Opuscula Math. 40, no. 2 (2020), 227–239
https://doi.org/10.7494/OpMath.2020.40.2.227 Opuscula Mathematica

ON THE ASYMPTOTIC BEHAVIOR OF NONOSCILLATORY SOLUTIONS OF CERTAIN FRACTIONAL DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE TERMS

John R. Graef, Said R. Grace, and Ercan Tunç

Communicated by Josef Diblík

Abstract. This paper is concerned with the asymptotic behavior of the nonoscillatory solutions of the forced fractional differential equation with positive and negative terms of the form

$$^{C}D_c^{\alpha}y(t) + f(t,x(t)) = e(t) + k(t)x^{\eta}(t) + h(t,x(t)),$$

where $t \geq c \geq 1$, $\alpha \in (0,1)$, $\eta \geq 1$ is the ratio of positive odd integers, and $^{C}D_c^{\alpha}y$ denotes the Caputo fractional derivative of y of order α. The cases

$$y(t) = \left(a(t)\left(x'(t)\right)^{\eta}\right)' \quad \text{and} \quad y(t) = a(t)\left(x'(t)\right)^{\eta}$$

are considered. The approach taken here can be applied to other related fractional differential equations. Examples are provided to illustrate the relevance of the results obtained.

Keywords: integro-differential equations, fractional differential equations, nonoscillatory solutions, boundedness, Caputo derivative.

Mathematics Subject Classification: 34E10, 34A34.

1. INTRODUCTION

We consider the forced fractional differential equation with positive and negative terms

$$^{C}D_c^{\alpha}y(t) + f(t,x(t)) = e(t) + k(t)x^{\eta}(t) + h(t,x(t)), \tag{1.1}$$

where $t \geq c > 1$, $\alpha \in (0,1)$, $\eta \geq 1$ is the ratio of positive odd integers, and $^{C}D_c^{\alpha}y$ denotes the Caputo fractional derivative of y of order α as defined by

$$^{C}D_c^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)}\int_c^t (t-s)^{n-\alpha-1}y^{(n)}(s)ds, \quad \alpha \in (n-1,n), n \in \mathbb{N}.$$

© 2020 Authors. Creative Commons CC-BY 4.0

If $\alpha \in (0,1)$, this definition was given by Caputo in [4]; for the definition of the Caputo derivative of order $\alpha \in (n-1,n)$, $n \geq 1$, see [1,5,6]. We will just consider the case $n = 1$, i.e., $\alpha \in (0,1)$, here, but our results can easily be generalized to values of n greater than 1 (see Section 3). We will be considering the following choices for the function y, namely,

$$y(t) = \left(a(t)\,(x'(t))^{\eta}\right)' \tag{1.2}$$

and

$$y(t) = a(t)\,(x'(t))^{\eta}. \tag{1.3}$$

Throughout the paper, we assume that:

(i) a, $k : [c,\infty) \to (0,\infty)$ and $e : [c,\infty) \to \mathbb{R}$ are continuous functions;

(ii) $f, h : [c,\infty) \times \mathbb{R} \to \mathbb{R}$ are real-valued continuous functions and there exist continuous functions b, $m : [c,\infty) \to (0,\infty)$ and positive real numbers λ and γ with $\lambda > \gamma$ such that

$$xf(t,x) \geq b(t)\,|x|^{\lambda+1} \quad \text{and} \quad 0 \leq xh(t,x) \leq m(t)\,|x|^{\gamma+1} \quad \text{for} \quad x \neq 0 \quad \text{and} \quad t \geq c.$$

A function $y : [c,\infty) \to \mathbb{R}$ is a *solution* of equation (1.1) and either (1.2) or (1.3) if $x \in C^1([c,\infty),\mathbb{R})$ and y satisfies (1.1). Only those solutions that are continuable and nontrivial in any neighborhood of ∞ are under consideration here. Such a solution is said to be *oscillatory* if there exists a sequence $\{t_n\} \subseteq [c,\infty)$ with $t_n \to \infty$ as $n \to \infty$ such that $x(t_n) = 0$, and it is *nonoscillatory* otherwise.

