, l=-h$			A_2
				2	$h=a, l=-\overline{b}$			C_3
				3	$l=a, h=-\overline{b}$			C_5
		2	$f=-\overline{a}, g=-e$	1	$e=-b, l=-h$			B_4
				2	$h=-\overline{b}, l=-\overline{e}$			D_6
				3	$l=-\overline{b}, h=-\overline{e}$	1	$b=-\overline{a}, m=e$	B_2
						2	$e=-a, m=\overline{b}$	C_3
						3	$m=a, e=-\overline{b}$	B_4
		3	$g=-\overline{a}, f=-e$					C_6

The 12 parameter free matrices are in fact conference matrices.[1)] The intersections are formed by picking one and only one matrix from each family. We list below the twelve intersections. In the notation used to denote the intersections, the superscript indicates the choice made. For instance, D_1^u means that we have chosen the matrix corresponding to the *upper* sign in matrix D_1. In a similar manner, D_1^l means that we have chosen the matrix in D_1 corresponding to the *lower* sign.

$$
\begin{aligned}
D_1^u &= A_2^u\,(1) = B_3^u\,(1) = C_4^u\,(1) = C_5^u\,(1) = C_6^u\,(1)\,,\\
D_1^l &= A_2^l\,(-1) = B_3^l\,(-1) = C_4^l\,(-1) = C_5^l\,(-1) = C_6^l\,(-1)\,,\\
D_2^l &= A_2^l\,(1) = C_3^l\,(-1) = B_4^l\,(-1) = B_5^l\,(-1) = B_6^l\,(-1)\,,\\
D_2^u &= A_2^u\,(-1) = C_3^u\,(1) = B_4^u\,(1) = B_5^u\,(1) = B_6^u\,(1)\,,\\
D_3^l &= B_2^l\,(-1) = A_3^l\,(-1) = B_4^u\,(-1) = A_5^l\,(-1) = C_6^l\,(1)\,,\\
D_3^u &= B_2^u\,(1) = A_3^u\,(1) = B_4^l\,(1) = A_5^u\,(1) = C_6^u\,(-1)\,,\\
D_4^l &= C_2^l\,(1) = B_3^l\,(1) = A_4^l\,(1) = A_5^u\,(-1) = B_6^u\,(-1)\,,\\
D_4^u &= C_2^u\,(-1) = B_3^u\,(-1) = A_4^u\,(-1) = A_5^l\,(1) = B_6^l\,(1)\,,\\
D_5^u &= C_2^u\,(1) = A_3^u\,(-1) = C_4^l\,(1) = B_5^u\,(-1) = A_6^l\,(1)\,,\\
D_5^l &= C_2^l\,(-1) = A_3^l\,(1) = C_4^u\,(-1) = B_5^l\,(1) = A_6^u\,(-1)\,,\\
D_6^u &= B_2^u\,(-1) = C_3^u\,(-1) = A_4^u\,(1) = C_5^l\,(1) = A_6^u\,(1)\,,\\
D_6^l &= B_2^1\,(1) = C_3^l\,(1) = A_4^l\,(-1) = C_5^u\,(-1) = A_6^l\,(-1)\,.
\end{aligned}
\tag{5.15}
$$

We now discuss the observations made from the twelve intersections. Suppose we consider these intersections as vertices of a discrete graph. An edge connecting any of the vertices is one and only one of the thirty matrices where one vertex corresponds to the matrix obtained by assigning the value $+1$ to the parameter, while the other vertex is obtained by assigning the value -1 to the parameter. For example the edge connecting the vertices D_1^u and D_2^u is the matrix $A_2\,(\alpha)$, where $D_1^u = A_2^u\,(1)$ and $D_2^u = A_2^u\,(-1)$. Figure 2 is the graph obtained where we have assigned a distinct colour to each family according to the classification in the notation (5.14). The graph obtained is bipartite and 5-regular. In the figure below each edge represents a one-parameter family (a loop). In fact each family is described by

1) A conference matrix is an $n \times n$ matrix C with diagonal entries 0 and off diagonal entries ± 1 which satisfies $CC^t = (n-1)\,I$.

6 parameters if one take into account the parameters $\theta_1, \ldots, \theta_5$ appearing in equation (5.1). The corresponding intersection are therefore 5-parameter families corresponding to six-dimensional conference matrices.

