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Maximization problem of three tasks operation process subject to constraint of availability in semi-Markov reliability model

Keywords

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Abstract

Semi-Markov decision processes theory delivers the methods which allow to control the operation processes of the systems. The infinite duration semi-Markov decision processes are presented in the chapter. The gain maximization problem of three tasks operation processes subject to constraint of an availability of the semi-Markov reliability model is discussed. The problem is transformed on some linear programing maximization problem.

1. Introduction

In many articles and books we can find applications of semi-Markov (SM) processes in the reliability theory. The most interesting and important books on these issues include monographs (Bernaciak, 2005; Howard, 1971; Jewell, 1963; Limnios & Oprisan, 2001). The semi-Markov decision processes theory delivers methods which give the opportunity to control an operation processes of the systems. We investigate the infinite duration SM decision processes. It was developed by Jewell (Jewell, 1963), Howard (Howard, 1960, 1964, 1971), Main and Osaki (Main & Osaki, 1970) and Gercbakh (Gercbakh, 1969). Those processes are also discussed in (Feinberg, 1994; Grabski, 2015, 2018; Korolyuk & Turbin, 1976) and (Boussemart & Limnios, 2004). The gain maximization problem subject to an availability constraint for a semi-Markov model of the operation in the reliability aspect is discussed in those papers. The problem is transformed on some maximization problem of linear programing. Very important and original scientific work concerning discussed here problem were published by (Boussemart et al., 2001; Boussemart & Limnios, 2004; Beutler & Ross, 1986) and also by (Feinberg,1994). It should be added that a similar problem but for the two task operation was presented at the ICNAAM 2018 conference and the extended abstract is published in the AIP Conference Proceedings (Grabski, 2018). The similar problem for the two task operation as a full paper is also published in AMSDA 2019 Conference Proceedings (Grabski, 2021).

2. Necessary concepts and properties from semi-Markov processes theory

We start from a brief presentation of concepts and properties of the semi-Markov processes theory that are essential in the chapter.

A stochastic $\{X(t): t \geq 0\}$ process with a finite or countable state space S, piecewise constant and right continuous trajectory is said to be a semi-Markov process if there exist non-negative random variables $\tau_0 = 0 < \tau_1 < \tau_2 < \cdots$ such that

$$P(\tau_{n+1} - \tau_n \leq t, X(\tau_{n+1}) = j \,|X(\tau_n)$$

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$= i, \tau_n - \tau_{n-1} \le t_n, \ldots, \tau_1 - \tau_0 \le t_1)$

$= P(\tau_{n+1} - \tau_n \le t, X(\tau_{n+1}) = j \mid X(\tau_n) = i),$

$t \ge 0, n = 1,2, \ldots .$ (1)

Two dimensional sequence

$\{(\tau_{n+1} - \tau_n, X(\tau_{n+1})) : n = 0,1,2, \ldots\}$

is said to be the Markov renewal process associated with the semi-Markov process.
The transition probabilities

$Q_{ij}(t) = P(\tau_{n+1} - \tau_n \le t, X(\tau_{n+1})$

$= j \mid X(\tau_n) = i), t \ge 0,$ (2)

form a matrix

$\boldsymbol{Q}(t) = [Q_{ij}(t) : i, j \in S],$ (3)

that is called the semi-Markov kernel.
To determine semi-Markov process as a model we have to define an initial distribution and all elements of its kernel.
It is easy to notice that the sequence $\{X(\tau_n) :\ n = 0,1, \ldots\}$ is a homogeneous Markov chain with transition probabilities

$p_{ij} = P(X(\tau_{n+1}) = j \mid X(\tau_n) = i) = \lim_{t\to\infty} Q_{ij}(t).$ (4)

This random sequence is called an embedded Markov chain in the semi-Markov process.
The function

$G_i(t) = P(T_i \le t) =$

$P(\tau_{n+1} - \tau_n \le t \mid X(\tau_n) = i) = \sum_{j\in S} Q_{ij}(t)$ (5)

is the CDF distribution of a waiting time T_i denoting the time spent in state i when the successor state is unknown, the function

$F_{ij}(t) =$

$P(\tau_{n+1} - \tau_n \le t \mid X(\tau_n) = i, X(\tau_{n+1}) = j)$

$= \frac{Q_{ij}(t)}{p_{ij}}$ (6)

is the CDF of a random variable T_{ij} that is called a holding time of a state i, if the next state will be j. It is easy to see that

$Q_{ij}(t) = p_{ij} F_{ij}(t).$ (7)

A set S represents the reliability states of the system. This set may be divided into two subset S_+ and S_- where the first contains the “up” states and the second one contains the failed states (“down” states). Those subset form a partition, i.e., $S = S_+ \cup S_-$ and $S_+ \cap S_- = \emptyset$.
Suppose that $i \in S_+$ is an initial state of the process. Conditional reliability functions of a system are defined by the rule

$R_i(t) = P(\forall u \in [0, t],\ X(u) \in S_+ \mid X(0) = i),$

$i \in S_+.$ (8)

The conditional reliability functions satisfy system of integral equations

$R_i(t) = 1 - G_i(t) + \sum_{j \in S_+} \int_0^t R_i(t - x) dQ_{ij}(x)$

$i \in S_+.$ (9)

Passing to the Laplace transforms, we obtain

$\tilde{R}_i(s) = \frac{1}{s} - \tilde{G}_i(s) + \sum_{j \in S_+} \tilde{R}_j(s) \tilde{q}_{ij}(s), i \in S_+,$ (10)

where

$\tilde{R}_i(s) = \int_0^\infty e^{-st} dR_j(t).$ (11)

