mgr Krystyna ŻYWUSZKO, dr Andrzej Antoni CZAJKOWSKI

Higher School of Technology and Economics in Szczecin, Faculty of Motor Transport, Technique and Informatics Education
Wyższa Szkoła Techniczno-Ekonomiczna w Szczecinie, Wydział Transportu Samochodowego, Kierunek Edukacja Techniczno-Informatyczna

MATHEMATICAL INDUCTION IN PROVING OF THEOREMS ABOUT NATURAL NUMBERS DIVISIBILITY

Abstract

Introduction and aims: This paper presents the concept of the division of mathematical expressions with natural variable related to the problem of divisibility. The paper shows some proofs of selected problem. The main aim of this paper is to show a few proofs of theorems about divisibility of expressions by using the method of mathematical induction.
Material and methods: In this paper have been solved examples from different sources. Considered problems contain: only polynomials, the sum of powers of different bases (and constant as a component), the sum of the products of powers with different bases (and constant as a component), the sum of the powers and polynomials, the sum of the products of powers and polynomials, the sum containing the power of (–1), Fibonacci sequence, the expression containing a power of the power and problems containing power in divider. In the paper has been used the method of mathematical induction.
Results: It has been shown 16 proofs of problems by using mathematical induction. In some examples have been used the additional lemmas which complete the main proof.
Conclusion: Using some properties of divisibility theorems and the theorem about mathematical induction allow to show proofs which refer to the divisibility by natural number of various mathematical expressions with natural variable n.

Keywords: Natural numbers, divisibility, proof, mathematical induction.
(Received: 05.06.2013; Reviewed: 15.07.2013; Accepted: 20.08.2013)

INDUKCJA MATEMATYCZNA W DOWODZENIU TWIERDZEŃ O PODZIELNOŚCI LICZB NATURALNYCH

Streszczenie

Wstęp i cele: *W pracy przedstawiono koncepcję podziału wyrażeń matematycznych ze zmienną naturalną odnoszących się do problemu podzielności a także przedstawiono dowody wybranych zadań. Głównym celem pracy jest pokazanie sposobu dowodzenia twierdzeń o podzielności wyrażeń przy zastosowaniu metody indukcji matematycznej.*
Materiał i metody: *W pracy rozwiązano przykłady z różnych źródeł. Rozważono zadania zawierające: tylko wielomiany, sumy potęg o różnych podstawach (i stałą w roli składnika), sumy iloczynów potęg o różnych podstawach (i stałą w roli składnika), sumy potęg i wielomianów, sumy iloczynów potęg i wielomianów, sumy zawierające potęgę (–1), ciąg Fibonacciego, wyrażenia zawierające potęgę potęgi oraz zadania zawierające potęgę w dzielniku. Zastosowano metodę indukcji matematycznej.*
Wyniki: *Przeprowadzono dowody 16 przykładów przy użyciu indukcji matematycznej. W niektórych przykładach zastosowano dodatkowo dowody lematów, które uzupełniają całość dowodu głównego.*
Wniosek: *Korzystanie z pewnych właściwości twierdzeń o podzielności i twierdzenia o indukcji matematycznej pozwala pokazać dowody, które odnoszą się do podzielności przez liczby naturalne różnych wyrażeń matematycznych ze zmienną naturalną.*

Słowa kluczowe: *Liczby naturalne, podzielność, dowód, indukcja matematyczna.*
(Otrzymano: 05.06.2013; Zrecenzowano:15.07.2013; Zaakceptowano: 20.08.2013)

1. Introduction

1.1. Divisibility of natural numbers

Theorem I. [1]-[5]

If each component of the sum is divisible by a number, the sum is divisible by that number.

Theorem II. [1]-[5]

If the product of one factor is divisible by the number, it is the product of a sharetion by that number.

1.2. The principle of induction

The principle of induction can be represented by the following theorem [5].

Theorem III.

If X is a set of numbers satisfying the following two conditions:

- 1 belongs to the set X,
- if n belongs to the set X, then n +1 belongs to the set X,
- then the set X contains all natural numbers.

Main idea of induction proofs lies in the ground of that in the first step the theorem is true, and that from a truth of theorem for the step n follows that the theorem is true for the step n+1.

Illustration often used for this type of problem is a domino effect. Imagine that we set the number of stones used to play dominoes so that they stand one behind the other on the short side.

We need to make sure that pushing one stone it will fall over, and assumes that any amount of stones set in a row fall over after pushing the first one.

Fig. 1. Illustration of the principle of mathematical induction by appropriately set dominoes
Photo: A.A.Czajkowski

We can now prove the induction step by proving that the number increased by one at the end of the stone also be overturned. The earliest known evidence of induction was given by Francesco Maurolico work *Arithmeticorum libri fuo* in 1575. Maurolico proved by induction that the sum of the first n odd integers is n^2 [9].

2. Classification of expressions divisible by natural numbers

Below is proposed some distribution of tasks according to construction of mathematical expressions containing natural variable.

