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Analysis of positive and stable fractional continuous-time linear systems by the use of Caputo-Fabrizio derivative

by

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Abstract: Using the Caputo-Fabrizio definition of fractional order derivative, the positivity and asymptotic stability of the fractional continuous-time linear systems are investigated. The solution to the matrix fractional differential state equations is derived. Necessary and sufficient conditions for the positivity and asymptotic stability of the fractional linear systems are established. Tests for checking of the asymptotic stability of the systems are provided.

Keywords: analysis, Caputo-Fabrizio definition, fractional, linear, positivity, stability, system

1. Introduction

A dynamical system is called positive if its trajectory, starting from any nonnegative initial condition state, remains forever in the positive orthant for all nonnegative inputs. An overview of the state of the art in positive system theory is given in the monographs by Farina and Rinaldi (2000), Kaczorek (2002), as well as in the papers by Kaczorek (2014a, 2014b, 1998, 2011b, 2015, 1997). The models that display positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc. The positive standard and descriptor systems, along with their stability, have been analyzed by Kaczorek (2002, 2014b, 1998, 2011b, 2015, 1997). The positive linear systems with different fractional orders have been addressed in Kaczorek (2011b, 2012), and the descriptor discrete-time linear systems in Kaczorek (1998). The descriptor positive discrete-time and continuous-time nonlinear systems have been analyzed in Kaczorek (2014a), and the positivity and linearization of nonlinear discrete-time systems by state-feedbacks in Kaczorek (2014b). New stability tests of positive standard and fractional linear systems have been investigated in Kaczorek (2011a). The stability and robust stabilization of discrete-time switched systems have been analyzed in Zhang (2014a,b). Recently, a new definition of the fractional derivative without singular kernel has been proposed in Losada

(2015). Using this new definition the positivity and asymptotic stability of the fractional continuous-time linear systems will be investigated in this paper.

The paper is organized as follows. In Section 2 the solution to the matrix fractional differential equations is derived. The necessary and sufficient conditions for the positivity are established in Section 3 and for the asymptotic stability – in Section 4, where also tests for checking the stability are provided. Concluding remarks are formulated in Section 5.

The following notation will be used in this paper. The set of real $n \times m$ matrices will be denoted by $\Re^{n\times m}$ and the set of $n \times m$ real matrices with nonnegative entries will be denoted by $\Re_+^{n\times m}$ $(\Re_+^n = \Re_+^{n\times 1})$. The set of $n \times n$ Metzler matrices will be denoted by M_n and the $n \times n$ identity matrix will be denoted by I_n.

2. Solution of fractional differential equations

The Caputo-Fabrizio definition of fractional derivative of order α of the function $f(t)$ for $0 < \alpha < 1$ has the following form (see Losada, 2015):

$$\begin{aligned} {}^{CF}D^\alpha f(t) &= \tfrac{1}{1-\alpha}\int_0^t \exp\Big(-\tfrac{\alpha}{1-\alpha}(t-\tau)\Big)\dot f(\tau)d\tau, \\ \dot f(\tau) &= \tfrac{df(\tau)}{d\tau}, \quad t \geqslant 0. \end{aligned} \tag{1}$$

Consider the matrix fractional differential state equations

$${}^{CF}D^\alpha x(t) = \frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \tag{2}$$

$$y(t) = Cx(t) + Du(t), \tag{3}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input and output vectors, respectively, and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

Theorem 1 *The solution $x(t)$ of the equation (2) for a given initial condition $x(0) = x_0$ and input $u(t)$ has the form*

$$x(t) = e^{\hat A t}(\hat x_0 + \hat B u_0) + \int_0^t e^{\hat A(t-\tau)}\hat B[\beta u(\tau) + \dot u(\tau)]d\tau, \quad \beta = \frac{\alpha}{1-\alpha}, \tag{4}$$

where

$$\begin{aligned} &\hat A = \alpha[I_n - (1-\alpha)A]^{-1}A, \quad \hat B = [I_n - (1-\alpha)A]^{-1}(1-\alpha)B, \\ &\hat x_0 = [I_n - (1-\alpha)A]^{-1}x_0, \quad e^{\hat A t} = L^{-1}\{[I_n s - \hat A]^{-1}\}, \\ &\dot u(\tau) = \tfrac{du(\tau)}{d\tau}, \quad u(0) = u_0. \end{aligned} \tag{5}$$

