Application of Rank Controlled Differential Quadrature Method for Solving an Infinite Steel Plate Cooling Problem

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Abstract

Rank Controlled Differential Quadrature method is a numerical method that allows to approximate the partial derivatives that appears in partial differential equations. Those equations with proper geometrical, physical, initial and boundary conditions make mathematical models of physical process. The heat transfer process is governed by Fourier – Kirchhoff equation, which is parabolic Partial Differential Equation.

In this paper authors present the steel plate cooling problem. At the beginning of the process plate is heated up to 450 °C and is cooled to ambient temperature. The cooling of the plate is basic heat transfer problem. If the plates dimensions has proper proportions such problem may be described as one dimensional and solved exactly. The mathematical model and exact solution is given in the work. Authors apply the Rank Controlled Differential Quadrature to approximate derivatives in Fourier – Kirchhoff equation and in boundary conditions. After changing derivatives into quadrature formulation set of algebraic equations is obtained. Substituting thermo-physical parameters numerical model is obtained. The computer program was prepared to solve the problem numerically. Results of simulation are confronted with the exact ones. Error value at each time step as well as error value increase rate for examined numerical method is analyzed.

Keywords: application of information technology to the foundry industry, solidification process, exact solution, Rank Controlled Differential Quadrature, numerical modelling.

1. Introduction

The heat transfer phenomenon is described by Fourier – Kirchhoff (F-K) Partial Differential Equation (PDE) with geometrical, physical, boundary and initial conditions [1-5]. In most cases it is impossible to find the analytical solution of this system, as the solving requires complex transformations which often leads to non-elementary functions. While modelling foundry processes numerical methods are used [4-8]. Basing on approximation methods coupled equations are being solved step by step. The obtained solution is laden with method error [1, 9, 10]. Developing of numerical methods aims on goal that is to define method that allows to find high accuracy solution in restricted time [10, 12].

Numerical methods are used for solving heat transfer problems that appears in industry. However in some cases it is possible to introduce mathematical model that describe simple heat transfer problem which can be solved exactly [11]. An example of such problem is cooling of infinite plate under assumption of constant thermo-physical parameters [1, 2]. Knowledge of exact solution allows one to analysis of numerical solution quality.

In this paper the application of a Rank Controlled Differential Quadrature (RCDQ) method for solving heat transfer problems is analyzed [10]. The RCDQ method is a numerical method for solving PDEs [9, 10]. This method come into being as the adaptation of Differential Quadrature method [12, 13] for solving heat transfer problems on dense, equidistant grids. The aim of this work is to apply the Rank Controlled Differential Quadrature method for solving an infinite steel plate cooling problem and confront numerical solution with the exact one.

2. Mathematical and numerical model. Exact solution

2.1. Mathematical model

The problem of cooling infinite plate made of uniform material is one of the problems that can be solved exactly. Under assumption of infinite geometry of the plate problem may be described with one-dimensional F-K equation [1, 2]. After this simplification the considered problem is symmetrical. In order to reduce the computation effort only half of the domain is taken under consideration. Mathematical model for this problem is:

$$
\begin{cases}
\dfrac{\partial T(x,\tau)}{\partial \tau} = \dfrac{\lambda}{c_p \rho} \dfrac{\partial^2 T(x,\tau)}{\partial x^2}, & \tau \geq 0, \quad x \in [0,\ L], \\[2mm]
\dfrac{\partial T(0,\tau)}{\partial x} = 0, \\[2mm]
-\lambda \dfrac{\partial T(L,\tau)}{\partial x} = \hat{\alpha}\big(T(L,\ \tau) - T_{amb}\big), \\[2mm]
T(x,0) = T_{init}, & x \in [0,\ L],
\end{cases}
\tag{1}
$$

where: λ [W m^{-1} K^{-1}] denotes thermal conductivity, c_p [J kg^{-1} K^{-1}] specific heat, ρ [kg m^{-3}] density, $\hat{\alpha}$ [W m^{-2} K^{-1}] heat transfer coefficient, $T(x,\tau)$ [K] function of temperature values, x [m] spatial variable, τ [s] time variable, $2L$ [m] thickness of plate, T_{init}, T_{amb} [K] initial and ambient temperature respectively.