The conditional means time to failure of the system can be calculated using equalities

$E(\Theta_i) = \lim_{s\to 0^+} \widetilde{R}_i(s), s > 0, i \in S_+.$ (12)

The matrix form of the equation system (9) is

$[\boldsymbol{I} - \widetilde{\boldsymbol{q}}_{S_+}(s)] \cdot \widetilde{\boldsymbol{R}}(s) = \widetilde{\boldsymbol{W}}_{S_+}(s),$ (13)

where

$\widetilde{\boldsymbol{R}}(s) = [\widetilde{R}_i(s) :\ i \in S_+]^T ,$

$$\widetilde{\boldsymbol{W}}_{S_+}(s) = \left[\frac{1}{s} - \tilde{G}_i(s) : i \in S_+\right]^T \tag{14}$$

are one-column matrices and

$$\widetilde{\boldsymbol{q}}_{S_+}(s) = [\tilde{q}_{ij}(s): i, j \in S_+],$$

$$\boldsymbol{I} = [\delta_{ij}: i, j \in S_+] \tag{15}$$

are square matrices.

A random variable Θ_{iS_-}, $i \in S_+$, denoting a first passage time from the state $i \in S_+$ to the subset S_- designates time to failure of a system with initial state $i \in S_+$. A cumulative distribution function of this random variables is denoted as

$$\Phi_{iS_-}(t) = P\left(\Theta_{iS_-} \le t\right),\ i \in S_+, t \ge 0. \tag{16}$$

Between functions $R_i(t)$ and $\Phi_{iS_-}(t)$, $i \in S_+$, $t \ge 0$, we have equalities

$$R_i(t) = 1 - \Phi_{iS_-}(t), i \in S_+, t \ge 0. \tag{17}$$

Under assumptions that are satisfied in considered here problem, the cumulative distribution functions are proper and they are the only solution of the system equations

$$\Phi_{iS_-}(t) = \sum_{j \in S_-} Q_{ij}(t) + \sum_{k \in S_+} \int_0^t \Phi_{iS_-}(t-x)\, dQ_{ik}(x),$$

$$i \in S_+, t \ge 0. \tag{18}$$

Passing to the Laplace-Stieltjes (L-S) transforms, we obtain

$$\tilde{\phi}_{iS_-}(s) = \sum_{j \in S_-} \tilde{q}_{ij}(s) + \sum_{k \in S_+} \tilde{q}_{ik}(s)\, \tilde{\phi}_{kS_-}(s),$$

$$i \in S_+, \tag{19}$$

system of linear equation, where L-S transforms $\tilde{\phi}_{iS_-}(s), i \in S_+$, are unknown.

From (17) it follows that

$$\tilde{R}_i(s) = \frac{1}{s} - \tilde{\phi}_{iS_-}(s), i \in S_+. \tag{20}$$

The system of linear equation (19) is equivalent to a matrix equation

$$[\boldsymbol{I} - \widetilde{\boldsymbol{q}}_{S_+}(s)] \cdot \widetilde{\boldsymbol{\varphi}}_{S_+}(s) = \widetilde{\boldsymbol{b}}_{S_+}(s), \tag{21}$$

where

$$\widetilde{\boldsymbol{q}}_{S_+}(s) = [\tilde{q}_{ij}(s): i, j \in S_+],$$

$$\widetilde{\boldsymbol{\varphi}}_{S_+}(s) = \left[\tilde{\phi}_{iS_-}(s): i \in S_+\right]^T,$$

$$\widetilde{\boldsymbol{b}}_{S_+}(s) = \left[\sum_{j \in S_-} \tilde{q}_{ij}(s): i \in S_+\right]^T.$$

From Theorem 3.2 (Boussemart & Limnios, 2004) it follows that there exist expectations $\left(\Theta_{iS_-}\right)$, $i \in S_+$, and they are unique solutions of the linear system of equations that is equivalent of the matrix equation

$$\left[\boldsymbol{I} - \boldsymbol{P}_{S_+}\right] \cdot \overline{\boldsymbol{Q}}_{S_+} = \overline{\boldsymbol{T}}_{S_+}, \tag{22}$$

where

$$\boldsymbol{P}_{S_+} = [p_{ij}: i, j \in S_+],$$

$$\overline{\boldsymbol{T}}_{S_+} = [E(T_i): i \in S_+],$$

$$\overline{\boldsymbol{\Theta}}_{S_+} = \left[E\left(\Theta_{iS_-}\right): i \in S_+\right]^T.$$

3. Semi-Markov decision process

The concept of Semi-Markov decision process (SMDP) is presented in many books and also in monograph (Grabski, 2015). Notations and definitions come from (Grabski, 2015).

The maximization problem considered in the chapter may be briefly described as finding a strategy $\delta \in D_1 \times D_2 \times \ldots \times D_N$ that maximized the criterion function $g(\delta)$ subject to availability constraint $k(\delta) > \alpha$, where $\alpha \in (0,1]$, $S = \{1,2,\ldots,N\}$ is a state set of considered semi-Markov decision process, $D_i, i \in S$, are sets of decisions and $g(\delta)$ denotes the gain per unit of time as a result of a long operation system.