Proof: By applying Laplace transform (L) to (2) and using the convolution theorem we obtain

$$L\left[\frac{d^\alpha x(t)}{dt^\alpha}\right]=\frac{1}{1-\alpha}L\left[\int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)\dot{f}(\tau)d\tau\right]=AL[x(t)]+BL[u(t)] \tag{6}$$

and

$$\frac{1}{1-\alpha}\left\{\frac{1}{s+\beta}[sX(s)-x_0]\right\}=AX(s)+BU(s), \tag{7}$$

where

$$\begin{aligned}&X(s)=L[x(t)]=\int_0^\infty x(t)e^{-st}dt,\ \ U(s)=L[u(t)],\quad L[e^{-\beta t}]=\tfrac{1}{s+\beta},\\ &L\left[\int_0^t \exp\left(-\tfrac{\alpha}{1-\alpha}(t-\tau)\right)\dot{x}(\tau)d\tau\right]=\tfrac{1}{s+\beta}[sX(s)-x_0],\\ &L[\dot{x}(t)]=sX(s)-x_0.\end{aligned} \tag{8}$$

From (7) we have

$$[s(I_n-\bar{A})-\beta\bar{A}]X(s)=x_0+(s+\beta)\bar{B}U(s), \tag{9}$$

where

$$\bar{A}=(1-\alpha)A,\ \ \bar{B}=(1-\alpha)B. \tag{10}$$

Note that for the asymptotically stable Metzler matrix A the matrix $[I_n-\bar{A}]$ is invertible. After premultiplication of (9) by $[I_n-\bar{A}]^{-1}$ we obtain

$$\begin{aligned}&[I_ns-\hat{A}]X(s)=[I_n-\bar{A}]^{-1}x_0+(s+\beta)[I_n-\bar{A}]^{-1}\bar{B}U(s)\\ &\quad=[I_n-\bar{A}]^{-1}x_0+\beta\hat{B}U(s)+\hat{B}[sU(s)-u_0]+\hat{B}u_0,\end{aligned} \tag{11}$$

where

$$\hat{A}=\beta[I_n-\bar{A}]^{-1}\bar{A}=\alpha[I_n-\bar{A}]^{-1}A,\quad \hat{B}=[I_n-\bar{A}]^{-1}\bar{B},\qquad u_0=u(0) \tag{12}$$

and

$$\begin{aligned}&X(s)=[I_ns-\hat{A}]^{-1}[I_n-\bar{A}]^{-1}x_0+[I_ns-\hat{A}]^{-1}\hat{B}u_0\\ &\quad+\beta[I_ns-\hat{A}]^{-1}\hat{B}U(s)+[I_ns-\hat{A}]^{-1}\hat{B}[sU(s)-u_0].\end{aligned} \tag{13}$$

Taking into account the fact that

$$L^{-1}\{[I_ns-\hat{A}]^{-1}\}=e^{\hat{A}t} \tag{14}$$

and using the inverse Laplace transform and the convolution theorem we obtain (4). □

By substituting (4) into (3) we obtain

$$y(t) = Ce^{\hat{A}t}(\hat{x}_0 + \hat{B}u_0) + \int_0^t Ce^{\hat{A}(t-\tau)}\hat{B}[\beta u(\tau) + \dot{u}(\tau)]d\tau + Du(t). \tag{15}$$

Using (15), we may find the output $y(t)$ for given initial conditions x_0 and input $u(t)$.

Lemma 1 *If λ_k, $k = 1, ..., n$ are the eigenvalues of the matrix A, then the eigenvalues of the matrix $\hat{A} = \alpha[I_n - (1-\alpha)A]^{-1}A$ are given by*

$$\hat{\lambda}_k = \alpha[1 - (1-\alpha)\lambda_k]^{-1}\lambda_k. \tag{16}$$

Proof: It is well-known (see Gantmacher, 1959) that if $f(\lambda_k)$ is well-defined on the spectrum λ_k, $k = 1, ..., n$ of the matrix A, then the eigenvalues of the matrix $f(A)$ are given by $f(\lambda_k)$, $k = 1, ..., n$. In this case $f(A) = \alpha[I_n - (1-\alpha)A]^{-1}A$. □

3. Positivity of the fractional linear systems

In this section the necessary and sufficient conditions for the positivity of the fractional linear systems described by the equations (2) will be established. It will be shown that the positivity of the systems depends on the first order derivative of the input $u(t)$.