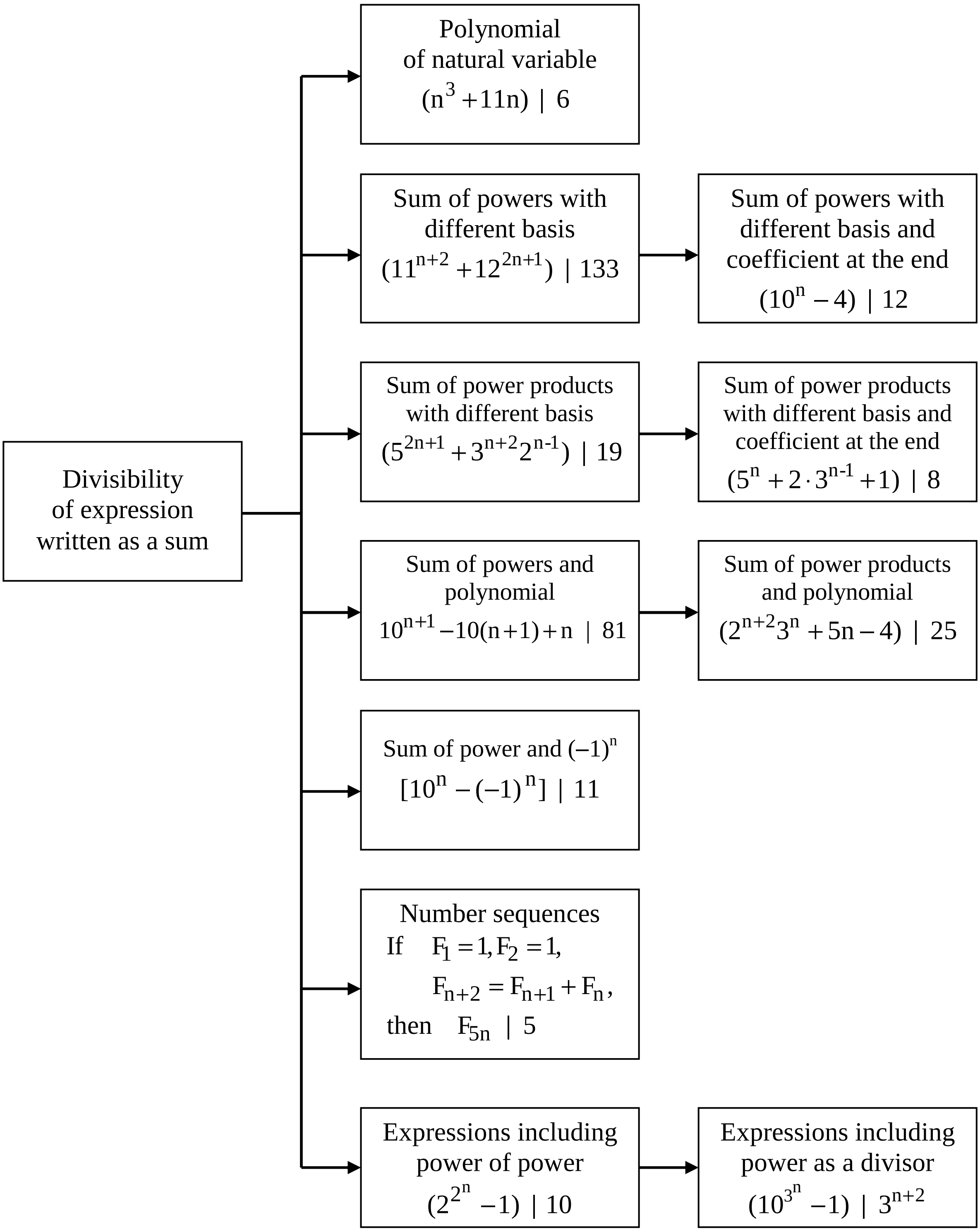

Fig. 2. Proposition of some classification for expressions divisible by natural numbers
Source: Illustration elaborated by the Authors

2. Proving theorems by mathematical induction

2.1. Theorems including only polynomials

Theorem 1. For each natural number $n \geq 1$ [6, p. 26, problem 22a]:

$$(n^3 + 11n) \mid 6. \tag{1}$$

The symbol "|" can be read as an expression *"is divisible by"* or *"is divided by"*.

Proof:

- Checking the truth of the theorem (1) for $n = 1$:

$$1^3 + 11 \cdot 1 = 1 + 11 + 12, \quad 12 \vdots 6. \tag{2}$$

- Checking the truth of the implication $k \Rightarrow k+1$:

$$\underbrace{(k^3 + 11k) \mid 6}_{A_{ind}} \Rightarrow \underbrace{[(k+1)^3 + 11(k+1)] \mid 6}_{T_{ind}} \tag{3}$$

where the symbols A_{ind} and T_{ind} mean the inductive assumption and thesis respectively.

We transform the following expression:

$$(k + 1)^3 + 11(k + 1) = k^3 + 3k^2 + 3k + 1 + 11k + 11 = (k^3 + 11k) + 3(k^2 + k + 4). \tag{4}$$

Let us analyse the obtained expression:

$k^3 + 11k$ is divisible by 6 from the inductive assumption, (5)

but $3(k^2 + k + 4)$ is divisible by 6 if and only if when $k^2 + k + 4$ is divisible by 2. (6)

Lemma 1. For each natural number $p \geq 1$:

$$(p^2 + p + 4) \mid 2. \tag{7}$$

Proof of the lemma 1:

Checking the truth of the lemma (1) for $p = 1$:

$$1^2 + 1 + 4 = 6, \quad 6 \mid 2. \tag{8}$$

Checking the truth of the implication $q \Rightarrow q+1$:

$$(q^2 + q + 4) \mid 2 \Rightarrow [(q+1)^2 + (q+1) + 4] \mid 2. \tag{9}$$

We transform the following expression:

$$(q + 1)^2 + (q + 1) + 4 = q^2 + 2q + 1 + q + 1 + 4 = (q^2 + q + 4) + 2(q + 1). \tag{10}$$

Let us analyse the obtained expression:

$q^2 + q + 4$ is divisible by 2 from the inductive assumption, (11)

$2(q + 1)$ is divisible by 2. (12)

We see that the lemma (1) is true for every natural number $p \geq 1$.