4. Decision semi-Markov model of operation

4.1. Description and assumption

The working object (device) can perform three types of tasks 1, 2 and 3. A duration of r type of a task is a non-negative random variable ξ_r, $r = 1,2,3$. The working object may be damaged. A time to failure of the object executing a task r

is a non-negative random variable ζ_r, $r = 1,2,3$, with a probability density function $f_{\zeta_r}(x)$, $x \geq 0$, $r = 1,2,3$.

A repair time of the object performing task r is a non-negative random variable η_r, $r = 1,2,3$, governed by a probability density function $f_{\eta_r}(x)$, $x \geq 0$, $r = 1,2,3$. Each repair is a renewal of the object. After the repair is completed, the object starts execution of the task 1 with a probability p_{1_r}, the task 2 with the p_{2_r} and the task 3 with the probability p_{3_r}

$$p_{1_r} + p_{2_r} + p_{3_r} = 1, r = 1,2,3.$$

A duration of an inspection after a task r is a non-negative random variable γ_r, having a PDF $f_{\gamma_r}(x)$, $x \geq 0$, $r = 1,2,3$. After the inspection is completed, the object starts execution of the task 1 with the probability q_{1_r}, the task 2 with probability q_{2_r} and the task 3 with the probability q_{3_r}

$$q_{1_r} + q_{2_r} + q_{3_r} = 1, r = 1,2,3.$$

Furthermore we assume that all random variables and their copies are independent and they have the finite and positive second moments.

4.2. Model construction

We start from introducing operation states of the process:

1 – an object repair after failure during executing of the task 1;
2 – an object repair after failure during executing of the task 2;
3 – an object repair after failure during executing of the task 3;
4 – an object operation, performing of the task 1;
5 – an object operation, performing of the task 2;
6 – an object operation, performing of the task 3;
7 – checking the object technical condition and renewal after the task 1 executing;
8 – checking the object technical condition and renewal after the task 2 executing;
9 – checking the object technical condition and renewal after the task 3 executing.

To construct a decision stochastic process we have to determine sets of decisions (alternatives) for every state.

*D*1: 1 – a normal repair after failure during executing of the task 1,
2 – an expencive repair after failure during executing of the task 1;

*D*2: 1 – a normal repair after failure during executing of the task 2,
2 – an expencive repair after failure during executing of the task 2;

*D*3: 1 – a normal repair after failure during executing of the task 3,
2 – an expencive repair after failure during executing the task 3;

*D*4: 1 – a normal profit per unit of time for the task 1 executing,
2 – a higher profit per unit of time for the task 1 executing;

*D*5: 1 – a normal profit per unit of time for the task 2 executing,
2 – a higher profit per unit of time for the task 2 executing;

*D*6: 1 – a normal profit per unit of time for the task 3 executing,
2 – a higher profit per unit of time for the task 3 executing;

*D*7: 1 – a normal inspection after performingof the task 1,
2 – an expencive inspection after performing of the task 1;

*D*8: 1 – a normal inspection after performing of the task 2,
2 – an expencive inspection after performing of the task 2;

*D*9: 1 – a normal inspection after performing of the task 3,
2 – an expencive inspection after performing of the task 3.

The possible state changes of the process are shown in Figure 1.

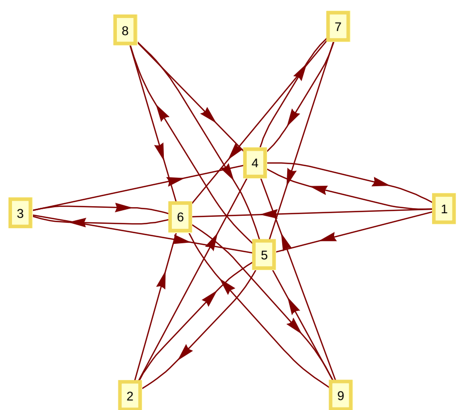

Figure 1. Possible state changes of the operation process.

A model of an object operation is a decision semi-Markov process with a state space $S = \{1,2,\dots,9\}$, sets of actions (decisions) D_1, $D_2,\dots, D_9$. This process is defined by a family of functions matrix that is called a kernel of the decision semi-Markov process. The kernel is determined by the matrix:

$$\boldsymbol{Q}^{(\delta)}(t) = \begin{bmatrix} 0 & 0 & 0 & Q_{14}^{(k)}(t) & Q_{15}^{(k)}(t) & Q_{16}^{(k)}(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{24}^{(k)}(t) & Q_{25}^{(k)}(t) & Q_{26}^{(k)}(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{34}^{(k)}(t) & Q_{35}^{(k)}(t) & Q_{36}^{(k)}(t) & 0 & 0 & 0 \\ Q_{41}^{(k)}(t) & 0 & 0 & 0 & 0 & 0 & Q_{47}^{(k)}(t) & 0 & 0 \\ 0 & Q_{52}^{(k)}(t) & 0 & 0 & 0 & 0 & 0 & Q_{58}^{(k)}(t) & 0 \\ 0 & 0 & Q_{63}^{(k)}(t) & 0 & 0 & 0 & 0 & 0 & Q_{69}^{(k)}(t) \\ 0 & 0 & 0 & Q_{74}^{(k)}(t) & Q_{75}^{(k)}(t) & Q_{76}^{(k)}(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{84}^{(k)}(t) & Q_{85}^{(k)}(t) & Q_{86}^{(k)}(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{94}^{(k)}(t) & Q_{95}^{(k)}(t) & Q_{96}^{(k)}(t) & 0 & 0 & 0 \end{bmatrix}$$