Definition 1 *The fractional system (2) is called (internally) positive if the state vector $x(t) \in \Re^n_+$ and the output vector $y(t) \in \Re^p_+$, $t \geqslant 0$, for all initial conditions and all inputs $u(t) \in \Re^m_+$, $\dot{u}(t) \in \Re^m_+$, $t \geqslant 0$.*

Definition 2 *A real matrix $A = [a_{ij}] \in \Re^{n\times n}$ is called Metzler matrix if its off-diagonal entries are nonnegative, i.e. $a_{ij} \geqslant 0$ for $i \neq j$; $i, j = 1, ..., n$.*

Lemma 2 *Let $\hat{A} \in M_n$ and $0 < \alpha < 1$. Then*

$$e^{\hat{A}t} \in \Re^{n\times n}_+ \quad \text{for} \quad t \geqslant 0. \tag{17}$$

Proof: The proof is similar to the one given in Kaczorek (2002).

Theorem 2 *The fractional system (2) is positive if and only if*

$$\hat{A} \in M_n, \quad \hat{B} \in \Re^{n\times m}_+, \quad C \in \Re^{p\times n}_+, \quad D \in \Re^{p\times m}_+. \tag{18}$$

Proof:
Sufficiency. If $\hat{A} \in M_n$ and $\hat{B} \in \Re^{n\times m}_+$ then from (4) we have $x(t) \in \Re^n_+$, $t \geqslant 0$, since, by Lemma 2, $e^{\hat{A}t} \in \Re^{n\times n}_+$ and $x_0 \in \Re^n_+$, $u(t) \in \Re^m_+$, $\dot{u}(t) \in \Re^m_+$, $t \geqslant 0$.

Necessity. Let $u(t) = 0$, $t \geqslant 0$ and $x_0 = e_i$ (i-th column of the identity matrix I_n). The trajectory does not leave the orthant $\Re_+^n$ only if $D^\alpha x(0) = \hat{A} e_i \geqslant 0$, what implies $\hat{a}_{ij} \geqslant 0$ for $i \neq j$; $i, j = 1, ..., n$ and $\hat{A} \in M_n$. If $x_0 = 0$, then $D^\alpha x(0) = Bu(0) \geqslant 0$ and this implies $B \in \Re_+^{n \times m}$, since $u(0) \in \Re_+^m$ is arbitrary. From (3) for $u(t) = 0$, $t \geqslant 0$ we have $y(0) = Cx(0)$ and $C \in \Re_+^{p \times n}$, since $x(0) = x_0 \in \Re_+^n$ is arbitrary. Assuming $x_0 = 0$ from (3) we have $y(0) = Du(0)$ and $D \in \Re_+^{p \times m}$ since $u(0) \in \Re_+^m$ is arbitrary. □

LEMMA 3 *The matrix* $\bar{A} = (1 - \alpha)A \in \Re^{n \times n}$ *for* $0 < \alpha < 1$ *is asymptotically stable if and only if the matrix* A *is asymptotically stable.*

PROOF: The eigenvalues $\bar{\lambda}_k$, $k = 1, ..., n$ of the matrix $\bar{A}$ are related with the eigenvalues λ_k, $k = 1, ..., n$ of the matrix A by

$$\bar{\lambda}_k = (1 - \alpha)\lambda_k, \quad k = 1, ..., n \tag{19}$$

since the characteristic polynomials of the matrices are related by the equality

$$\begin{aligned} &\det[I_n \bar{\lambda}_k - \bar{A}] = \det[I_n \bar{\lambda}_k - (1 - \alpha)A] = \\ &= (1 - \alpha)^n \det\left[I_n \tfrac{\bar{\lambda}_k}{1-\alpha} - A\right] = (1 - \alpha)^n \det[I_n \lambda_k - A]. \end{aligned} \tag{20}$$

Therefore, from (19) it follows that $Re\bar{\lambda}_k < 0$, $k = 1, ..., n$ if and only if $Re\lambda_k < 0$, $k = 1, ..., n$. □

LEMMA 4 *The matrix*

$$\hat{A} = \alpha[I_n - (1 - \alpha)A]^{-1} A \in M_n \tag{21}$$

is asymptotically stable if and only if the eigenvalues $\lambda_k = -\alpha_k + j\beta_k$, $k = 1, ..., n$ *of the matrix* A *satisfy the condition* $[1 + (1 - \alpha)\alpha_k]\alpha_k + (1 - \alpha)\beta_k^2 = n(k) > 0$.