- We conclude that theorem (1) is true for every natural number $n \geq 1$. ■

Theorem 2. For each natural number n ≥ 1 [4, p. 23, problem 1.34.3]:

$$(n^5 - n) \mid 30\,. \tag{13}$$

Proof:

- Checking the truth of the theorem (2) for n = 1 and n = 2:

$$1^5 - 1 = 1 - 1 = 0, \quad 0 \mid 30\,, \tag{14}$$

$$2^5 - 2 = 32 - 2 = 30, \quad 30 \mid 30\,. \tag{15}$$

- Checking the truth of the implication k ⇒ k+1:

$$\underbrace{(k^5 - k) \mid 30}_{A_{ind}} \quad \Rightarrow \quad \underbrace{[(k+1)^5 - (k+1)] \mid 30}_{T_{ind}} \tag{16}$$

We transform the following expression:

$$\begin{aligned}(k + 1)^5 - (k + 1) &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 = \\ &= (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)\,.\end{aligned} \tag{17}$$

Let us analyse the obtained expression:

$k^5 - k$	is divisible by 30 from the inductive assumption,	(18)
but $5(k^4 + 2k^3 + 2k^2 + k)$	is divisible by 30 if and only if when $(k^4 + 2k^3 + 2k^2 + k)$ is divisible by 6.	(19)

Lemma 2. For each natural number p ≥ 1:

$$(p^4 + 2p^3 + 2p^2 + p) \mid 6\,. \tag{20}$$

Proof of the lemma 2:

Checking the truth of the lemma (2) for p = 1:

$$1^4 + 2\cdot 1^3 + 2\cdot 1^2 + 1 = 1 + 2 + 2 + 1 = 6, \quad 6 \mid 6\,. \tag{21}$$

Checking the truth of the implication q ⇒ q+1:

$$(q^4 + 2q^3 + 2q^2 + q) \mid 6 \quad \Rightarrow \quad [(q+1)^4 + 2(q+1)^3 + 2(q+1)^2 + (q+1)] \mid 6\,. \tag{22}$$

We transform the following expression:

$$\begin{aligned}&(q + 1)^4 + 2(q + 1)^3 + 2(q + 1)^2 + (q + 1) = \\ &= q^4 + 4q^3 + 6q^2 + 4q + 1 + 2q^3 + 6q^2 + 6q + 2 + 2q^2 + 4q + 2 + q + 1 = \\ &= (q^4 + 2q^3 + 2q^2 + q) + 2(2q^3 + 6q^2 + 7q) + 6\,.\end{aligned} \tag{23}$$

Let us analyse the obtained expression:

$q^4 + 2q^3 + 2q^2 + q$	is divisible by 6 from the inductive assumption,	(24)
$2(2q^3 + 6q^2 + 7q)$	is divisible by 6 if and only if when $(2q^3 + 6q^2 + 7q) \mid 3$	(25)
6	is divisible by 6.	(26)

<u>Lemma 3</u>. For each natural number q ≥ 1:

$$(2q^3 + 6q^2 + 7q) \mid 3 . \qquad (27)$$

<u>Proof of the lemma 3</u>:

Checking the truth of the lemma (3) for q = 1:

$$2 \cdot 1^3 + 6 \cdot 1^2 + 7 \cdot 1 = 2 + 6 + 7 = 15, \quad 15 \mid 3 . \qquad (28)$$

Checking the truth of the implication m ⇒ m+1:

$$(2m^3 + 6m^2 + 7m) \mid 3 \quad \Rightarrow \quad 2(m+1)^3 + 6(m+1)^2 + 7(m+1) \mid 3 \qquad (29)$$

We transform the following expression:

$$2(m + 1)^3 + 6(m + 1)^2 + 7(m + 1) = 2qm^3 + 6m^2 + 6m + 2 + 6m^2 + 12m + 6 + 7m + 7 = \\ = (2m^3 + 6m^2 + 7m) + 3(2m^2 + 6m + 3) . \qquad (30)$$

Let us analyse the obtained expression:

$2m^3 + 6m^2 + 7m$ is divisible by 3 from the inductive assumption, (31)

$3(2m^2 + 6m + 3)$ is divisible by 3. (32)

We see that the lemma (3) is true for every natural number q ≥ 1.

We see that the lemma (2) is true for every natural number p ≥ 1.

➢ We conclude that theorem (2) is true for every natural number n ≥ 1. ■

<u>Theorem 3</u>. For each natural number n ≥ 1 [3, p. 23, problem 1]:

$$(n^3 + 3n^2 + 5n + 3) \mid 3 . \qquad (33)$$

<u>Proof</u>:

➢ Checking the truth of the theorem (3) for n = 1:

$$1^3 + 3 \cdot 1^2 + 5 \cdot 1 + 3 = 1 + 3 + 5 + 3 = 12, \quad 12 \mid 3 , \qquad (34)$$

➢ Checking the truth of the implication k ⇒ k+1:

$$\underbrace{(k^3 + 3k^2 + 5k + 3) \mid 3}_{A_{ind}} \quad \Rightarrow \quad \underbrace{[(k+1)^3 + 3(k+1)^2 + 5(k+1) + 3] \mid 3}_{T_{ind}} \qquad (35)$$

We transform the following expression:

$$(k + 1)^3 + 3(k + 1)^2 + 5(k+1) + 3 = k^3 + 3k^2 + 3k + 1 + 3k^2 + 6k + 3 + 5k + 5 + 3 = \\ = (k^3 + 3k^2 + 5k + 3) + 3(k^2 + 3k + 3) . \qquad (36)$$

