

Seeking realistic upper-bounds for internal reliability of systems with uncorrelated observations

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Abstract: From the theory of reliability it follows that the greater the observational redundancy in a network, the higher is its level of internal reliability. However, taking into account physical nature of the measurement process one may notice that the planned additional observations may increase the number of potential gross errors in a network, not raising the internal reliability to the theoretically expected degree. Hence, it is necessary to set realistic limits for a sufficient number of observations in a network. An attempt to provide principles for finding such limits is undertaken in the present paper. An empirically obtained formula (Adamczewski 2003) called there the law of gross errors, determining the chances that a certain number of gross errors may occur in a network, was taken as a starting point in the analysis. With the aid of an auxiliary formula derived on the basis of the Gaussian law, the Adamczewski formula was modified to become an explicit function of the number of observations in a network. This made it possible to construct tools necessary for the analysis and finally, to formulate the guidelines for determining the upper-bounds for internal reliability indices. Since the Adamczewski formula was obtained for classical networks, the guidelines should be considered as an introductory proposal requiring verification with reference to modern measuring techniques.

Keywords: internal reliability, upper-bounds, law of gross errors, probability-derived formula, binomial distribution

1. Introduction

The internal reliability criteria based on the relationships between the network responses to a single gross error or multiple gross errors (Prószyński, 1997) settle lower-bounds that ensure effective detection of k gross errors ($k = 1, 2, 3, 4$). For each value of k , the bounds expressed in terms of the internal reliability indices imply a necessary number of observations in a network. It follows from the theory of internal reliability that, the greater the number of observations in a network, the

greater the values of internal reliability indices, each asymptotically approaching 1 in systems with minimal constraints on parameters. A specially high importance of some engineering survey tasks may require high network reliability which, according to theory, implies the use of excessively great number of observations to effectively detect outliers. There are, however, the following two counteracting factors that are the arguments in favour of introducing certain limits for increasing the number of observations beyond those resulting from the lower-bounds:

- i. the additional observations may contain new gross errors, thus increasing the total number of gross errors;
- ii. the increase in the number of observations will result in a rise of the total cost of fieldwork (due to increase in measurement time, involvement of staff and instrumentation).

Hence, there appears a problem of determining the upper-bounds for the values of internal reliability indices that would take into account the influence of the above mentioned two factors, and be a sort of compromise between theory and practice. Such upper-bounds would specify sufficient number of observations in a network. In the case of the first factor (i) we would need a knowledge on frequency of occurrence of gross errors. The problem is that gross errors (mistakes being one of their forms) are rather unpredictable due to their very nature, so they do not obey any law, including Gaussian law; they behave differently under different conditions, such as for example type of observation or degree of automation of observation process. An attempt to formulate a rule that would describe behaviour of gross errors in networks was made in (Adamczewski, 2003), where a formula called the law of gross errors was proposed. The proposal was taken as a starting point in the present analysis. However, the law did not operate explicitly with the number of observations in a network. The inspiration for the present author how to approach this problem were the introductory comments in (Knight et al., 2010), concerning the number of potential gross errors that may occur in a network of specified number of observations.

To take into account the second factor (ii) we need to know the cost of network measurement expressed as a function of the number of observations. Due to the lack of practical examples of such a function, the economical aspect will only be signalized.

The problem of realistic upper-bounds for network internal reliability, which is of vital importance especially for engineering surveying tasks, has so far not been discussed in geodetic professional literature, so the approach presented in this paper is a proposal requiring further studies.

2. Modified form of the Adamczewski law of gross errors

Assuming a generalized definition of *gross error* covering measurement errors both random and systematic, as well as mistakes made in measurement (e.g. false identification of the measured quantity), Adamczewski formulated on the basis of

practical experiments a law of gross errors, defined by the probability $p(k)$ that k gross errors occur in a network (Adamczewski 2003), i.e.

$$p(k) = a(1-a)^k \quad k = 0, 1, 2, 3, \dots \quad (1)$$

where

$$a = p(0); \quad \sum_{k=0}^{\infty} p(k) = 1 .$$

The material for the experiment were the observation sets of distances and of directions formed on the basis of post-adjustment documentation for 63 2nd-order networks of different size, measured in the period 1979 ÷ 1987. No mention was made about magnitudes of the detected outliers.

Although having realistic setting, the law does not operate explicitly with a number of observations in a network. It cannot be used directly for the purpose of the present research since the upper-bounds of network reliability sought for should be expressed in terms of the number of observations.