PROOF: From (16) for $\hat{\lambda}_k = -\hat{\alpha}_k + j\hat{\beta}_k$ and $\lambda_k = -\alpha_k + j\beta_k$, $k = 1, ..., n$ we have

$$\begin{aligned} &\hat{\lambda}_k = -\hat{\alpha}_k + j\hat{\beta}_k = \alpha[1 - (1 - \alpha)\lambda_k]^{-1}\lambda_k = \\ &= \alpha[1 - (1 - \alpha)(-\alpha_k + j\beta_k)]^{-1}(-\alpha_k + j\beta_k) \\ &\quad = \alpha \tfrac{1 + (1-\alpha)\alpha_k + j(1-\alpha)\beta_k}{[1 + (1-\alpha)\alpha_k]^2 + [(1-\alpha)\beta_k]^2}(-\alpha_k + j\beta_k) \\ &\qquad = \alpha \left(\tfrac{-[1 + (1-\alpha)\alpha_k]\alpha_k - (1-\alpha)\beta_k^2}{[1 + (1-\alpha)\alpha_k]^2 + [(1-\alpha)\beta_k]^2} \right. \\ &\qquad \left. + j \tfrac{[1 + (1-\alpha)\alpha_k]\beta_k - (1-\alpha)\alpha_k\beta_k}{[1 + (1-\alpha)\alpha_k]^2 + [(1-\alpha)\beta_k]^2} \right) \end{aligned} \tag{22}$$

and

$$\hat{\alpha}_k = \alpha \left(\frac{[1 + (1 - \alpha)\alpha_k]\alpha_k + (1 - \alpha)\beta_k^2}{[1 + (1 - \alpha)\alpha_k]^2 + [(1 - \alpha)\beta_k]^2} \right) = \alpha \frac{n(k)}{d(k)}, \quad k = 1, ..., n. \tag{23}$$

From (23) it follows that $\hat{\alpha}_k > 0$, $k = 1, ..., n$ if and only if $n(k) > 0$, $k = 1, ..., n$. $\square$

For a particular case, from Lemma 4, we have the following corollary.

COROLLARY 1 *The matrix (21) is asymptotically stable if the matrix A is asymptotically stable, since it is possible that $\hat{\alpha}_k > 0$ if $\alpha_k < 0$.*

LEMMA 5 *The matrices*

$$\begin{aligned} \hat{A} &= \alpha[I_n - (1-\alpha)A]^{-1}A \in M_n, \\ \hat{B} &= [I_n - (1-\alpha)A]^{-1}(1-\alpha)B \in \Re_+^{n\times m} \end{aligned} \tag{24}$$

if $A \in M_n$ is asymptotically stable and $B \in \Re_+^{n\times m}$.

PROOF: The matrix $[I_n-(1-\alpha)A]^{-1} \in \Re_+^{n\times n}$ if the matrix $A \in M_n$ is asymptotically stable (see Kaczorek, 2002). Therefore, by Lemma 3 and $(1-\alpha)B \in \Re_+^{n\times m}$ for $0 < \alpha < 1$ (24) holds if $A \in M_n$ is asymptotically stable. $\square$

From Lemma 4 and Theorem 2 we have the following.

THEOREM 3 *The fractional system (2) is positive if $A \in M_n$ is asymptotically stable and $B \in \Re_+^{n\times m}, C \in \Re_+^{p\times n}, D \in \Re_+^{p\times m}$.*

EXAMPLE 1 *Consider the fractional system described by the equation (2) for $\alpha = 0.5$ and*

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad u(t) = 1(t) = \begin{cases} 0 & \textit{for} \quad t < 0, \\ 1 & \textit{for} \quad t \geqslant 0. \end{cases} \tag{25}$$