Let us analyse the obtained expression:

$k^3 + 3k^2 + 5k + 3$ is divisible by 3 from the inductive assumption, (37)

but $3(k^2 + 3k + 3)$ is divisible by 3. (38)

➢ We conclude that theorem (3) is true for every natural number n ≥ 1. ■

2.2. Theorems including powers with different basis

Theorem 4. For each natural number n ≥ 0 [1, p. 71, problem 8.1(4)]:

$$(11^{n+2}+12^{2n+1}) \mid 133. \quad (39)$$

Proof:

- Checking the truth of the theorem (4) for n = 0:

$$11^2+12^1=121+12=133, \quad 133|133. \quad (40)$$

- Checking the truth of the implication k ⇒ k+1:

$$(11^{k+2}+12^{2k+1}) \mid 133 \quad \Rightarrow \quad (11^{k+3}+12^{2k+3}) \mid 133 \quad (41)$$

We transform the following expression:

$$\begin{aligned}&11^{k+3}+12^{2k+3}=11\cdot 11^{k+2}+12^2\cdot 12^{2k+1}=11\cdot 11^{k+2}+144\cdot 12^{2k+1}=\\&=11\cdot 11^{k+2}+11\cdot 12^{2k+1}+133\cdot 12^{2k+1}=11(11^{k+2}+12^{2k+1})+133\cdot 12^{2k+1}.\end{aligned} \quad (42)$$

Let us analyse the obtained expression:

$11(11^{k+2}+12^{2k+1})$ is divisible by 133 from the inductive assumption, (43)

but $133\cdot 12^{2k+1}$ is divisible by 133. (44)

- We conclude that theorem (4) is true for every natural number n ≥ 0. ■

Theorem 5. For each natural number n ≥ 0 [3, p. 24, problem 1.36.6]:

$$(5^{5n+1}+4^{5n+2}+3^{5n}) \mid 11. \quad (45)$$

Proof:

- Checking the truth of the theorem (5) for n = 0:

$$5^1+4^2+3^0=5+16+3=22, \quad 22|11. \quad (46)$$

- Checking the truth of the implication k ⇒ k+1:

$$(5^{5k+1}+4^{5k+2}+3^{5k}) \mid 11 \quad \Rightarrow \quad (5^{5k+6}+4^{5k+7}+3^{5k+5}) \mid 11. \quad (47)$$

We transform the following expression:

$$\begin{aligned}&5^{5k+6}+4^{5k+7}+3^{5k+5}=5^5\cdot 5^{5k+1}+4^5\cdot 4^{5k+2}+3^5\cdot 3^{5k}=\\&=3125\cdot 5^{5k+1}+1024\cdot 4^{5k+2}+243\cdot 3^{5k}=\\&=(5^{5k+1}+4^{5k+2}+3^{5k})+11\cdot(284\cdot 5^{5k+1}+93\cdot 4^{5k+2}+22\cdot 3^{5k}).\end{aligned} \quad (48)$$

Let us analyse the obtained expression:

$5^{5k+1}+4^{5k+2}+3^{5k}$ is divisible by 11 from the inductive assumption, (49)

$11\cdot(284\cdot 5^{5k+1}+93\cdot 4^{5k+2}+22\cdot 3^{5k})$ is divisible by 11. (50)

- We conclude that theorem (5) is true for every natural number n ≥ 0. ■

2.3. Theorems including powers with different basis and coefficient at the end

Theorem 6. For each natural number n ≥ 2 [4, p. 24, problem 1.35.2]:

$$(10^n - 4) \mid 12. \quad (51)$$

Proof:

- Checking the truth of the theorem (6) for n = 2:

$$10^2 - 4 = 100 - 4 = 96 = 8 \cdot 12, \quad 96 | 12. \quad (52)$$

- Checking the truth of the implication k ⇒ k+1:

$$(10^k - 4) \mid 12 \quad \Rightarrow \quad (10^{k+1} - 4) \mid 12. \quad (53)$$

We transform the following expression:

$$10^{k+1} - 4 = 10 \cdot 10^k - 4 = (10^k - 4) + 9 \cdot 10^k. \quad (54)$$

Let us analyse the obtained expression:

$10^k - 4$ is divisible by 12 from the inductive assumption, (55)

$9 \cdot 10^k$ is divisible by 12 = 3·4 because $9 | 3$ and $10^k | 4$ for k ≥ 2. (56)

- We conclude that theorem (6) is true for every natural number n ≥ 2. ■

Theorem 7. For each natural number n ≥ 0 [7, p. 30, problem 325]:

$$(2^{6n+1} + 3^{6n+1} + 5^{6n} + 1) \mid 7. \quad (57)$$

Proof:

- Checking the truth of the theorem (7) for n = 0:

$$2^1 + 3^1 + 5^0 + 1 = 2 + 3 + 1 + 1 = 7, \quad 7 | 7. \quad (58)$$

- Checking the truth of the implication k ⇒ k+1:

$$(2^{6k+1} + 3^{6k+1} + 5^{6k} + 1) \mid 7 \quad \Rightarrow \quad (2^{6k+7} + 3^{6k+7} + 5^{6k+6} + 1) \mid 7. \quad (59)$$