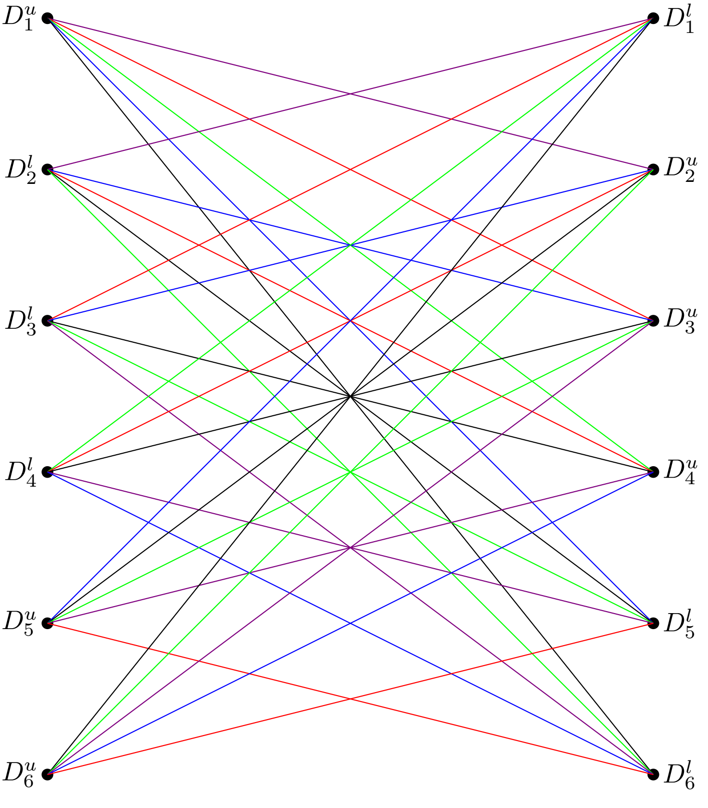

Fig. 2. Five- and six-parameter families of 6×6 RET-matrices

Conclusion 5.5. After our work was accomplished Ingemar Bengtsson pointed out, that similar matrices appear in the coding theory [7]. In fact 4-dimensional RET-matrices (4.4) can be found there as well as the 6-dimensional matrix D_1^l. But no description of all possible matrices is given there either.

Symmetric conference matrices have been discussed recently in [3], where an example of such matrices dimension 10 can be found. It is claimed that it is not known whether such matrix in dimension 8 exists or not.

We have constructed all RET-matrices in dimensions 2, 4 and 6. It is natural to continue studies by constructing RET-matrices for $n = 8$ and for arbitrary values of n.

6. APPENDIX: DEFINITION OF QUANTUM GRAPHS

A graph Γ is an ordered pair $(\boldsymbol{V}, \boldsymbol{E})$ of the set of vertices $\boldsymbol{V}$ and the set of edges $\boldsymbol{E}$. A metric graph is obtained by assigning positive length to the edges. Consider N compact and semi-infinite intervals E_n, each one considered as a subset of an individual copy of $\mathbb{R}$:

$$E_n = \begin{cases} [x_{2n-1}, x_{2n}], & n = 1, \ldots, N_c, \\ [x_{2n-1}, \infty), & n = N_c + 1, \ldots, N_c + N_i = N, \end{cases}$$

where N_c (respectively N_i) denotes the number of compact (respectively semi-infinite) intervals. Let $\boldsymbol{V} = \{x_j\}$ be the set of all endpoints. Its arbitrary partition into M equivalence classes V_m, $m = 1, \ldots, M$, gives the set of vertices.

A metric graph Γ is the union of the edges with endpoints belonging to the same vertex identified

$$\Gamma = \bigcup_{n=1}^{N} e_n/_{x \sim y},$$

where the equivalence relation is defined as

$$x \sim y \Longleftrightarrow \begin{cases} x = y, \\ x \neq y \quad \Rightarrow \exists V_m : x, y \in V_m. \end{cases}$$

The number v_m of elements in the class V_m is called the valence (or degree) of V_m. We note that points in a metric graph also include intermediate points on the edges, besides the vertices. Each compact edge has the length $l_n = x_{2n} - x_{2n-1}$, and the total length of a compact metric graph is defined as

$$\mathcal{L} = \sum_{n=1}^{N} l_n.$$

A quantum graph is obtained when we consider the magnetic Schrödinger operator $L_{q,a} = \left(i\frac{d}{dx} + a(x)\right)^2 + q(x)$ on functions defined on the edges. However often it is the Laplace operator $-\frac{d^2}{dx^2}$ that is used. The magnetic and electric potentials, $a(x)$ and $q(x)$ respectively, satisfy certian conditions. Both are real-valued. The magnetic potential is continuously differentiable on the edges, i.e. $a \in C^1(\Gamma \setminus \boldsymbol{V})$ and the electric potential is integrable, that is, $q \in L_1(\Gamma)$. The domain of the operator is the Sobolev space $W_2^2(\Gamma) = \oplus W_2^2(E_n)$ with the functions satisfying appropriate matching/boundary conditions so that the operator is self-adjoint.

With every vertex V_m we associate a unitary irreducible $v_m \times v_m$ matrix S^m. Then the matching conditions of the vertex V_m can be written as

$$i(S^m - I)\vec{u} = (S^m + I)\partial\vec{u},$$

where $\vec{u}$ is the vector of limit values at V_m of the function u along the edges meeting there and $\partial\vec{u}$ is the corresponding vector of normal derivatives. The matrix S^m determines the vertex scattering matrix $S_v^m(k)$ which describes how waves are transmitted at the vertex V_m.

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