In recent years, integro-differential and fractional differential equations have gained considerably more attention due to their applicability to problems in engineering and other scientific disciplines. For example, they arise as mathematical models for systems and processes in physics, mechanics, chemistry, aerodynamics, electrodynamics, and more recently in social networking (see [13]). Additional examples can be found in [1,15–17,22–25].

Our aim here is to obtain some new results on the asymptotic behavior of nonoscillatory solutions of equation (1.1). We note that this equation is equivalent to the Volterra type equation

$$y(t) = c_0 + \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1}[e(s) + k(s)x^{\eta}(s) + h(s,x(s)) - f(s,x(s))]ds, \tag{1.4}$$

where $c \geq 1$, $\alpha \in (0,1)$, $c_0 = \frac{y(c)}{\Gamma(1)} = y(c)$, and c_0 is a real constant.

Oscillation and asymptotic properties of solutions of integro-differential equations and fractional differential equations are relatively scarce in the literature; some results can be found in [2,7,9,10,12,17–21]. The only results to date for forced fractional differential equations with positive and negative terms of the type (1.1) appear to be those in [11] where sufficient conditions for boundedness of nonoscillatory solutions are obtained. In that paper, $\eta = 1$ and, as remarked below, additional conditions were needed on the function $a(t)$. As a consequence, in the present paper, we are able to obtain growth estimates on the nonoscillatory solutions of (1.1) (see (2.6) and (2.23) below).

The idea to transform a fractional differential equation into a Volterra integral equation is not new. For example, Medved' did this for a much simpler equation in [20, Lemma 2.5]; in this regard also see [21, Lemma 1]. We make use of Young's inequality which is not the case for example in [20] and [21]. It is not difficult to see that the approach we use in this paper can be useful in the study of other types of fractional differential equations as well.

2. MAIN RESULTS

We will make use of the following two lemmas in the proofs of our main results.

Lemma 2.1 ([3]). *Let α and p be positive constants such that $p(\alpha-1)+1>0$. Then*

$$\int_0^t (t-s)^{p(\alpha-1)} e^{ps} ds \le Q e^{pt}, \quad t \ge 0,$$

where

$$Q = \frac{\Gamma(1+p(\alpha-1))}{p^{1+p(\alpha-1)}},$$

and

$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds, \quad x > 0,$$

is the usual Euler-Gamma function.

Lemma 2.2 (Young's inequality, [14]). *If X and Y are nonnegative, $\delta > 1$, and $1/\delta + 1/\beta = 1$, then*

$$XY \le \frac{1}{\delta} X^\delta + \frac{1}{\beta} Y^\beta, \tag{2.1}$$

where equality holds if and only if $Y = X^{\delta-1}$.

For notational purpose, we put

$$g(t) = \left(\frac{\lambda-\gamma}{\gamma}\right)\left(\frac{\gamma}{\lambda} m(t)\right)^{\lambda/(\lambda-\gamma)} (b(t))^{\gamma/(\gamma-\lambda)},$$

$$R(t,c) = \int_c^t a^{-1/\eta}(s) ds,$$

and we assume that

$$R(t,c) \to \infty \quad \text{as} \quad t \to \infty. \tag{2.2}$$

We now give sufficient conditions under which any nonoscillatory solution x of equation (1.1) with (1.2) satisfies

$$|x(t)| = O\left(t^{1/\eta} e^{t/\eta} R(t,c)\right) \quad \text{as} \quad t \to \infty.$$