$t \geq 0.$ (23)

The model is constructed if all kernel elements are determined. According to assumptions we calculate elements of the matrix $\boldsymbol{Q}^{(\delta)}(t), t \geq 0$,

$$\begin{aligned}
&Q_{14}^{(k)}(t) = p_{1_1}^{(k)} F_{\eta_1^{(k)}}(t),\ Q_{15}^{(k)}(t) = p_{2_1}^{(k)} F_{\eta_1^{(k)}}(t),\\
&Q_{16}^{(k)}(t) = p_{3_1}^{(k)} F_{\eta_1^{(k)}}(t),\ Q_{24}^{(k)}(t) = p_{1_2}^{(k)} F_{\eta_2^{(k)}}(t),\\
&Q_{25}^{(k)}(t) = p_{2_2}^{(k)} F_{\eta_2^{(k)}}(t),\ Q_{26}^{(k)}(t) = p_{3_2}^{(k)} F_{\eta_2^{(k)}}(t),\\
&Q_{34}^{(k)}(t) = p_{1_3}^{(k)} F_{\eta_3^{(k)}}(t),\ Q_{35}^{(k)}(t) = p_{2_3}^{(k)} F_{\eta_3^{(k)}}(t),\\
&Q_{36}^{(k)}(t) = p_{3_3}^{(k)} F_{\eta_3^{(k)}}(t),\\
&Q_{41}^{(k)}(t) = \int_0^t \left[1 - F_{\zeta_1^{(k)}}(x)\right] dF_{\xi_1^{(k)}}(x)\ ,\\
&Q\,_{47}^{(k)}(t) = \int_0^t \left[1 - F_{\xi_1^{(k)}}(x)\right] dF_{\zeta_1^{(k)}}(x),\\
&\mathrm{Q}_{52}^{(k)}(t) = \int_0^t \left[1 - F_{\zeta_2^{(k)}}(x)\right] dF_{\xi_2^{(k)}}(x),\\
&\mathrm{Q}_{58}^{(k)}(t) = \int_0^t [1 - F_{\xi_2^{(k)}}(x)] dF_{\zeta_2^{(k)}}(x),\\
&\mathrm{Q}_{63}^{(k)}(t) = \int_0^t \left[1 - F_{\zeta_3^{(k)}}(x)\right] dF_{\xi_3^{(k)}}(x),\\
&\mathrm{Q}_{69}^{(k)}(t) = \int_0^t [1 - F_{\xi_3^{(k)}}(x)] dF_{\zeta_3^{(k)}}(x),\\
&Q_{74}^{(k)}(t) = q_{1_1}^{(k)} F_{\gamma_1^{(k)}}(t),\ Q_{15}^{(k)}(t) = q_{2_1}^{(k)} F_{\gamma_1^{(k)}}(t),\\
&Q_{16}^{(k)}(t) = q_{3_1}^{(k)} F_{\gamma_1^{(k)}}(t),\ Q_{84}^{(k)}(t) = q_{1_2}^{(k)} F_{\gamma_2^{(k)}}(t),\\
&Q_{25}^{(k)}(t) = q_{2_2}^{(k)} F_{\gamma_2^{(k)}}(t),\ Q_{26}^{(k)}(t) = q_{3_2}^{(k)} F_{\gamma_2^{(k)}}(t),\\
&Q_{84}^{(k)}(t) = q_{1_3}^{(k)} F_{\gamma_3^{(k)}}(t),\ Q_{35}^{(k)}(t) = q_{2_3}^{(k)} F_{\gamma_3^{(k)}}(t),\\
&Q_{36}^{(k)}(t) = q_{3_3}^{(k)} F_{\gamma_3^{(k)}}(t).
\end{aligned} \tag{24}$$

The transition probability matrix of the embedded Markov chain $\{X(\tau_n): n \in \mathbb{N}_0\}$ we obtain using equality

$$p_{ij}^{(k)} = \lim_{t\to\infty} Q_{ij}^{(k)}(t). \tag{25}$$

The below matrix represents the transition probability matrix of the embedded Markov chain

$$\boldsymbol{P}^{(\delta)} = \begin{bmatrix} 0 & 0 & 0 & p_{14}^{(k)} & p_{15}^{(k)} & p_{16}^{(k)} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{24}^{(k)} & p_{25}^{(k)} & p_{26}^{(k)} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{34}^{(k)} & p_{35}^{(k)} & p_{36}^{(k)} & 0 & 0 & 0 \\ p_{41}^{(k)} & 0 & 0 & 0 & 0 & 0 & p_{47}^{(k)} & 0 & 0 \\ 0 & p_{52}^{(k)} & 0 & 0 & 0 & 0 & 0 & p_{58}^{(k)} & 0 \\ 0 & 0 & p_{63}^{(k)} & 0 & 0 & 0 & 0 & 0 & p_{69}^{(k)} \\ 0 & 0 & 0 & p_{74}^{(k)} & p_{75}^{(k)} & p_{76}^{(k)} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{84}^{(k)} & p_{85}^{(k)} & p_{86}^{(k)} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{94}^{(k)} & p_{95}^{(k)} & p_{96}^{(k)} & 0 & 0 & 0 \end{bmatrix} \tag{26}$$