Seeking an auxiliary formula free from the above disadvantage, that might approximate the Adamczewski law, we shall concentrate on the Gaussian-type random errors of excessive magnitudes, i.e.

$$|\varepsilon| > r \cdot \sigma \quad r = 3 \quad (2)$$

where σ is a standard deviation of measurement of a certain quantity in a network.

Obviously, these errors represent only a certain part of a broad spectrum of gross errors. Nevertheless they have a clear probability model.

Hence, in our approach the set of all possible random errors that may occur in a single measurement can be split into the following two, mutually complementary, subsets of Gaussian-type random errors:

$$OE = \{ \varepsilon : |\varepsilon| \leq r \cdot \sigma \} \quad \text{ordinary errors}$$

$$GE = \{ \varepsilon : |\varepsilon| > r \cdot \sigma \} \quad \text{gross errors}$$

Assigning 0 to each element of OE and 1 to each element of GE, we may consider a variable X with a 0,1 distribution, defined by

$$P(X = 0) = 1 - p, \quad P(X = 1) = p$$

where p is determined from the normal distribution $N(0, \sigma)$ for a specified value of the coefficient r as in (2). For $r = 3$, we get $p = 0.0027$.

To obtain the probability that k gross errors occur in n observations, we shall introduce a variable Y having binomial distribution, defined by

$$P(Y = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad k = 0, 1, 2, \dots, n \quad (3)$$

where

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$; $P(Y = 0) = (1-p)^n$, which corresponds to parameter a in the formula (1).

The formula (3) can be termed the *probability-derived formula for occurrence of Gaussian-type gross errors*.

To enable comparison of the formulas (1) and (3) we determine the values of parameter a in (1) from the relationship

$$a = (1-p)^n \quad (4)$$

already given in explanations under the formula (3). Graphs for both the formulas are shown in Fig. 1 for four chosen values of n , i.e. 50, 150, 300, 500 and $p = 0.0027$.

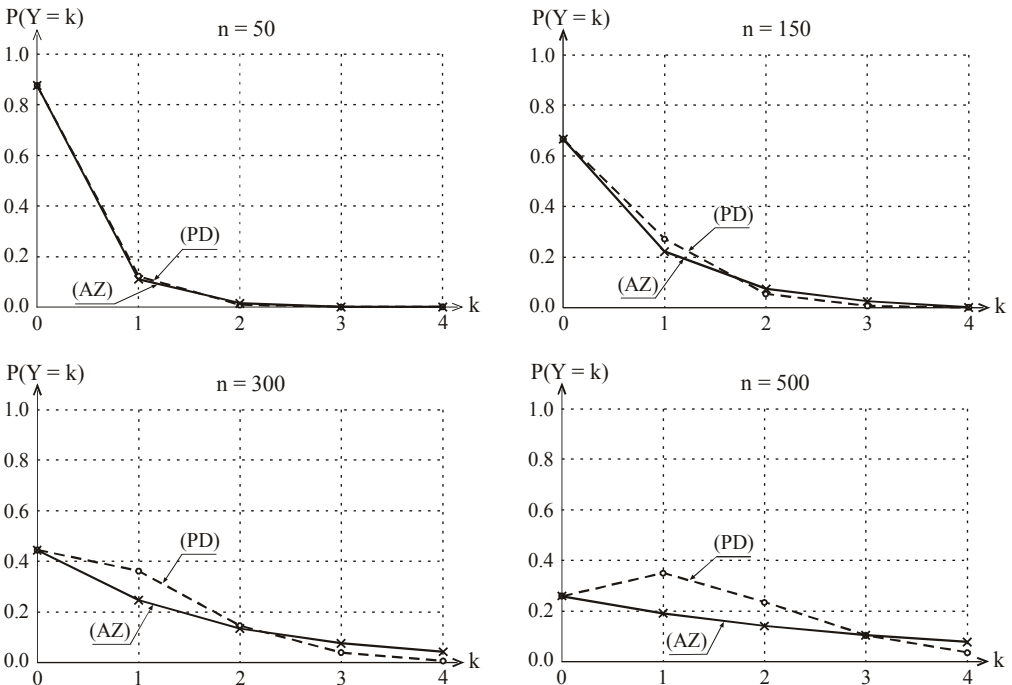


Fig. 1. Comparison of the probability-derived formula (3) with the Adamczewski law (1), denoted by (PD) and (AZ) respectively