Theorem 2.3. *Let conditions* (i)–(ii) *and* (2.2) *hold and assume that there exist real numbers* $p > 1$ *and* $0 < \alpha < 1$ *such that* $p(\alpha - 1) + 1 > 0$. *If*

$$\int_c^\infty k^q(s)s^q R^{\eta q}(s,c)ds < \infty, \quad \text{where} \quad q = \frac{p}{p-1}, \tag{2.3}$$

$$\lim_{t\to\infty} \int_c^t (t-s)^{\alpha-1} |e(s)|\, ds < \infty, \tag{2.4}$$

and

$$\lim_{t\to\infty} \int_c^t (t-s)^{\alpha-1} g(s) ds < \infty, \tag{2.5}$$

then every nonoscillatory solution $x(t)$ *of equation* (1.1) *with* (1.2) *satisfies*

$$\limsup_{t\to\infty} \frac{|x(t)|}{t^{1/\eta} e^{t/\eta} R(t,c)} < \infty. \tag{2.6}$$

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1) with (1.2), say $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq c$. Setting $F(t) = h(t, x(t)) - f(t, x(t))$, it follows from (i)–(ii) and (1.1) that

$$\begin{aligned}\left(a(t)\left(x'(t)\right)^\eta\right)' \leq c_0 &+ \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} |F(s)|\, ds + \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} |e(s)|\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} \left[m(s)x^\gamma(s) - b(s)x^\lambda(s)\right] ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} k(s) \left|x^\eta(s)\right| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} k(s) x^\eta(s) ds.\end{aligned} \tag{2.7}$$

Applying (2.1) to $\left[m(t)x^\gamma(t) - b(t)x^\lambda(t)\right]$ with

$$\delta = \frac{\lambda}{\gamma}, \quad X = x^\gamma(t), \quad Y = \frac{\gamma}{\lambda}\frac{m(t)}{b(t)}, \quad \text{and} \quad \beta = \frac{\lambda}{\lambda - \gamma},$$

we obtain

$$
\begin{aligned}
m(t)x^{\gamma}(t) - b(t)x^{\lambda}(t) &= \frac{\lambda}{\gamma} b(t) \left[x^{\gamma}(t) \frac{\gamma}{\lambda} \frac{m(t)}{b(t)} - \frac{\gamma}{\lambda} (x^{\gamma}(t))^{\lambda/\gamma} \right] \\
&= \frac{\lambda}{\gamma} b(t) \left[XY - \frac{1}{\delta} X^{\delta} \right] \leq \frac{\lambda}{\gamma} b(t) \left(\frac{1}{\beta} Y^{\beta} \right) \\
&= \left(\frac{\lambda - \gamma}{\gamma} \right) \left[\frac{\gamma}{\lambda} m(t) \right]^{\lambda/(\lambda-\gamma)} (b(t))^{\gamma/(\gamma-\lambda)} := g(t).
\end{aligned} \tag{2.8}
$$

Using (2.8) in (2.7) gives

$$
\begin{aligned}
\left(a(t) \left(x'(t) \right)^{\eta} \right)' \leq c_0 &+ \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t_1 - s)^{\alpha-1} |F(s)| \, ds + \frac{1}{\Gamma(\alpha)} \int_c^{t} (t - s)^{\alpha-1} |e(s)| \, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t - s)^{\alpha-1} g(s) ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{c}^{t_1} (t_1 - s)^{\alpha-1} k(s) \left| x^{\eta}(s) \right| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t - s)^{\alpha-1} k(s) x^{\eta}(s) ds.
\end{aligned} \tag{2.9}
$$