In this case the matrix

$$\begin{aligned} \hat{A} &= \alpha[I_2 - (1-\alpha)A]^{-1}A = 0.5 \begin{bmatrix} 2 & -0.5 \\ -0.5 & 2.5 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix} \\ &= \tfrac{1}{4.75} \begin{bmatrix} 2.5 & 0.5 \\ 0.5 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0.5 \\ 0.5 & -1.5 \end{bmatrix} = \tfrac{1}{4.75} \begin{bmatrix} -2.25 & 0.5 \\ 0.5 & -2.75 \end{bmatrix} \end{aligned} \tag{26}$$

is also an asymptotically stable Metzler matrix, since its characteristic polynomial

$$\det[I_2 s - \hat{A}] = \begin{vmatrix} s + \frac{2.25}{4.75} & -\frac{0.5}{4.75} \\ -\frac{0.5}{4.75} & s + \frac{2.75}{4.75} \end{vmatrix} = s^2 + \frac{5}{4.75}s + \frac{1.25}{4.75} \tag{27}$$

has positive coefficients and its roots are $s_1 = -0.644$, $s_2 = -0.4086$.

Note that the matrix

$$\hat{B} = [I_n-(1-\alpha)A]^{-1}(1-\alpha) = \begin{bmatrix} 2 & -0.5 \\ -0.5 & 2.5 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \frac{1}{4.75} \begin{bmatrix} 1.5 \\ 1.25 \end{bmatrix} \tag{28}$$

has positive entries.

Using the Sylvester theorem we may find the matrix

$$e^{\hat{A}t} = Z_1 e^{s_1 t} + Z_2 e^{s_2 t}, \quad Z_1 = \frac{\hat{A} - I_2 s_2}{s_1 - s_2}, \quad Z_2 = \frac{\hat{A} - I_2 s_1}{s_2 - s_1}. \tag{29}$$

From (29) and (26) we obtain

$$e^{\hat{A}t} = \begin{bmatrix} 0.2765 & -0.4472 \\ -0.4472 & 0.7237 \end{bmatrix} e^{-0.644t} + \begin{bmatrix} 0.7235 & 0.4472 \\ 0.4472 & 0.2763 \end{bmatrix} e^{-0.4086t} \tag{30}$$

Using (4) with (25)-(30) we can find the desired solution

$$x(t) = \begin{bmatrix} 0.0737(e^{-0.4086t} + e^{-0.644t}) & +0.1177(e^{-0.4086t} - e^{-0.644t}) + 0.8 \\ 0.0947(e^{-0.4086t} + e^{-0.644t}) & +0.0235(e^{-0.4086t} - e^{-0.644t}) + 0.6 \end{bmatrix}. \tag{31}$$

The solutions of the state equation (2) with (25) and $\alpha = \{0.4, \ 0.6, \ 0.8, \ 1\}$ are shown in Figs. 1 and 2. Note that for $\alpha = 1$ we use the standard solution