Although for $n = 50$ and $n = 150$ the graphs in Figure 1 run close to each other, the discrepancies between them appear for greater n (see $n = 300$ and $n = 500$). With the increase in the number of observations a trend can be clearly observed, that the formula (3) assigns higher probability for $k = 1$ ($n = 300$) or for $k = 1$ and $k = 2$ ($n = 500$), and lower probability for $k = 3$ and $k = 4$ ($n = 300$) and for $k = 4$ ($n = 500$). The explanation of these differences is very difficult, since both the formulas are hardly comparable. The formula (3) does not cover mistakes of any type as well as gross errors that may undergo non-Gaussian distributions. On the other hand, with the range of magnitudes as in (2) the formula (3) takes into account additionally the gross errors falling within the interval $3\sigma < |\varepsilon| < 6\sigma$ that are undetectable according to the MDB formula (Baarda 1968) with $h = 0.5$, and thus could not be recorded in empirical tests carried out by Adamczewski.

The closeness of the graph (PD) to the graph (AZ) up to n being around 300, enables us to use the formula (3) for modification of the formula (1) so that the latter becomes a function of n , i.e. assumes the form necessary for determining the upper-bounds of network internal reliability.

Substituting into (1) the relationship (4) as was used for constructing the graphs in Fig. 1, we obtain the modified form of Adamczewski formula, as shown below

$$p(k) = (1 - p)^n [1 - (1 - p)^n]^k \quad k = 0, 1, 2, \dots, n \tag{5}$$

For a given value of k the expression in (5) is a geometric series, with the sum of r terms, where $r = k + 1$, being

$$S_r = 1 - \left[1 - (1 - p)^n \right]^{k+1} \tag{6}$$

e.g. for $n = 1$, we have $k + 1 = 2$ terms: for $k = 0$ and for $k = 1$.

The relationship (6) can be obtained immediately on basis of the well known formula for S_r , in this analysis taking the form as $S_r = \frac{a(1 - q^r)}{1 - q}$, where: $a = (1 - p)^n$; $q = 1 - (1 - p)^n$; $r = k + 1$.

From the properties of geometric series it follows that with $1 - (1 - p)^n < 1$, the sum S_r as in (6) converges with $k \rightarrow \infty$ to 1. Since S_r corresponds to $P(Y \leq k) = F(k)$, it satisfies the requirement for cumulative function, i.e. $F(k) = 1$, but it is achieved beyond the limit for k being $k = n$. However, the numerical tests show a high convergence rate of S_r , such that even for $n = 1$, $k = 1$ where $S_r = 0.999993$, the discrepancy from 1 can be neglected.

3. Guidelines for determining the upper-bounds for internal reliability indices

Strict relationships defining the upper-bounds for network internal reliability cannot be formulated since the input factors are all burdened with uncertainty. So, only the guidelines will be given instead, providing general orientation as regards the relation between the increase in the number of observations and the growth of the number of potential gross errors. The economical aspect will only be signaled.

3.1. Preliminaries

To simplify formulation of the guidelines and make them unambiguous we introduce the following notation:

k_h – the number of gross errors corresponding to reliability level h

k_1 – the number of potential gross errors resulting from the Adamczewski modified formula (5)

n^* , h^* – the increased number of observations and the increased reliability level

\gtrsim , \lesssim – the symbols meaning here “slightly greater” and “slightly smaller”

In the analysis we shall use a global reliability index defined by $\bar{h} = 1 - \frac{u}{n}$, where u is the number of model parameters, n is the number of observations in a network. We shall assume that the indices for individual observations are all equal, i.e. $h_i \cong \bar{h}$, $i = 1, 2, \dots, n$.

The following properties of internal reliability criteria, related to the number of gross errors in a network, should be taken into account in formulating the guidelines:

(1) numerical tests show that the lower-bound $h \gtrsim 0.5$ ensures effective detection of a single gross error ($k = 1$), or even more gross errors ($k = 2, 3$) provided that they are of magnitudes not smaller than those computed from MDB and reside in the observations being in the 3rd or a higher degree of coexistence with each other (Prószyński and Kwaśniak 2002). Except for levelling networks the lower bound $h \gtrsim 0.5$ will be considered as a basic level of internal reliability in the design of networks;

(2) the numbers of observations corresponding to $h = 0.75$ (for $k = 2$), $h = 0.83$ (for $k = 3$) and $h = 0.88$ (for $k = 4$) are excessively high, since these values of h were derived assuming the worst scenario as regards signs and location of gross errors (Prószyński 1997). Numerical tests indicate that in each of these cases we can detect one or two more gross errors, especially if they reside in the observations of distant mutual location as specified above.