In view of (2.4) and (2.5), it follows from (2.9) that

$$
\left(a(t) \left(x'(t) \right)^{\eta} \right)' \leq C_1 + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t - s)^{\alpha-1} k(s) x^{\eta}(s) ds, \tag{2.10}
$$

for some constant $C_1 > 0$. Integrating (2.10) from t_1 to t gives

$$
a(t) \left(x'(t) \right)^{\eta} \leq a(t_1) \left| \left(x'(t_1) \right)^{\eta} \right| + C_1 (t - t_1) + \frac{1}{\alpha \Gamma(\alpha)} \int_{t_1}^{t} (t - s)^{\alpha} k(s) x^{\eta}(s) ds := w(t), \tag{2.11}
$$

which can be written as

$$
x'(t) \leq \left(\frac{w(t)}{a(t)} \right)^{1/\eta}. \tag{2.12}
$$

Noting that $w(t)$ is an increasing function, it follows from (2.12) that

$$\begin{aligned} x(t) &\le x(t_1) + w^{1/\eta}(t) \int_{t_1}^{t} a^{-1/\eta}(s)ds \\ &= x(t_1) + w^{1/\eta}(t) R(t, t_1) \\ &= \left[\frac{x(t_1)}{R(t, t_1)} + w^{1/\eta}(t) \right] R(t, t_1) \\ &\le \left[\frac{x(t_1)}{R(t_2, t_1)} + w^{1/\eta}(t) \right] R(t, t_1) \end{aligned} \tag{2.13}$$

for $t \ge t_2$ and any fixed $t_2 > t_1$. From (2.13), we see that

$$\frac{x(t)}{R(t, t_1)} \le C_2 + w^{1/\eta}(t) \quad \text{for } t \ge t_2, \tag{2.14}$$

where $C_2 = x(t_1)/R(t_2, t_1) > 0$. Applying the elementary inequality

$$(A + B)^{\mu} \le 2^{\mu - 1}(A^{\mu} + B^{\mu}), \quad A, B \ge 0 \quad \text{and} \quad \mu \ge 1, \tag{2.15}$$

to (2.14) gives

$$\left(\frac{x(t)}{R(t, t_1)} \right)^{\eta} \le 2^{\eta - 1}(C_2)^{\eta} + 2^{\eta - 1} w(t) \quad \text{for } t \ge t_2. \tag{2.16}$$

In view of the definition of $w(t)$, it follows from (2.16) that

$$\begin{aligned} \left(\frac{x(t)}{R(t, t_1)} \right)^{\eta} &\le 2^{\eta - 1}(C_2)^{\eta} + 2^{\eta - 1} a(t_1) \left| \left(x'(t_1)\right)^{\eta} \right| + 2^{\eta - 1} C_1 (t - t_1) \\ &\quad + \frac{2^{\eta - 1}}{\alpha \Gamma(\alpha)} \int_{t_1}^{t} (t - s)^{\alpha} k(s) x^{\eta}(s) ds \\ &\le 2^{\eta - 1}(C_2)^{\eta} + 2^{\eta - 1} a(t_1) \left| \left(x'(t_1)\right)^{\eta} \right| + 2^{\eta - 1} C_1 t \\ &\quad + \frac{2^{\eta - 1} t}{\alpha \Gamma(\alpha)} \int_{t_1}^{t} (t - s)^{\alpha - 1} k(s) x^{\eta}(s) ds. \end{aligned} \tag{2.17}$$

From (2.17), we arrive at

$$\left(\frac{x(t)}{t^{1/\eta} R(t, t_1)} \right)^{\eta} \le C_3 + \frac{2^{\eta - 1}}{\alpha \Gamma(\alpha)} \int_{t_1}^{t} (t - s)^{\alpha - 1} k(s) x^{\eta}(s) ds \tag{2.18}$$

for some constant $C_3 > 0$. Applying Hölder's inequality and Lemma 2.1 to the integral on the right in (2.18) yields