2.2. Exact solution

Given model (eq. 1) under assumption of constant thermo-physical parameters can be solved exactly [2, 11]. The solution is function of temperature in selected point of spatial and time domain (x, τ):

$$
T(x,\tau) = T_{amb} +
$$
$$
2(T_0 - T_{amb}) \sum_{n=1}^{\infty} \frac{2\sin(\mu_n)}{\mu_n + \sin(\mu_n)\cos(\mu_n)} \cos\left(\mu_n \frac{x}{L}\right) \exp(-\mu_n Fo),
\tag{2}
$$

where: $Fo = \dfrac{\lambda \tau}{c_p \rho L^2}$ [K m^{-2}] is Fourier's number and μ_n is nth solution of equation:

$$
ctg(\mu_n) = \frac{\mu_n}{Bi}, \quad (n-1)\pi < \mu_n (n-0,5)\pi, \ i = 1,2,\ldots
\tag{3}
$$

To find selected values of T in points (x, τ) computer program was written. Procedures that allow to find solutions of equation 3, and approximate value of series that appears in equation 2 were implemented. The level of sum (eq. 2) accuracy may be controlled with selection of addends number. Their number was selected to be big enough to reach machine numerical representation of floating point numbers accuracy.

2.3. The Rank Controlled Differential Quadrature Method

The f function defined in $[a, b]$ interval is considered, a set of N discrete coordinates x_i:

$$
S = \left\{ x_i : a = x_1 < x_2 < \ldots < x_N = b \right\}
\tag{4}
$$

is the grid introduced in the computational domain.

The Rank Controlled Differential Quadrature is numerical method for spatial partial derivative of unknown field function f approximation as a linear weighted sum of the function values at grid points:

$$
\frac{\partial^n f}{\partial x^n}(x_i) = \sum_{j=S1(i)}^{j=S2(i)} \omega_{i,j}^{(n)} f(x_j)
\tag{5}
$$

where: $\dfrac{\partial^n f}{\partial x^n}(x_i)$ denotes the nth derivative of the function f with respect to spatial derivative x in point x_i; $\omega_{i,j}^{(n)}$ are the quadrature weighting coefficients and $f(x_j)$ are the function values at the x_j.

The RCDQ method is the modification of DQ method [10, 12]. Improving the DQ leads to increase computation accuracy, lowering the computational effort and overcome the high Lebesgue constant problem [9, 10] that may cause numerical instability.

For each point, i, of the grid S (Eq. 4) the rank distribution representation is defined, $R_{DQ}(i)$. With this function the limitations for grid points indices $S1$, $S2$ are calculated as follows:

$$
\begin{cases}
i < \frac{N}{2} \\[1mm]
S1(i) = \max\left(i - \tfrac{1}{2}\big(2 + R_{DQ}(i)\big), 1\right) \\[1mm]
S2(i) = S1(i) + R_{DQ}(i) + 2
\end{cases}
\tag{6}
$$

$$
\begin{cases}
i \geq \frac{N}{2} \\[1mm]
S1(i) = \min\left(i - \tfrac{1}{2}\big(2 + R_{DQ}(i)\big), N - R_{DQ}(i) - 1\right) \\[1mm]
S2(i) = S1(i) + R_{DQ}(i) + 2
\end{cases}
\tag{7}
$$