We transform the following expression:

$$\begin{aligned} &2^{6k+7} + 3^{6k+7} + 5^{6k+6} + 1 = 2^6 \cdot 2^{6k+1} + 3^6 \cdot 3^{6k+1} + 5^6 \cdot 5^{6k} + 1 = \\ &= 64 \cdot 2^{6k+1} + 729 \cdot 3^{6k+1} + 15625 \cdot 5^{6k} + 1 = \\ &= (2^{6k+1} + 3^{6k+1} + 5^{6k} + 1) + 7 \cdot (9 \cdot 2^{6k+1} + 104 \cdot 3^{6k+1} + 2232 \cdot 5^{6k}). \end{aligned} \quad (60)$$

Let us analyse the obtained expression:

$2^{6k+1} + 3^{6k+1} + 5^{6k} + 1$ is divisible by 7 from the inductive assumption, (61)

$7(9 \cdot 2^{6k+1} + 104 \cdot 3^{6k+1} + 2232 \cdot 5^{6k})$ is divisible by 7 for k ≥ 0. (62)

- We conclude that theorem (7) is true for every natural number n ≥ 0. ■

2.4. Sum of powers products with different basis

Theorem 8. For each natural number n ≥ 1 [7, p. 44, problem 478]:

$$(5^{2n+1} + 3^{n+2} \cdot 2^{n-1}) \mid 19. \tag{63}$$

Proof:

- Checking the truth of the theorem (8) for n = 1:

$$5^3 + 3^3 2^0 = 125 + 27 \cdot 1 = 152 = 8 \cdot 19, \quad 152 \mid 19. \tag{64}$$

- Checking the truth of the implication k ⇒ k+1:

$$(5^{2k+1} + 3^{k+2} \cdot 2^{k-1}) \mid 19 \quad \Rightarrow \quad (5^{2k+3} + 3^{k+3} \cdot 2^{k}) \mid 19. \tag{65}$$

We transform the following expression:

$$\begin{aligned} &5^{2k+3} + 3^{k+3} \cdot 2^k = 25 \cdot 5^{2k+1} + 6 \cdot 3^{k+2} \cdot 2^{k-1} = \\ &= (25 \cdot 5^{2k+1} + 6 \cdot 3^{k+2} \cdot 2^{k-1}) + 19 \cdot 3^{k+2} 2^{k-2} - 19 \cdot 3^{k+2} \cdot 2^{k-1} = \\ &= 25(5^{2k+1} + 3^{k+2} \cdot 2^{k-1}) - 19 \cdot 3^{k+2} \cdot 2^{k-1}. \end{aligned} \tag{66}$$

Let us analyse the obtained expression:

$25 \cdot (5^{2k+1} + 3^{k+2} \cdot 2^{k-1})$ is divisible by 19 from the inductive assumption, (67)

$(-19) \cdot 3^{k+2} \cdot 2^{k-1}$ is divisible by 19 for k ≥ 1. (68)

- We conclude that theorem (8) is true for every natural number n ≥ 1. ■

Theorem 9. For each natural number n ≥ 1 [1, p. 72, problem 8.1(7)]:

$$(5^{2n+1} \cdot 2^{n+2} + 3^{n+2} \cdot 2^{2n+1}) \mid 19. \tag{69}$$

Proof:

- Checking the truth of the theorem (9) for n = 1:

$$5^3 \cdot 2^3 + 3^3 \cdot 2^3 = 125 \cdot 8 + 27 \cdot 8 = 1000 + 216 = 1216 = 19 \cdot 64, \quad 1216 \mid 19. \tag{70}$$

- Checking the truth of the implication k ⇒ k+1:

$$(5^{2k+1} \cdot 2^{k+2} + 3^{k+2} \cdot 2^{2k+1}) \mid 7 \quad \Rightarrow \quad (5^{2k+3} \cdot 2^{k+3} + 3^{k+3} \cdot 2^{2k+3}) \mid 7. \tag{71}$$

We transform the following expression:

$$\begin{aligned} &5^{2k+3} \cdot 2^{k+3} + 3^{k+3} \cdot 2^{2k+3} = 50 \cdot 5^{2k+1} \cdot 2^{k+2} + 12 \cdot 3^{k+2} \cdot 2^{2k+1} = \\ &= 12 \cdot 5^{2k+1} \cdot 2^{k+2} + 38 \cdot 5^{2k+1} \cdot 2^{k+2} + 12 \cdot 3^{k+2} \cdot 2^{2k+1} = \\ &= 12 \cdot (5^{2k+1} \cdot 2^{k+2} + 3^{k+2} \cdot 2^{2k+1}) + 38 \cdot 5^{2k+1} \cdot 2^{k+2}. \end{aligned} \tag{72}$$

Let us analyse the obtained expression:

$12 \cdot (5^{2k+1} \cdot 2^{k+2} + 3^{k+2} \cdot 2^{2k+1})$ is divisible by 19 from the inductive assumption, (73)

$38 \cdot 5^{2k+1} \cdot 2^{k+2}$ is divisible by 19 for k ≥ 1. (74)

- We conclude that theorem (9) is true for every natural number n ≥ 1. ■

2.5. Sum of powers products with different basisand coefficient at the end

Theorem 10. For each natural number n ≥ 1 [7, p. 49, problem 542]:

$$(5^{n}+2\cdot 3^{n-1}+1)\ |\ 8. \tag{75}$$

Proof:

- Checking the truth of the theorem (10) for n = 1:

$$5^{1}+2\cdot 3^{0}+1=5+2\cdot 1+1=8,\quad 8|8. \tag{76}$$

- Checking the truth of the implication k ⇒ k+1:

$$(5^{k}+2\cdot 3^{k-1}+1)\ |\ 8 \Rightarrow\quad (5^{k+1}+2\cdot 3^{k}+1)\ |\ 8. \tag{77}$$