$$\begin{aligned}
&\int_{t_1}^{t} \left[(t-s)^{\alpha-1} e^{s}\right] \left[e^{-s} k(s) x^{\eta}(s)\right] ds \\
&\le \left(\int_{t_1}^{t} (t-s)^{p(\alpha-1)} e^{ps} ds\right)^{1/p} \left(\int_{t_1}^{t} e^{-qs} k^{q}(s) x^{\eta q}(s) ds\right)^{1/q} \\
&\le \left(\int_{0}^{t} (t-s)^{p(\alpha-1)} e^{ps} ds\right)^{1/p} \left(\int_{t_1}^{t} e^{-qs} k^{q}(s) x^{\eta q}(s) ds\right)^{1/q} \qquad (2.19) \\
&\le (Qe^{pt})^{1/p} \left(\int_{t_1}^{t} e^{-qs} k^{q}(s) x^{\eta q}(s) ds\right)^{1/q} \\
&= Q^{1/p} e^{t} \left(\int_{t_1}^{t} e^{-qs} k^{q}(s) x^{\eta q}(s) ds\right)^{1/q}.
\end{aligned}$$

Using (2.19) in (2.18), we obtain

$$z(t) := \left(\frac{x(t)}{t^{1/\eta} e^{t/\eta} R(t,t_1)}\right)^{\eta} \le 1 + C_4 + K \left(\int_{t_1}^{t} e^{-qs} k^{q}(s) x^{\eta q}(s) ds\right)^{1/q}, \qquad (2.20)$$

where C_4 is an upper bound for $C_3 e^{-t}$ and $K = 2^{\eta-1} Q^{1/p}/\alpha\Gamma(\alpha)$. Employing again inequality (2.15), we obtain from (2.20) that

$$z^{q}(t) \le 2^{q-1}(1+C_4)^{q} + 2^{q-1} K^{q} \int_{t_1}^{t} e^{-qs} k^{q}(s) x^{\eta q}(s) ds,$$

which can be written as

$$z^{q}(t) \le 2^{q-1}(1+C_4)^{q} + 2^{q-1} K^{q} \int_{t_1}^{t} k^{q}(s) s^{q} R^{\eta q}(s,t_1) z^{q}(s) ds. \qquad (2.21)$$

Setting $P_1 = 2^{q-1}(1+C_4)^{q}$, $Q_1 = 2^{q-1}K^{q}$, and $w(t) = z^{q}(t)$ so that $z(t) = w^{1/q}(t)$, inequality (2.21) becomes

$$w(t) \le P_1 + Q_1 \int_{t_1}^{t} k^{q}(s) s^{q} R^{\eta q}(s,t_1) w(s) ds.$$

The conclusion then follows from Gronwall's inequality and condition (2.3), that is,

$$\limsup_{t\to\infty} \frac{x(t)}{t^{1/\eta} e^{t/\eta} R(t,t_1)} < \infty.$$

The proof in case $x(t)$ is an eventually negative solution is similar. This completes the proof of the theorem. □

We next give sufficient conditions under which any nonoscillatory solution x of equation (1.1) with (1.3) satisfies

$$|x(t)| = O\left(e^{t/\eta} R(t,c)\right) \quad \text{as } t \to \infty.$$

Theorem 2.4. *Let conditions* (i)–(ii) *and* (2.2) *hold and assume that there exist real numbers $p > 1$ and $\alpha \in (0,1)$ such that $p(\alpha-1)+1 > 0$. If*

$$\int_c^\infty k^q(s) R^{\eta q}(s,c) ds < \infty, \quad \textit{where} \;\; q = \frac{p}{p-1}, \tag{2.22}$$

and conditions (2.4)–(2.5) *hold, then every nonoscillatory solution $x(t)$ of equation* (1.1) *with* (1.3) *satisfies*

$$\limsup_{t\to\infty} \frac{|x(t)|}{e^{t/\eta} R(t,c)} < \infty. \tag{2.23}$$

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1) with (1.3), say $x(t) > 0$ for $t \ge t_1$ for some $t_1 \ge c$. As in the proof of Theorem 2.3, we again let

$$F(t) = h(t,x(t)) - f(t,x(t)).$$

Then, in view of (i)–(ii), it follows from equation (1.1) that

$$\begin{aligned} a(t)\left(x'(t)\right)^\eta \le c_0 &+ \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} |F(s)|\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_c^{t} (t-s)^{\alpha-1} |e(s)|\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} \left[m(s)x^\gamma(s) - b(s)x^\lambda(s)\right] ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} k(s) \left|x^\eta(s)\right| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} k(s) x^\eta(s) ds. \end{aligned} \tag{2.24}$$