With such defined limits $S1$, $S2$ formulations for weighting coefficients in formula for approximation of first derivative (eq. 5) can be given as follows:

$$
\begin{cases}
\omega_{i,j}^{(1)} = \dfrac{1}{x_j - x_i} \prod_{\substack{k=S1 \\ k \neq i,j}}^{S2} \dfrac{(x_i - x_k)}{(x_j - x_k)}, & j \in [S1, S2], i \neq j, \\[3mm]
\omega_{i,j}^{(1)} = 0, & j \notin [S1, S2], \\[3mm]
\omega_{i,i}^{(1)} = -\sum_{k=S1}^{S2} \omega_{i,k}^{(1)}.
\end{cases}
\tag{8}
$$

$$\begin{cases} \omega_{i,j}^{(2)} = 2 \cdot \omega_{i,i}^{(1)} \cdot \left(\omega_{i,i}^{(1)} - \frac{1}{x_i - x_j} \right), & j \in [S1, S2], i \neq j, \\ \omega_{i,j}^{(2)} = 0, & j \notin [S1, S2], \\ \omega_{i,i}^{(2)} = -\sum_{k=0}^{N} \omega_{i,k}^{(2)}. \end{cases} \quad (9)$$

and for second derivative approximation weighted coefficients can be calculated with formulas:

Further derivatives can be approximated recursively in the same way as it was done for DQ [12, 13].

The most important parameter of this method is rank distribution function. Setting it correctly leads to very quick and accurate method that can be applied to heat transfer problems.

In this paper rank distribution grows in 10 nodes per side at the both sides near to boundary of the computational domain (eq. 6, 7). The lowest rank, R_{min}, is at the domain boundary node. Rank grows to the centre to reach the highest rank, R_{max}. In the remaining nodes in the centre of domain rank is set to be constant, equal to R_{max}.

2.4. Numerical model

Numerical approximation of infinite steel plate problem (eq. 1) with RCDQ method was performed. The thermo-physical data after [1] is gathered in Table 1.

Table 1. Thermo-physical, initial, boundary and geometry parameters of chosen steel [1]

Parameter	Value	
λ	35	W m^{-1} K^{-1}
c_p	690	J kg^{-1} K^{-1}
ρ	7500	kg m^{-3}
$\hat{\alpha}$	250	W m^{-2} K^{-1}
T_{init}	450	°C
T_{amb}	23	°C
L	0.1	m

Numerical model base on the mathematical model with defined thermo-physical, initial, boundary and geometric parameters (Table 1). Next stage is to introduce set of discrete points – grid points. In presented paper $N = 300$ nodes equidistant grid (eq. 3) was chosen. On the grid RCDQ method coefficients were calculated. Time domain was divided into equidistant time steps $\Delta \tau = 0.5 \, 10^{-7}$ s. For such discretization parameters set of algebraic equations define the numerical model:

$$\begin{cases} \dfrac{T_i^{k+1} - T_i^k}{\Delta \tau} = \dfrac{\lambda}{c_p \rho} \sum_{j=S1(i)}^{j=S2(i)} \omega_{i,j}^{(2)} T_j^k, & i = 1, 2, \dots, N, \\ \sum_{j=1}^{j=S2(1)} \omega_{1,j}^{(1)} T_j^k = 0, \\ -\lambda \sum_{j=S1(N)}^{N} \omega_{N,j}^{(1)} T_j^k = \hat{\alpha} \left(T_N^k - T_{amb} \right), \\ T_i^0 = T_{init}, & i = 1, 2, \dots, N. \end{cases} \quad (10)$$

After substituting parameters gathered in Table 1, set of algebraic equations is obtained. Their solution is matrix of temperature values at discrete grid points at each layer of time.

3. Results and discussion

Basing on the numerical model (eq. 10, Table 1) computer program was prepared. Results of numerical simulations are shown in this chapter.