From (24) and (25) we obtain

$p_{14}^{(k)} = p_{1_1}, p_{15}^{(k)} = p_{2_1}, p_{16}^{(k)} = p_{3_1}, p_{24}^{(k)} = p_{1_2},$
$p_{25}^{(k)} = p_{2_2}, p_{26}^{(k)} = p_{3_2}, p_{34}^{(k)} = p_{1_3}, p_{35}^{(k)} = p_{2_3},$
$p_{36}^{(k)} = p_{3_3}, p_{41}^{(k)} = \int_0^\infty \left[1 - F_{\zeta_1^{(k)}}(x)\right] dF_{\xi_1^{(k)}}(x)\ ,$

$p_{47}^{(k)} = \int_0^\infty [1 - F_{\xi_1^{(k)}}(x)]dF_{\zeta_1^{(k)}}(x),$
$p_{52}^{(k)} = \int_0^\infty \left[1 - F_{\zeta_2^{(k)}}(x)\right] dF_{\xi_2^{(k)}}(x),$
$p_{58}^{(k)} = \int_0^\infty [1 - F_{\xi_2^{(k)}}(x)]dF_{\zeta_2^{(k)}}(x),$
$p_{63}^{(k)} = \int_0^\infty \left[1 - F_{\zeta_3^{(k)}}(x)\right] dF_{\xi_3^{(k)}}(x),$
$p_{69}^{(k)} = \int_0^\infty [1 - F_{\xi_3^{(k)}}(x)]dF_{\zeta_3^{(k)}}(x),$
$p_{74}^{(k)} = q_{1_1}, p_{75}^{(k)} = q_{2_1}, p_{76}^{(k)} = q_{3_1},$
$p_{84}^{(k)} = q_{1_2}, p_{85}^{(k)} = q_{2_3}, p_{86}^{(k)} = q_{3_2},$
$p_{94}^{(k)} = q_{1_3}, p_{95}^{(k)} = q_{2_3}, p_{96}^{(k)} = q_{3_3}.$ (27)

5. Linear programing method

Mine and Osaki (Mine & Osaki, 1970) presented linear programming method for solving the problem of optimization without additional constraints. The problem of optimization with an object availability constraints is investigated in this chapter.

Stationary probabilities $\pi_j(\delta), j \in S$, for every decision $k \in D_i$ satisfy the following linear system of equations

$$\sum_{i \in S} \pi_i(\delta) p_{ij}^{(k)} = \pi_j(\delta), \sum_{i \in S} \pi_i(\delta) = 1,$$

$$\pi_j(\delta) > 0, j \in S, \quad (28)$$

where

$$p_{ij}^{(k)} = \lim_{t \to \infty} Q_{ij}^{(k)}(t), i, j \in S. \quad (29)$$

Let $a_j^{(k)}$ be a probability that in the state $j \in S$ has been taken decision $k \in D_j$. It is obvious that

$$\sum_{k \in D_j} a_j^{(k)} = 1, 0 \le a_j^{(k)} \le 1, j \in S. \quad (30)$$

The criterion function and constraints can be written as

$$g(\delta) = \frac{\sum_{j \in S} \sum_{k \in D_j} a_j^{(k)} \pi_j(\delta) m_j^{(k)} r_j^{(k)}}{\sum_{j \in S} \sum_{k \in D_j} a_j^{(k)} \pi_j(\delta) m_j^{(k)}}, \quad (31)$$

$$k(\delta) = \frac{\sum_{j \in S_+} \sum_{k \in D_j} a_j^{(k)} \pi_j(\delta) m_j^{(k)}}{\sum_{j \in S} \sum_{k \in D_j} a_j^{(k)} \pi_j(\delta) m_j^{(k)}} > \alpha, \quad (32)$$

where $m_j^{(k)}$ is a mean value of the waiting time in state j under decision k and $r_j^{(k)}$ is a reward received by staying at a state j per one unit of time. Finally, we obtain the following problem of linear programming:

Find stationary strategy δ maximizing the function

$$g(\delta) = \sum_{i \in S} \sum_{k \in D_i} u_i^{(k)} y_i^{(k)} \quad (33)$$

under constraints

$$\sum_{j \in S_+} \sum_{k \in D_j} m_j^{(k)} y_j^{(k)} \ge \alpha, \quad (34)$$

(or $\sum_{j \in S_-} \sum_{k \in D_j} m_j^{(k)} y_j^{(k)} \le 1 - \alpha, S_- = S - S_+$) (35)

$$\sum_{k \in D_j} y_j^{(k)} - \sum_{i \in S} \sum_{k \in D_i} p_{ij}^{(k)} y_i^{(k)} = 0, \quad (36)$$

$$\sum_{j \in S} \sum_{k \in D_j} m_j^{(k)} y_j^{(k)} = 1, \quad (37)$$

$$y_j^{(k)} = \frac{a_j^{(k)} \pi_j(\delta)}{\sum_{j \in S} \sum_{k \in D_j} a_j^{(k)} \pi_j(\delta) m_j^{(k)}} \ge 0, \quad (38)$$

$j \in S, k \in D_j.$

The optimal stationary strategy consists of decisions determined by probabilities

$$a_j^{(k)} = \frac{y_j^{(k)}}{\sum_{k \epsilon D_j} y_j^{(k)}}. \quad (39)$$

In the model the set of "up" states is $S_+ = \{4, 5,6\}$ and the "down" states set is $S_- = S - S_+ = \{1,2,3,7,8,9\}$.