Proceeding as in the proof of Theorem 2.3, it follows from (2.24) that

$$a(t)\left(x'(t)\right)^{\eta} \leq C_5 + \frac{1}{\Gamma(\alpha)} \int\limits_{t_1}^{t} (t-s)^{\alpha-1} k(s) x^{\eta}(s) ds := \varphi(t), \tag{2.25}$$

for some constant $C_5 > 0$. Again as in Theorem 2.3, inequality (2.25) can be written as

$$x'(t) \leq \left(\frac{\varphi(t)}{a(t)}\right)^{1/\eta}. \tag{2.26}$$

The remainder of the proof is similar to that of Theorem 2.3 and so we omit the details. □

We conclude this paper with two examples to illustrate our results.

Example 2.5. Consider the equation

$${}^{C}D_8^{1/2}\left(t(x'(t))^3\right)' + f(t,x(t)) = e^{-2t}\cos t + \frac{1}{1+t^4}x^3(t) + h(t,x(t)), \quad t \geq 8. \tag{2.27}$$

Here we have $y(t) = \left(t(x'(t))^3\right)'$, $\alpha = 1/2$, $c = 8$, $\eta = 3$, $a(t) = t$, $e(t) = e^{-2t}\cos t$, $k(t) = 1/(1+t^4)$, and

$$R(t,c) = R(t,8) = \int\limits_{8}^{t} s^{-1/3} ds = \frac{3}{2}(t^{2/3} - 4).$$

Then, it is easy to see that conditions (i) and (2.2) hold. Letting $p = 3/2$, we see that $q = 3$, and $p(\alpha - 1) + 1 = 1/4 > 0$. Letting $f(t,x(t)) = b(t)\left|x(t)\right|^{\lambda-1} x(t)$ and $h(t,x(t)) = m(t)\left|x(t)\right|^{\gamma-1} x(t)$ with $\lambda > \gamma$, and taking $b(t) = m(t) = e^{-t}$, we see that condition (ii) holds. Since

$$\int\limits_{c}^{\infty} k^q(s) s^q R^{\eta q}(s,c) ds \leq \left(\frac{3}{2}\right)^9 \int\limits_{8}^{\infty} \frac{s^9}{(1+s^4)^3} ds < \infty,$$

condition (2.3) holds. To see that (2.4) holds, note that letting $u = t - s + 8$, the integral becomes

$$\begin{aligned}
\int_8^t (t-s)^{-1/2}|e^{-2s}\cos s|ds &= -\int_t^8 (u-8)^{-1/2}e^{2u-2t-16}|\cos(t-u+8)|du \\
&\le \frac{1}{e^{2t+16}}\int_8^t (u-8)^{-1/2}e^{2u}du \\
&= \frac{1}{e^{2t+16}}\left[\int_8^{16}(u-8)^{-1/2}e^{2u}du + \int_{16}^t (u-8)^{-1/2}e^{2u}du\right] \\
&= \frac{1}{e^{2t+16}}\left[\lim_{b\to 8^+}\int_b^{16}(u-8)^{-1/2}e^{2u}du\right] \\
&\quad + \frac{1}{e^{2t+16}}\left[\int_{16}^t (u-8)^{-1/2}e^{2u}du\right] \\
&= \frac{e^{32}}{e^{2t+16}}\lim_{b\to 8^+}\int_b^{16}(u-8)^{-1/2}du + \frac{(16-8)^{-1/2}}{e^{2t+16}}\int_{16}^t e^{2u}du \\
&= \frac{2^{5/2}e^{32}}{e^{2t+16}} + \frac{2^{-5/2}}{e^{2t+16}}\left(e^{2t}-e^{32}\right) < \infty \quad \text{as } t\to\infty.
\end{aligned}$$