In the figure 1 comparison of numerical and exact solution of infinite steel plate cooling problem is shown. It can be observed that difference between those two solutions is very small. This confirm that RCDQ method is numerical method of very high accuracy.
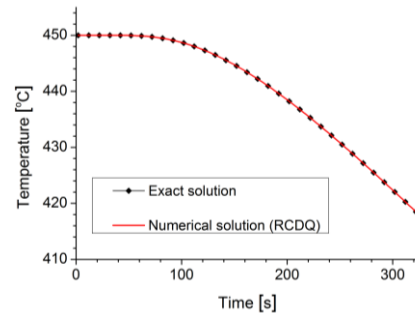

Fig. 1. Comparison of numerical and exact solution of infinite plate cooling problem, $R_{min} = 5$, $R_{max} = 11$ as numerical method was chosen.

Further analyze of the method accuracy was performed on the base of obtained numerical data. Knowing the accurate solution percentage relative error may be introduced:

$$\varepsilon_i = \frac{\left| T_i^{approx} - T_i^{exact} \right|}{\left| T_i^{exact} \right|} \cdot 100\%. \quad (11)$$

where i denotes the grid node number.

For each time step average value of percentage relative error was calculated and divided by the nodes number. In result mean relative percentage error was found at each computation step, which may be observed in figure 2. The observed parameter shows the dynamic of error growth during simulation. This curve is affected by the summation of errors committed at each time step of the computations.
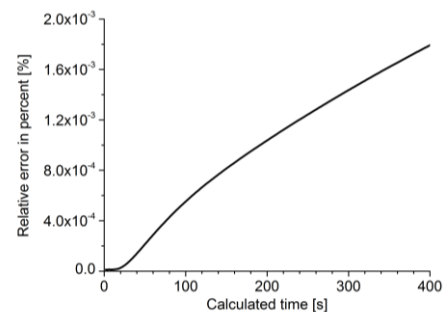

Fig. 2. Mean relative percentage error increase during numerical simulation of infinite steel plate cooling. Error values are shown in calculated moments of time

Rate of error growth may be determine on the base of the mean relative percentage error by differentiate with respect to time.
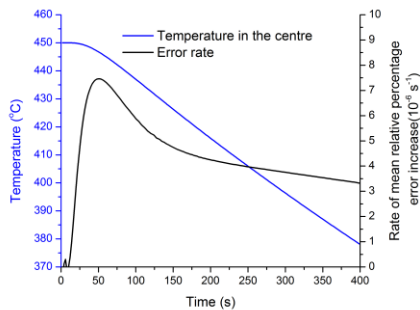

Fig. 3. Rate of mean relative percentage error increase with respect to calculated time of a process with actual calculated temperature at the centre of steel plate

In the figure 3 the increase rate of mean relative percentage error can be seen. It may be observed that at the early stage of simulation rate of error change is very small, but then it starts to accelerate. This process ends as the temperature drops in the plate is stable (around 50 s). Than as temperature decrease is stable error change rate drops quite fast until it reaches (150 s) stable value around $3 - 4 \ 10^{-6} \ s^{-1}$. Since this moment rate of mean relative error change still decrease, but very slowly.

At the early stage inertia of numerical method may cause so fast error increase. Boundary condition at the interface that connects two domains that differs with temperatures highly produces high errors also. This problem refers to all numerical methods. Later when temperature distribution in both domains is smooth less numerical approximation errors appears. Observed error is connected only to numerical approximation.

4. Conclusions

Knowledge of exact solution of problems described with PDEs allows to test quality of numerical solution. This procedure is especially important for the newly developed methods. Experimental results may be affected by processes that wasn't taken into account in mathematical model or may be laden with measurement errors. Exact solution gives values which should be reached by the numerical one.

The RCDQ method can be used to solving the heat transfer problems. The numerical solution given by the simulation that base on the RCDQ method approximation is of high accuracy.

Numerical approximation error increase during simulation. This may be caused by summing errors that appear at each time step. Errors that cause the error increase have different origins. It is effect of dynamic change of simulation condition during solving.

Whenever it is possible it is important to use relative error as it shows the fraction of the error that is included in numerical solution.

Because of its accuracy the Rank Controlled Differential Quadrature method may be used to solve more complicated foundry problems that include solidification.

Acknowledgements

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