We transform the following expression:

$$\begin{aligned}&5^{k+1}+2\cdot 3^{k}+1=5\cdot 5^{k}+6\cdot 3^{k-1}+1=(5^{k}+2\cdot 3^{k-1}+1)+4\cdot 5^{k}+4\cdot 3^{k-1})\\&=(5^{k}+2\cdot 3^{k-1}+1)+4(5^{k}+3^{k-1}).\end{aligned} \tag{78}$$

Let us analyse the obtained expression:

$(5^{k}+2\cdot 3^{k-1}+1)$ is divisible by 8 from the inductive assumption, (79)

$4(5^{k}+3^{k-1})$ is divisible by 8 if and only if when $5^{k}+3^{k-1}$ is divisible by 2. (80)

Lemma 4. For each natural number p ≥ 1:

$$(5^{p}+3^{p-1})\ |\ 2. \tag{81}$$

Checking the truth of the lemma for p = 1:

$$5^{1}+3^{0}=5+1=6,\quad 6\ |\ 2. \tag{82}$$

Checking the truth of the implication q ⇒ q+1:

$$(5^{q}+3^{q-1})\ |\ 3\quad\Rightarrow\quad (5^{q+1}+3^{q})\ |\ 2. \tag{83}$$

We transform the following expression:

$$\begin{aligned}&5^{q+1}+3^{q}=5\cdot 5^{q}+3\cdot 3^{q-1}=\\&=(5^{q}+3^{q-1})+4\cdot 5^{q}+2\cdot 3^{q-1}=\\&=(5^{q}+3^{q-1})+2(2\cdot 5^{q}+3^{q-1}).\end{aligned} \tag{84}$$

Let us analyse the obtained expression:

$(5^{q}+3^{q-1})$ is divisible by 2 from the inductive assumption, (85)

$2(2\cdot 5^{q}+3^{q-1})$ is divisible by 2. (86)

We see that the lemma (4) is true for every natural number p ≥ 1.

- We conclude that theorem (10) is true for every natural number n ≥ 1. ■

2.6. Sum of powers and polynomial

Theorem 11. For each natural number n ≥ 1 [3, p. 23 problem 4]:

$$10^{n+1} - 10(n+1) + n \mid 81. \quad (87)$$

Proof:

- Checking the truth of the theorem for n = 1:

$$10^{1+1} - 10 \cdot (1+1) + 1 = 100 - 20 + 1 = 81, \quad 81|81. \quad (88)$$

- Checking the truth of the implication k ⇒ k+1:

$$10^{k+1} - 10(k+1) + k \mid 81 \quad \Rightarrow \quad 10^{k+2} - 10(k+2) + (k+1) \mid 81. \quad (89)$$

We transform the following expression:

$$\begin{aligned} 10^{k+2} - 10(k+2) + (k+1) &= 10 \cdot 10^{k+1} - 10k - 20 + k + 1 = \\ &= [10^{k+1} - 10(k+1) + k] + 9 \cdot (10^{k+1} - 9) = \\ &= [10^{k+1} - 10(k+1) + k] + 9 \cdot (9 \cdot 10^{k} + 10^{k} - 1) = \\ &= [10^{k+1} - 10(k+1) + k] + 81 \cdot 10^{k} + 9 \cdot (10^{k} - 1). \end{aligned} \quad (90)$$

Let us analyse the obtained expression:

$[10^{k+1} - 10(k+1) + k]$ is divisible by 81 from the inductive assumption, (91)

$81 \cdot 10^{k}$ is divisible by 81, (92)

$9 \cdot (10^{k} - 1)$ is divisible by 81 if and only if when $10^{k} - 1$is divisible by 9. (93)

Lemma 5. For each natural number p ≥ 1:

$$10^{p} - 1 \mid 9. \quad (94)$$

Checking the truth of the lemma for p = 1:

$$10^{1} - 1 = 9, \quad 9 \mid 9. \quad (95)$$

Checking the truth of the implication q ⇒ q+1:

$$10^{q} - 1 \mid 9 \quad \Rightarrow \quad 10^{q+1} - 1 \mid 9. \quad (96)$$

We transform the following expression:

$$10^{q+1} - 1 = 10 \cdot 10^{q} - 1 = (10^{q} - 1) + 9 \cdot 10^{q}. \quad (97)$$

Let us analyse the obtained expression:

$(10^{q} - 1)$ is divisible by 9 from the inductive assumption, (98)

$9 \cdot 10^{q}$ is divisible by 9. (99)

We see that the lemma (5) is true for every natural number p ≥ 1.

- We conclude that theorem (11) is true for every natural number n ≥ 1. ■

2.7. Sum of power products and polynomial

Theorem 12. For each natural number n ≥ 1 [4, p. 24, problem 1.36.4]:

$$2^{n+2}3^{n} + 5n - 4 \mid 25. \qquad (100)$$

Proof:

- Checking the truth of the theorem for n = 1:

$$2^{1+2}3^{1} + 5 \cdot 1 - 4 = 8 \cdot 3 + 5 - 4 = 25, \quad 25 | 25. \qquad (101)$$

- Checking the truth of the implication k ⇒ k+1:

$$2^{k+2}3^{k} + 5k - 4 \mid 25 \quad \Rightarrow \quad 2^{k+3}3^{k+1} + 5(k+1) - 4 \mid 25. \qquad (102)$$

We transform the following expression:

$$\begin{aligned} &2^{k+2}\cdot 3^{k+1} + 5(k+1) - 4 = 2\cdot 2^{k+2}\cdot 3\cdot 3^{k} + 5k + 5 - 4 = \\ &= [2^{k+2}\cdot 3^{k} + 5k - 4] + 5\cdot 2^{k+2}\cdot 3^{k} + 5 = \\ &= [2^{k+2}\cdot 3^{k} + 5k - 4] + 5(2^{k+2}\cdot 3^{k} + 1) . \end{aligned} \qquad (103)$$