Hence, (2.4) and similarly (2.5) hold. Since all conditions of Theorem 2.3 are satisfied, every nonoscillatory solution $x(t)$ of equation (2.27) satisfies (2.6), that is,

$$\limsup_{t\to\infty}\frac{|x(t)|}{\frac{3}{2}e^{t/3}(t-4t^{1/3})} < \infty. \tag{2.28}$$

Example 2.6. Consider the equation

$$^{C}D_4^{3/5}\left(t^{1/2}x'(t)\right) + f(t,x(t)) = e^{-3t}\sin t + \frac{1}{t^2}x(t) + h(t,x(t)), \quad t \ge 4. \tag{2.29}$$

Here we have $y(t) = t^{1/2}x'(t)$, $\alpha = 3/5$, $c = 4$, $\eta = 1$, $a(t) = t^{1/2}$, $e(t) = e^{-3t}\sin t$, $k(t) = 1/t^2$, and

$$R(t,c) = R(t,4) = \int_4^t s^{-1/2}ds = 2t^{1/2} - 4.$$

Then, it is easy to see that conditions (i) and (2.2) hold. Letting $p = 2$, we see that $q = 2$, and $p(\alpha - 1) + 1 = 1/5 > 0$. Letting $f(t,x(t)) = b(t)\,|x(t)|^{\lambda-1}\,x(t)$ and

$h(t,x(t)) = m(t)\,|x(t)|^{\gamma-1}\,x(t)$ with $\lambda > \gamma$ and $b(t) = m(t) = e^{-3t}$, we see that (ii) holds. Since

$$\int_c^\infty k^q(s)R^{\eta q}(s,c)ds \le 4\int_4^\infty \frac{1}{s^3}ds < \infty,$$

condition (2.22) holds. As in Example 2.5, it is easy to see that conditions (2.4)–(2.5) hold. Since all conditions of Theorem 2.4 are fulfilled, we have that every nonoscillatory solution $x(t)$ of equation (2.29) satisfies (2.23), i.e.,

$$\limsup_{t\to\infty} \frac{|x(t)|}{2e^t(t^{1/2}-2)} < \infty. \tag{2.30}$$

3. REMARKS AND CONCLUSIONS

The results in this paper can also be obtained for higher fractional differential equations of order $\alpha \in (n-1,n)$ with $n \ge 1$. For example, if we take

$$y(t) = \big(a(t)|x'(t)|^{\nu-1}x'(t)\big)'$$

with $\nu \ge 1$, then our equation would include the equation considered in [8] as a special case. Another possibility is to replace the fractional derivative ${}^C D_c^\alpha y(t)$ by ${}^C D_c^r y(t)$ where $r = \alpha+n-1$ with $\alpha \in (0,1)$ which would then include the problem considered by Medveď [20, Eq. (24)]. These would make for interesting problems for future research.

We would also like to point out that if $\eta \equiv 1$ in (1.1), then the results in this paper generalize those in [11] where an additional condition on the function a was needed (see (2.3) and (2.18) in [11]). Another possible direction for future investigations is to consider equation (1.1) with $\eta < 1$.

Acknowledgements

The authors would like to thank the referee who gave a careful reading of the paper and made several suggestions for improvements in the presentation of our results.

REFERENCES

[1] D. Băleanu, J.A.T. Machado, A.C.J. Luo, *Fractional Dynamics and Control*, Springer, New York, 2012.

[2] M. Bohner, S.R. Grace, N. Sultana, *Asymptotic behavior of nonoscillatory solutions of higher-order integro-dynamic equations*, Opuscula Math. **34** (2014), 5–14.