$$\bar{x}(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau. \tag{32}$$

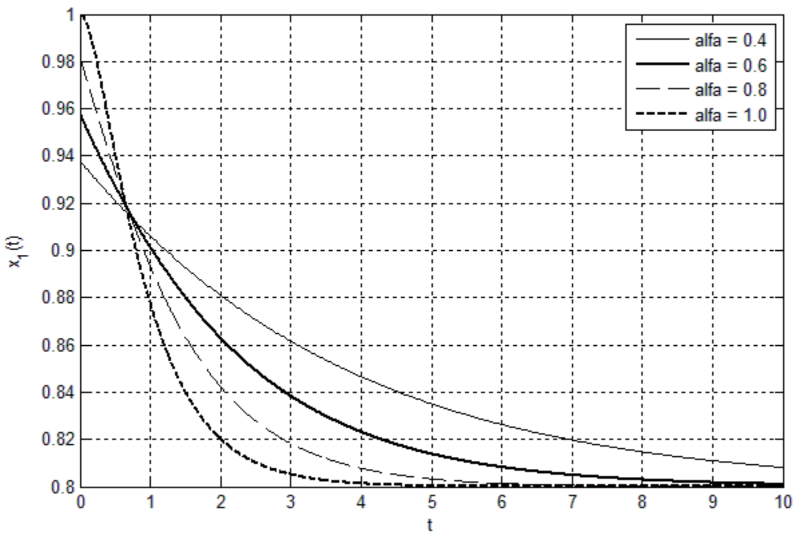

Figure 1. State variable x_1

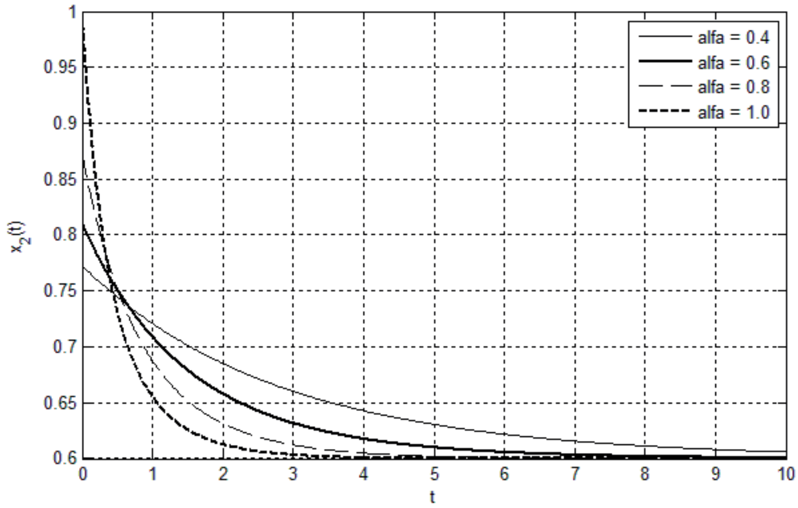

Figure 2. State variable x_2

4. Stability of positive systems

Consider the autonomous positive fractional system (obtained from (2) for $u(t) = 0$, $t \geqslant 0$

$$\frac{d^\alpha x}{dt^\alpha} = Ax(t), \quad 0 < \alpha < 1, \ \ x(t) \in \Re_+^n, \quad t \geqslant 0, \quad A \in M_n. \tag{33}$$

Definition 3 *The positive fractional system (33) is called asymptotically stable (shortly stable) if*

$$\lim_{t\to\infty} x(t) = 0 \quad \textit{for all} \quad x_0 \in \Re_+^n. \tag{34}$$

Theorem 4 *The positive fractional system (33) is (asymptotically) stable if and only if one of the following equivalent conditions is satisfied:*

- *All coefficients of the polynomial*

$$\det[I_n s - \hat{A}] = s^n + \hat{a}_{n-1}s^{n-1} + ... + \hat{a}_1 s + \hat{a}_0 \tag{35}$$

are positive, i.e. $\hat{a}_k > 0$ *for* $k = 0, 1, ..., n-1$.

- *All principal minors* $M_k, k = 1, ..., n$ *of the matrix* $-\hat{A}$ *are positive, i.e.*

$$M_1 = |-a_{11}| > 0, \quad M_2 = \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0, \quad ... \ , \\ M_n = \det[-A] > 0. \tag{36}$$

- *The diagonal entries of the matrices*

$$\hat{A}_{n-k}^{(k)} \quad for \quad k = 1, ..., n-1 \tag{37}$$

are negative, where $\hat{A}_{n-k}^{(k)}$ *are defined as follows:*

$$\hat{A}_n^{(0)} = \hat{A} = \begin{bmatrix} \hat{a}_{11}^{(0)} & \cdots & \hat{a}_{1,n}^{(0)} \\ \vdots & \ddots & \vdots \\ \hat{a}_{n,1}^{(0)} & \cdots & \hat{a}_{n,n}^{(0)} \end{bmatrix} = \begin{bmatrix} \hat{a}_{11}^{(0)} & \hat{b}_{n-1}^{(0)} \\ \hat{c}_{n-1}^{(0)} & \hat{A}_{n-1}^{(0)} \end{bmatrix},$$

$$\hat{A}_{n-1}^{(0)} = \begin{bmatrix} \hat{a}_{22}^{(0)} & \cdots & \hat{a}_{2,n}^{(0)} \\ \vdots & \ddots & \vdots \\ \hat{a}_{n,2}^{(0)} & \cdots & \hat{a}_{n,n}^{(0)} \end{bmatrix}, \quad \hat{b}_{n-1}^{(0)} = [\ \hat{a}_{12}^{(0)} \ \ \cdots \ \ \hat{a}_{1,n}^{(0)}\ ], \quad \hat{c}_{n-1}^{(0)} = \begin{bmatrix} \hat{a}_{21}^{(0)} \\ \vdots \\ \hat{a}_{n,1}^{(0)} \end{bmatrix} \tag{38}$$