Let us analyse the obtained expression:

$[2^{k+2}\cdot 3^{k} + 5k - 4]$ is divisible by 25 from the inductive assumption, (104)

$5(2^{k+2}\cdot 3^{k} + 1)$ is divisible by 25 if and only if when $2^{k+2}\cdot 3^{k} + 1$ is divisible by 5. (105)

Lemma 6. For each natural number p ≥ 1:

$$2^{p+2}3^{p} + 1 \mid 5. \qquad (106)$$

Checking the truth of the lemma for p = 1:

$$2^{1+2}3^{1} + 1 = 2^{3}3 + 1 = 8 \cdot 3 + 1 = 25, \quad 25 \mid 25. \qquad (107)$$

Checking the truth of the implication q ⇒ q+1:

$$2^{q+2}3^{q} + 1 \mid 5 \quad \Rightarrow \quad 2^{q+3}3^{q+1} + 1 \mid 5. \qquad (108)$$

We transform the following expression:

$$2^{q+3}3^{q+1} + 1 = 2 \cdot 2^{q+2}3 \cdot 3^{q} + 1 = 6 \cdot 2^{q+2}3^{q} + 1 = (2^{q+2}3^{q} + 1) + 5 \cdot 2^{q+2}3^{q}. \qquad (109)$$

Let us analyse the obtained expression:

$(2^{q+2}3^{q} + 1)$ is divisible by 5 from the inductive assumption, (110)

$5 \cdot 2^{q+2}3^{q}$ is divisible by 5. (111)

We see that the lemma (6) is true for every natural number p ≥ 1.

- We conclude that theorem (12) is true for every natural number n ≥ 1. ■

2.8. Sum of power and $(-1)^n$

Theorem 13. For each natural number n ≥ 1 [4, p. 24, problem 1.35.4]:

$$10^n - (-1)^n \mid 11. \tag{112}$$

Proof:

- Checking the truth of the theorem for n = 1:

$$10^1 - (-1)^1 = 10 + 1 = 11 \quad 11 \mid 11. \tag{113}$$

- Checking the truth of the implication k ⇒ k+1:

$$10^k - (-1)^k \mid 11 \quad \Rightarrow \quad 10^{k+1} - (-1)^{k+1} \mid 11. \tag{114}$$

We transform the following expression:

$$\begin{aligned} &10^{k+1} - (-1)^{k+1} = 10 \cdot 10^k + (-1)^k = 10 \cdot 10^k + 10^k - 10^k + (-1)^k = \\ &= 11 \cdot 10^k - 10^k + (-1)^k = 11 \cdot 10^k - [10^k - (-1)^k]. \end{aligned} \tag{115}$$

Let us analyse the obtained expression:

$[10^k - (-1)^k]$ is divisible by 11 from the inductive assumption, (116)

$11 \cdot 10^k$ is divisible by 11. (117)

- We conclude that theorem (13) is true for every natural number n ≥ 1. ■

2.9. Number sequences

Theorem 14. For each natural number n ≥ 1 [4, p. 24, problem 1.37.3]:

$$F_{5n} \mid 5 \tag{118}$$

where Fibonacci sequence is defined by the following recurrence form:

$$F_1 = 1, \quad F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n. \tag{119}$$

Proof:

- Checking the truth of the theorem for n = 1:

$$F_{5 \cdot 1} = F_5 = 5, \quad 5 \mid 5. \tag{120}$$

- Checking the truth of the implication k ⇒ k+1:

$$F_{5n} \mid 5 \quad \Rightarrow \quad F_{5(n+1)} \mid 5. \tag{121}$$

We transform the following expression:

$$\begin{aligned} &F_{5(n+1)} = F_{5n+5} = F_{5n+4} + F_{5n+3} = F_{5n+3} + F_{5n+2} + F_{5n+2} + F_{5n+1} = F_{5n+2} + F_{5n+1} + F_{5n+1} + F_{5n} + \\ &+ F_{5n+1} + F_{5n} + F_{5n+1} = F_{5n+1} + F_{5n} + F_{5n+1} + F_{5n+1} + F_{5n} + F_{5n+1} + F_{5n} + F_{5n+1} = 3 \cdot F_{5n} + 5 \cdot F_{5n+1}. \end{aligned} \tag{122}$$

Let us analyse the obtained expression:

$3 \cdot F_{5n}$ is divisible by 5 from the inductive assumption, (123)

$5 \cdot F_{5n+1}$ is divisible by 5. (124)

- We conclude that theorem (14) is true for every natural number n ≥ 1. ■

2.10. Expressions including power of power

Theorem 15. For each natural number n ≥ 2 [4, p. 24, problem 1.36.2]:

$$(2^{2^n}-6)\ |\ 10. \tag{125}$$

Proof:

➢ Checking the truth of the theorem for n = 2:

$$2^{2^2}-6=2^4-1=16-6=10,\quad 10|\ 10. \tag{126}$$

➢ Checking the truth of the implication k ⇒ k+1:

$$(2^{2^k}-6)\ |\ 10\quad\Rightarrow\quad (2^{2^{k+1}}-6)\ |\ 10. \tag{127}$$

We transform the following expression:

$$\begin{aligned}2^{2^{k+1}}-6&=2^{2\cdot 2^k}-6=(2^{2^k})^2-6=[(2^{2^k})^2-12\cdot 2^{2^k}+36]+12\cdot 2^{2^k}-36-6=\\&=(2^{2^k}-6)^2+12\cdot 2^{2^k}-42+72-72=(2^{2^k}-6)^2+12(2^{2^k}-6)+30.\end{aligned} \tag{128}$$