[3] E. Brestovanská, M. Medveď, *Asymptotic behavior of solutions to second-order differential equations with fractional derivative perturbations*, Electron. J. Differ. Eq. **2014** (2014), 1–10.

[4] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent II*, Geophys. J. Royal Astronom. Soc. **13** (1967), 529–535.

[5] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Berlin, 2010.

[6] K.M. Furati, N.-E. Tatar, *Power-type estimates for a nonlinear fractional differential equations*, Nonlinear Anal. **62** (2005), 1025–1036.

[7] S.R. Grace, A. Zafer, *Oscillatory behavior of integro-dynamic and integral equations on time scales*, Appl. Math. Lett. **28** (2014), 47–52.

[8] S.R. Grace, A. Zafer, *On the asymptotic behavior of nonoscillatory solutions of certain fractional differential equations*, Eur. Phys. J. Special Topics **226** (2017), 3657–3665.

[9] S.R. Grace, R.P. Agarwal, P.J.Y. Wong, A. Zafer, *On the oscillation of fractional differential equations*, Fract. Calc. Appl. Anal. **15** (2012), 222–231.

[10] S.R. Grace, J.R. Graef, S. Panigrahi, E. Tunç, *On the oscillatory behavior of Volterra integral equations on time-scales*, PanAmerican Math. J. **23** (2013), 35–41.

[11] S.R. Grace, J.R. Graef, E. Tunç, *On the boundedness of nonoscillatory solutions of certain fractional differential equations with positive and negative terms*, Appl. Math. Lett. **97** (2019), 114–120.

[12] S.R. Grace, J.R. Graef, A. Zafer, *Oscillation of integro-dynamic equations on time scales*, Appl. Math. Lett. **26** (2013), 383–386.

[13] J.R. Graef, L. Kong, A. Ledoan, M. Wang, *Modeling online social network dynamics using fractional-order epidemiological models*, to appear.

[14] G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1988, Reprint of the 1952 edition.

[15] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204, Elsevier, Amsterdam, 2006.

[16] V. Lakshmikantham, S. Leela, J. Vaaundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, 2009.

[17] Q.-H. Ma, J. Pečarić, J.-M. Zhang, *Integral inequalities of systems and the estimate for solutions of certain nonlinear two-dimensional fractional differential systems*, Comput. Math. Appl. **61** (2011), 3258–3267.

[18] M. Medved', *A new approach to an analysis of Henry type integral inequalities and their Bihari type versions*, J. Math. Anal. Appl. **214** (1997), 349–366.

[19] M. Medved', *Integral inequalities and global solutions of semilinear evolution equations*, J. Math. Anal. Appl. **267** (2002), 643–650.

[20] M. Medved', *Asymptotic integration of some classes of fractional differential equations*, Tatra Mt. Math. Publ. **54** (2013), 119–132.

[21] M. Medved', M. Pospíšil, *Asymptotic integration of fractional differential equations with integrodifferential right-hand side*, Math. Modelling Anal. **20** (2015), 471–489.

[22] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, Wiley, New York, 1993.

[23] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering, vol. 198, Academic Press, San Diego, 1999.

[24] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, *Integral and Series: Elementary Functions*, vol. 1, Nauka, Moscow, 1981 [in Russian].

[25] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, New York, 1993.

John R. Graef (corresponding author)
John-Graef@utc.edu

University of Tennessee at Chattanooga
Department of Mathematics
Chattanooga, TN 37403, USA

Said R. Grace
saidgrace@yahoo.com

Cairo University
Faculty of Engineering
Department of Engineering Mathematics
Orman, Giza 12221, Egypt

Ercan Tunç
ercantunc72@yahoo.com

Gaziosmanpasa University
Department of Mathematics
Faculty of Arts and Sciences
60240, Tokat, Turkey

Received: August 6, 2019.
Revised: October 9, 2019.
Accepted: November 3, 2019.