6. Numerical example

Decision variables:

$y_1^{(1)}, y_1^{(2)}, y_2^{(1)}, y_2^{(2)}, y_3^{(1)}, y_3^{(2)}, y_4^{(1)}, y_4^{(2)}, y_5^{(1)}, y_5^{(2)}, y_6^{(1)}, y_6^{(2)}, y_7^{(1)}, y_7^{(2)}, y_8^{(1)}, y_8^{(2)}, y_9^{(1)}, y_9^{(2)}.$

Known parameters:

$u_1^{(1)}, u_1^{(2)}, u_2^{(1)}, u_2^{(2)}, u_3^{(1)}, u_3^{(2)}, u_4^{(1)}, u_4^{(2)}, u_5^{(1)}, u_5^{(2)}, u_6^{(1)}, u_6^{(2)}, u_7^{(1)}, u_7^{(2)}, u_8^{(1)}, u_8^{(2)}, u_9^{(1)}, u_9^{(2)}.$

$m_1^{(1)}, m_1^{(2)}, m_2^{(1)}, m_2^{(2)}, m_3^{(1)}, m_3^{(2)}, m_4^{(1)}, m_4^{(2)}, m_5^{(1)}, m_5^{(2)}, m_6^{(1)}, m_6^{(2)}, m_7^{(1)}, m_7^{(2)}, m_8^{(1)}, m_8^{(2)}, m_9^{(1)}, m_9^{(2)}$.

In this case the formulas (33)–(38) take the following form.
Criterion function:

$$g(\delta) = u_1^{(1)}y_1^{(1)} + u_1^{(2)}y_1^{(2)} + u_2^{(1)}y_2^{(1)} + u_2^{(2)}y_2^{(2)} + u_3^{(1)}y_3^{(1)} + u_3^{(2)}y_3^{(2)} + u_4^{(1)}y_4^{(1)} + u_4^{(2)}y_4^{(2)} + u_5^{(1)}y_5^{(1)} + u_5^{(2)}y_5^{(2)} + u_6^{(1)}y_6^{(1)} + u_6^{(2)}y_6^{(2)} + u_7^{(1)}y_7^{(1)} + u_7^{(2)}y_7^{(2)} + u_8^{(1)}y_8^{(1)} + u_8^{(2)}y_8^{(2)} + u_9^{(1)}y_9^{(1)} + u_9^{(2)}y_9^{(2)}.$$

Constraints:

$j = 1$:
$$y_1^{(1)} + y_1^{(2)} - \left(p_{41}^{(1)}y_4^{(1)} + p_{41}^{(2)}y_4^{(2)}\right) = 0,$$

$j = 2$:
$$y_2^{(1)} + y_2^{(2)} - \left(p_{52}^{(1)}y_5^{(1)} + p_{52}^{(2)}y_5^{(2)}\right) = 0,$$

$j = 3$:
$$y_3^{(1)} + y_3^{(2)} - \left(p_{63}^{(1)}y_6^{(1)} + p_{63}^{(2)}y_6^{(2)}\right) = 0,$$

$j = 4$:
$$y_4^{(1)} + y_4^{(2)} - (p_{14}^{(1)}y_1^{(1)} + p_{14}^{(2)}y_1^{(2)} + p_{24}^{(1)}y_2^{(1)} + p_{24}^{(2)}y_2^{(2)} + p_{34}^{(1)}y_3^{(1)} + p_{34}^{(2)}y_3^{(2)} + p_{74}^{(1)}y_7^{(1)} + p_{74}^{(2)}y_7^{(2)} + p_{84}^{(1)}y_8^{(1)} + p_{84}^{(2)}y_8^{(2)} + p_{94}^{(1)}y_9^{(1)} + p_{94}^{(2)}y_9^{(2)}) = 0,$$

$j = 5$:
$$y_5^{(1)} + y_5^{(2)} - (p_{15}^{(1)}y_1^{(1)} + p_{15}^{(2)}y_1^{(2)} + p_{25}^{(1)}y_2^{(1)} + p_{25}^{(2)}y_2^{(2)} + p_{35}^{(1)}y_3^{(1)} + p_{35}^{(2)}y_3^{(2)} + p_{75}^{(1)}y_7^{(1)} + p_{75}^{(2)}y_7^{(2)} + p_{85}^{(1)}y_8^{(1)} + p_{85}^{(2)}y_8^{(2)} + p_{95}^{(1)}y_9^{(1)} + p_{95}^{(2)}y_9^{(2)}) = 0,$$

$j = 6$:
$$y_6^{(1)} + y_6^{(2)} - (p_{16}^{(1)}y_1^{(1)} + p_{16}^{(2)}y_1^{(2)} + p_{26}^{(1)}y_2^{(1)} + p_{26}^{(2)}y_2^{(2)} + p_{36}^{(1)}y_3^{(1)} + p_{36}^{(2)}y_3^{(2)} + p_{76}^{(1)}y_7^{(1)} + p_{76}^{(2)}y_7^{(2)} + p_{86}^{(1)}y_8^{(1)} + p_{86}^{(2)}y_8^{(2)} + p_{96}^{(1)}y_9^{(1)} + p_{96}^{(2)}y_9^{(2)}) = 0,$$

$j = 7$:
$$y_7^{(1)} + y_7^{(2)} - \left(p_{47}^{(1)}y_4^{(1)} + p_{47}^{(2)}y_4^{(2)}\right) = 0,$$

$j = 8$:
$$y_8^{(1)} + y_8^{(2)} - \left(p_{58}^{(1)}y_5^{(1)} + p_{58}^{(2)}y_5^{(2)}\right) = 0,$$