and

$$\hat{A}_{n-k}^{(k)} = \hat{A}_{n-k}^{(k-1)} - \frac{\hat{c}_{n-k}^{(k-1)}\hat{b}_{n-k}^{(k-1)}}{\hat{a}_{k+1,k+1}^{(k-1)}} = \begin{bmatrix} \hat{a}_{k+1,k+1}^{(k)} & \cdots & \hat{a}_{k+1,n}^{(0)} \\ \vdots & \ddots & \vdots \\ \hat{a}_{n,k+1}^{(0)} & \cdots & \hat{a}_{n,n}^{(0)} \end{bmatrix} =$$

$$= \begin{bmatrix} \hat{a}_{k+1,k+1}^{(k)} & \hat{b}_{n-k-1}^{(k)} \\ \hat{c}_{n-k-1}^{(k)} & \hat{A}_{n-k-1}^{(k)} \end{bmatrix},$$

$$\hat{A}_{n-k-1}^{(k)} = \begin{bmatrix} \hat{a}_{k+2,k+2}^{(k)} & \cdots & \hat{a}_{k+2,n}^{(0)} \\ \vdots & \ddots & \vdots \\ \hat{a}_{n,k+2}^{(0)} & \cdots & \hat{a}_{n,n}^{(0)} \end{bmatrix}, \quad \hat{b}_{n-k-1}^{(0)} = [\ \hat{a}_{k+1,k+2}^{(k)} \ \ \cdots \ \ \hat{a}_{k+1,n}^{(k)}\ ],$$

$$\hat{c}_{n-k-1}^{(k)} = \begin{bmatrix} \hat{a}_{k+2,k+1}^{(k)} \\ \vdots \\ \hat{a}_{n,k+1}^{(k)} \end{bmatrix} \tag{39}$$

for $k = 1, ..., n-1$.

- *All diagonal entries of the upper (lower) triangular matrix*

$$\tilde{A}_u = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1,n} \\ 0 & \tilde{a}_{22} & \cdots & \tilde{a}_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{a}_{n,n} \end{bmatrix}, \quad \tilde{A}_l = \begin{bmatrix} \tilde{a}_{11} & 0 & \cdots & 0 \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n,1} & \tilde{a}_{n,2} & \cdots & \tilde{a}_{n,n} \end{bmatrix} \tag{40}$$

are negative, i.e. $\tilde{a}_{kk} < 0$ *for* $k = 1, ..., n$ *and the matrices* $\tilde{A}$ *have been obtained from the matrix* $\hat{A}$ *by the use of elementary row operation.*

- *There exists a strictly positive vector* $\lambda = [\ \lambda_1 \quad \cdots \quad \lambda_n]^T$ $, \lambda_k > 0, k = 1, ..., n$ *such that*

$$\hat{A}\lambda < 0. \tag{41}$$

Proof: By substituting in (4) $u(t) = 0$, $t \geqslant 0$ we obtain the solution of the equation (33) in the form

$$x(t) = e^{\hat{A}t} x_0. \tag{42}$$

The system (33) is stable if and only if

$$\lim_{t \to \infty} e^{\hat{A}t} = 0 \quad \text{for all} \quad x_0 \in \Re_+^n. \tag{43}$$

The condition (43) is satisfied if and only if $\hat{A} \in M_n$. Kaczorek (2002) has shown that the system (33) with $\hat{A} \in M_n$ is asymptotically stable if and only if one of the conditions 1) − 4) is satisfied. If the system is asymptotically stable, then from condition 1) we have $\hat{a}_0 = \det[-\hat{A}] > 0$ and $-\hat{A}^{-1} \in \Re_+^{n \times n}$ (see Kaczorek, 2002). Then, using (41) we obtain $(-\hat{A}^{-1})(-\hat{A})\lambda > 0$ and $\lambda > 0$ if and only if the system is asymptotically stable. □

5. Concluding remarks

Using the Caputo-Fabrizio definition of fractional order derivative, the positivity and asymptotic stability of the fractional continuous-time linear systems have been investigated. The solution to the matrix fractional differential state equations has been derived (Theorem 1). Necessary and sufficient conditions for the positivity (Theorems 2 and 3) and asymptotic stability of the fractional linear systems have been established. Tests for checking the asymptotic stability of the systems (Theorem 4) have been also given. The considerations have been illustrated by a numerical example. The considerations can be extended to the descriptor fractional continuous-time linear systems.

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