Let us analyse the obtained expression:

$(2^{2^k}-6)$ is divisible by 10 from the inductive assumption, (129)

30 is divisible by 10. (130)

➢ We conclude that theorem (15) is true for every natural number n ≥ 2. ■

2.11. Expressions including power with natural variable as a divisor

Theorem 16. For each natural number n ≥ 0 [1, p. 26, problem 22в]:

$$(10^{3^n}-1)\ |\ 3^{n+2}. \tag{131}$$

Proof:

➢ Checking the truth of the theorem for n = 0:

$$(10^{3^0}-1)\ |3^{0+2}\ \Leftrightarrow\ (10^1-1)\ |\ 3^2\ \Leftrightarrow\ (10-1)\ |\ 3^2\ \Leftrightarrow\ 9\ |\ 9. \tag{132}$$

➢ Checking the truth of the implication k ⇒ k+1:

$$(10^{3^k}-1)\ |\ 3^{k+2}\quad\Rightarrow\quad (10^{3^{(k+1)}}-1)\ |\ 3\cdot 3^{k+2}. \tag{133}$$

We transform the following expression:

$$\begin{aligned}10^{3^{(k+1)}}-1&=10^{3\cdot 3^k}-1=(10^{3^k})^3-1^3=(10^{3^k}-1)[(10^{3^k})^2+10^{3^k}+1]=\\&=(10^{3^k}-1)\{[(10^{3^k})^2-2\cdot 10^{3^k}+1]+3\cdot 10^{3^k}\}=(10^{3^k}-1)[(10^{3^k}-1)^2+3\cdot 10^{3^k}].\end{aligned} \tag{134}$$

Let us analyse the obtained expression:

$(10^{3^k}-1)$ is divisible by 3^{k+2} from the inductive assumption, (135)

$3\cdot 10^{3^k}$ is divisible by 3. (136)

➢ We conclude that theorem (16) is true for every natural number n ≥ 0. ■

3. Conclusion

- Using some chosen properties of divisibility theorems and the theorem on mathematical induction is possible to make some proofs which refer to the divisibility by natural number of various mathematical expressions with natural variable n.

4. Another way of editing mathematical induction theorem

The principle of mathematical induction can be expressed in the following theorem [8]:

If

✓ *exists a natural number n_0 such that $T(n_0)$ is a true sentence,*

✓ *for each natural number $n \geq n_0$ the implication $T(n) \Rightarrow T(n+1)$,*

then T (n) is a true sentence for each natural number $n \geq n_0$.

Proof done by mathematical induction is called inductive proof consists of two steps:

✓ check that $T(n_0)$ is true,

✓ that, for all $n \geq n_0$ if T(n) is true, then T(n+1) is true.

This second step is called the inductive step. It assumes that the natural number n ≥ n0 sentence T (n) is true *(i.e. induction hypothesis),* and on this basis we prove the truth of the sentence T (n+1).

As you can see from the foregoing considerations theorem of mathematical induction can be compared to the reasoning *step by step,* where the *steps* are numbered natural.

The crux of inductive proofs is that in the first step, the theorem is true, and that for each $n \in N$ from the truth of a theorem for step n implies the truth of the theorem for step n+1.

Literature

[1] Башвмаков М.Н., Беккер Б.М., Гольховой В.М.: *Задачи по математике. Алгебра и анализ,* Главная Редакция Физико-Математической Литературы, Издательство «Наука» под редакцией Д.К. Фаддеева, Библиотека «Квант», выпуск 22, Москва 1982, стр. 71,72.

[2] Czajkowski A.A., Żywuszko K.: *Metoda indukcji matematycznej w dowodzeniu sum potęg kolejnych liczb naturalnych,* Dydaktyka Nauk Stosowanych Tom 7: Informatyka i Media, Elektrotechnika, Biomechanika i Neonatologia, Rozwój Nauki i Techniki, (red. A.A. Czajkowski), Uniwersytet Szczeciński, Wydz. Matematyczno-Fizyczny, Kat. Edukacji Informatycznej i Technicznej, Szczecin 2011, s. 33-44.

[3] Говоров В.М., Дыбов П.Т., Мирошин Н.В., Стринова С.Ф.: *Зборник конкурсных задач по математике с методическим указаням и решеням,* Главная Редакция Физико-Математической Литературы, Издательство «Наука», Москва 1986, издание второе, стереотипное стр. 23.

[4] Jeśmianowicz L., Łoś J.: *Zbiór zadań z algebry,* PWN, Warszawa 1975, s. 23, 24, w. VI.

[5] Mostowski A., Stark M.: *Elements of higher algebra,* PWN, Warsaw 1970, p. 11, 5-th edition.

[6] Воробъев Н.Н.: *Признаки деолимости*, Главная Редакция Физико-Математической Литературы, Издательство «Наука», Москва 1974, издание второе, исправленное, стр. 26.

[7] Вышенский В.А., Карташов Н.В., Михайловский В.И., Ядренко М.И.: *Зборник задач кивеских математических олимпиад*, Издательство при Кивеском государственном университете издательского объединения «Вуща Школа», Киев 1984, стр. 44, 49.

[8] *http://www.math.us.edu.pl/pgladki/faq/node67.html (access 2013-09-29).*

[9] *http://pl.wikipedia.org/wiki/Indukcja_matematyczna (access 2013-09-29).*