$j = 9$:
$$y_9^{(1)} + y_9^{(2)} - \left(p_{69}^{(1)}y_6^{(1)} + p_{69}^{(2)}y_6^{(2)}\right) = 0,$$

$$m_1^{(1)}y_1^{(1)} + m_1^{(2)}y_1^{(2)} + m_2^{(1)}y_2^{(1)} + m_2^{(2)}y_2^{(2)} + m_3^{(1)}y_3^{(1)} + m_3^{(2)}y_3^{(2)} + m_4^{(1)}y_4^{(1)} + m_4^{(2)}y_4^{(2)} + m_5^{(1)}y_5^{(1)} + m_5^{(2)}y_5^{(2)} + m_6^{(1)}y_6^{(1)} + m_6^{(2)}y_6^{(2)} + m_7^{(1)}y_7^{(1)} + m_7^{(2)}y_7^{(2)} + m_8^{(1)}y_8^{(1)} + m_8^{(2)}y_8^{(2)} + m_9^{(1)}y_9^{(1)} + m_9^{(2)}y_9^{(2)} = 1,$$

$$m_4^{(1)}y_4^{(1)} + m_4^{(2)}y_4^{(2)} + m_5^{(1)}y_5^{(1)} + m_5^{(2)}y_5^{(2)} + m_6^{(1)}y_6^{(1)} + m_6^{(2)}y_6^{(2)} \geq \alpha,$$

$$y_i^{(k)} \geq 0, i = 1,2,\dots,9, k = 1,2.$$

We assume $\alpha = 0.742$.

The gain parameters and transition probabilities of the semi-Markov decision process for this example are given in Tables 1–2.

Table 1. The gain parameters

State i	Decision k	$m_i^{(k)}$	$r_i^{(k)}$	$u_i^{(k)}$
1	1	22.5	-150	-3375
	2	24.0	-180	-4320
2	1	20.5	-150	-3075
	2	22.5	-180	-4050
3	1	20	-150	-3000
	2	22.5	-180	-4050
4	1	72	1200	86400
	2	66	1600	105600
5	1	56	1250	95000
	2	52	1400	100800
6	1	48	1200	93600
	2	54	148	106560
7	1	5.2	-150	-780
	2	6.8	-220	-1496
8	1	7.2	-120	-864
	2	6.6	-135	-891
9	1	6.0	-128	-768
	2	6.5	-140	-910

Table 2. The transition probabilities of the semi-Markov decision process

State i	Decision k	$p_{i1}^{(k)}$	$p_{i2}^{(k)}$	$p_{i3}^{(k)}$	$p_{i4}^{(k)}$	$p_{i5}^{(k)}$	$p_{i6}^{(k)}$	$p_{i7}^{(k)}$	$p_{i8}^{(k)}$	$p_{i9}^{(k)}$
1	1	0	0	0	0.42	0.30	0.28	0	0	0
	2	0	0	0	0.44	0.28	0.28	0	0	0
2	1	0	0	0	0.38	0.40	0.22	0	0	0
	2	0	0	0	0.40	0.30	0.30	0	0	0
3	1	0	0	0	0.29	0.33	0.38	0	0	0
	2	0	0	0	0.32	0.33	0.35	0	0	0
4	1	0.97	0	0	0	0	0	0.03	0	0
	2	0.99	0	0	0	0	0	0.01	0	0
5	1	0	0.96	0	0	0	0	0	0.04	0
	2	0	0.98	0	0	0	0	0	0.02	0
6	1	0	0	0.97	0	0	0	0	0	0.03
	2	0	0	0.99	0	0	0	0	0	0.01
7	1	0	0	0	0.41	0.31	0.28	0	0	0
	2	0	0	0	0.43	0.29	0.28	0	0	0
8	1	0	0	0	0.37	0.41	0.22	0	0	0
	2	0	0	0	0.42	0.30	0.28	0	0	0
9	1	0	0	0	0.29	0.33	0.38	0	0	0
	2	0	0	0	0.32	0.33	0.35	0	0	0

Using MATHEMATICA computer system and the data from Tables 1–2 we obtain solution of the problem:

$y_1^{(1)} = 0.00445005,\ y_1^{(2)} = 0;$
$y_2^{(1)} = 0.00416844,\ y_2^{(2)} = 0;$
$y_3^{(1)} = 0.00354532,\ y_3^{(2)} = 0;$
$y_4^{(1)} = 0.00441825,\ y_4^{(2)} = 0.000166008;$
$y_5^{(1)} = 0,\ y_5^{(2)} = 0.00425351;$
$y_6^{(1)} = 0.000332827,\ y_6^{(2)} = 0.00325502;$
$y_7^{(1)} = 0.000134208,\ y_7^{(2)} = 0;$
$y_8^{(1)} = 0,\ y_8^{(2)} = 0.0000850701;$
$y_9^{(1)} = 0.0000425351,\ y_9^{(2)} = 0.$

From (39) we obtain probabilities

$a_1^{(1)} = 1,\ a_1^{(2)} = 0;$
$a_2^{(1)} = 1,\ a_2^{(2)} = 0;$
$a_3^{(1)} = 1,\ a_3^{(2)} = 0;$
$a_4^{(1)} = 0.963787378,\ a_4^{(2)} = 0.036212622;$
$a_5^{(1)} = 0,\ a_5^{(2)} = 1;$
$a_6^{(1)} = 0.092765104,\ a_6^{(2)} = 0.907234896;$
$a_7^{(1)} = 1,\ a_7^{(2)} = 0;$
$a_8^{(1)} = 0,\ a_8^{(2)} = 1;$
$a_9^{(1)} = 1,\ a_9^{(2)} = 0.$

The vector of the optimal action in each step is

$$\delta = (1,1,1,\gamma_4,2,\gamma_6,1,2,1),$$

where

$$\gamma_4 = \begin{cases} 1 \text{ with probability } 0.963787378 \\ 2 \text{ with probability } 0.036212622 \end{cases}$$

and

$$\gamma_6 = \begin{cases} 1 \text{ with probability } 0.092765104 \\ 2 \text{ with probability } 0.907234896. \end{cases}$$

In MATHEMATICA computer system, linear programing is determined only for the minimum problem. The minimum of the expected cost for one step of the operation in this case is

$$\begin{aligned} c(\delta) &= 3375 \cdot 0.00445005 + 4320 \cdot 0.00416844 \\ &+ 3075 \cdot 0.00354532 + 4050 \cdot 0.00441825 \\ &+ 3000 \cdot 0.036212622 + 4050 \cdot 0.00354532 \\ &- 86400 \cdot 0.963787378 \\ &- 10560 \cdot 0.036212622 \\ &- 95000 \cdot 0.036212622 \\ &- 100800 \cdot 0.00425351 \\ &- 93600 \cdot 0.092765104 \\ &- 106560 \cdot 0.907234896 \\ &+ 780 \cdot 0.000134208 + 1496 \cdot 0.000134208 \\ &+ 864 \cdot 0.0000850701 + 891 \cdot 0.000134208 \\ &+ 768 \cdot 0.0000850701 \\ &+ 910 \cdot 0.0000425351 = -10550.7. \end{aligned}$$

The maximum of the expected gain for one step of the operation in this case is expressed by a number opposite to the corresponding minimum expected cost

$$g(\delta) = -c(\delta) = 10550.7.$$

For availability parameter $\alpha = 0.743$ no solution subject to above constraints.

It should be mentioned that the Laplace transform of reliability function can be found for an optimal stationary strategy. A matrix equation (13) gives this possiblity.

Consider the model with the set of "up" states $S_+ = \{4,5,6,7,8,9\}$ and the set of "down" states $S_- = S - S_+ = \{1, 2, 3\}$. For simplicity we accept $\delta = (1, 1, 1, 1, 2, 2, 1, 2, 1)$.

From Theorem 3.2 (Boussemart & Limnios, 2004) it follows that there exist expectations $E(\Theta_{iS_-}), i \in S_+$, and they are unique solutions of the linear system of equations that are equivalent to the matrix equation

$$\left[\boldsymbol{I} - \boldsymbol{P}_{S_+}^{(\delta)}\right] \cdot \overline{\boldsymbol{Q}}_{S_+} = \overline{\boldsymbol{T}}_{S_+}$$

where

$$\boldsymbol{P}_{S_+}^{(\delta)} = \begin{bmatrix} 0 & 0 & 0 & p_{47}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{58}^{(2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & p_{69}^{(2)} \\ p_{74}^{(1)} & p_{75}^{(1)} & p_{76}^{(1)} & 0 & 0 & 0 \\ p_{84}^{(2)} & p_{85}^{(2)} & p_{86}^{(2)} & 0 & 0 & 0 \\ p_{94}^{(1)} & p_{95}^{(1)} & p_{96}^{(1)} & 0 & 0 & 0 \end{bmatrix}$$

$$\overline{\boldsymbol{\Theta}}_{S_+}^{(\delta)} = \left[E\left(\Theta_{iS_-}^{(k)}\right): i \in S_+\right]^T,$$

$$\overline{\boldsymbol{T}}_{S_+} = \left[E\left(T_i^{(k)}\right): i \in S_+\right] = \left[m_i^{(k)}: i \in S_+\right]$$

$$= [72, 52, 54, 5.2, 6.6, 6.0]^T \text{ [h]}.$$

Then, after substituting accordingly numbers,

$$\overline{\boldsymbol{\Theta}}_{S_+}^{(\delta)} = \left[\boldsymbol{I} - \boldsymbol{P}_{S_+}^{(\delta)}\right]^{-1} \cdot \overline{\boldsymbol{T}}_{S_+}$$

$$= [74.022, 53.38, 54.659, 67.401, 69.008, 65.852]^T \text{[h]}.$$

Under asumption that initial state is 4, the conditional expected value

$$E(\Theta_{4S_-}) = 74.022 \text{ [h]}$$

means expectation of the time to failure of the operation process.

7. Conclusion

The semi-Markov decision processes theory provides the possibility to formulate and solve the optimization problems that can be modelled by SM processes. In such kind of problems we choose the process that brings the largest profit or smallest cost. If the semi-Markov process describing the evolution of the real system in a long time satisfies the assumptions of the limit theorem, we can use the results of the infinite duration SM decision processes theory. An algorithm that allows finding the best strategy is equivalent to the some problem of linear programing. The gain optimization problem subject to the availability constraint for the semi-Markov model of operation is considered and solved.

From Theorem 5.5 (Mine & Osaki, 1970) for the problem without additional constraints it follows, that for every $j \in S$ exists exactly one $k \in D_j$ such that $y_j^{(k)} > 0$. For the gain optimization problem subject to constraint of availability this theorem is not true. The optimal stationary strategy can contain the vectors with mixed decisions. This fact extends the previously known results.

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