(3) fixing the values of the upper-bounds should be based on the law of gross errors (here – the modified Adamczewski formula) and also on a cost analysis if suitable data are available.

3.2. Preparation of tools for the analysis

Using the formula (5) with $p = 0.0027$, we shall construct a step-function (Fig. 2a), showing a relationship between the number of potential gross errors k_1 and the number of observations n .

The value of n separating the i -th and the $(i+1)$ -th region of k_1 , denoted by $n_{i,max}$ is taken such that $P(Y = k_{1,i+1})$, being a growing function of n , reaches a significance level assumed as 0.05. The numbers of observations marking the steps are rounded off to the nearest tens. The original lengths of k_1 – regions, i.e. before rounding-off, were not exact multiples of 110.

With the lower-bound being $h \gtrsim 0.5$, we shall form the intervals for model parameters u (where $u = 0.5n$), corresponding to successive regions of k_1 in a step-function (Fig. 2b). Each u -interval is characterized by $k_h = 1$.

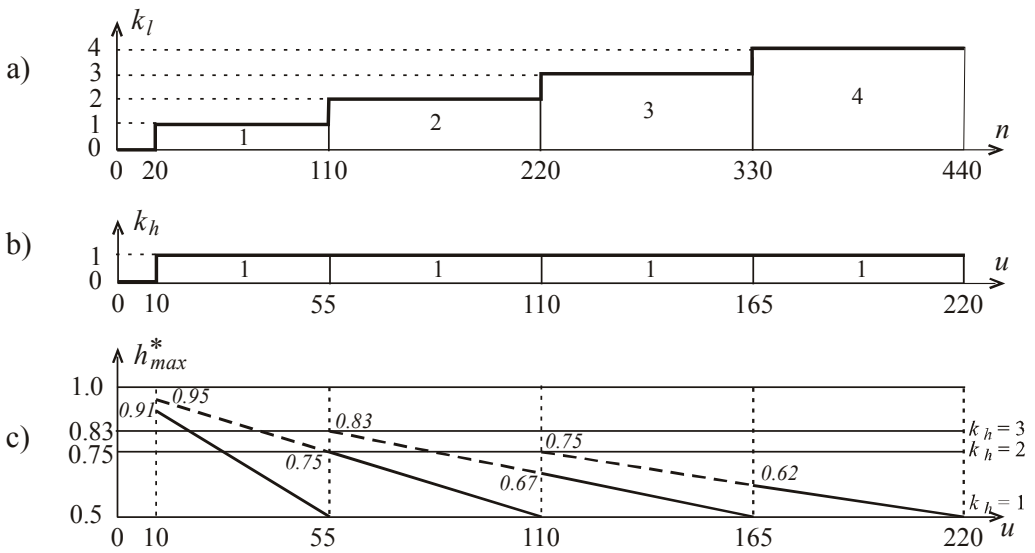


Fig. 2. a) potential number of gross errors as a function of the number of observations, b) corresponding intervals for model parameters, c) max. possible values of internal reliability (staying within $k_{1,i}$ – region, solid line; entering into $k_{1,i+1}$ – region, dashed line)

Comparing the graphs a) and b) in Fig. 2 we can see that the difference between $k_h = 1$ (corresponding to the lower-bound $h \gtrsim 0.5$) and $k_{1,i}$ where $i = 1, 2, 3, 4$ (resulting from the applied law of gross errors) grows systematically. This weakens to a great extent a theory-supported positive role of observation redundancy in a network.

For each u -interval there is a possibility to raise the reliability of a network over the lower-bound $h \gtrsim 0.5$ by increasing the number of observations over $2u$.

The increased reliability (h_i^*) corresponding to the increased number of observations (n_i^*) is obtained from the basic formula (see Sect. 3.1)

$$h_i^* = 1 - \frac{u_i}{n_i^*} = 1 - \frac{0.5n_i}{n_i^*} \quad (7)$$

the subscript “ i ” denotes the i -th u -interval;

We shall consider the following two cases for the value of n_i^* :

- a) $n_i^* \leq n_{i,\max}$; we do not enter into the region of higher value of k_1 .
- b) $n_{i,\max} < n_i^* \leq n_{i+1,\max}$; we enter into the region of higher value of k_1 .

On the basis of formula (7) we can express the maximum achievable reliability ($h_{i,\max}^*$) as a function of u_i and maximum value of n_i^* ($n_{i,\max}^*$), i.e.

$$h_{i,\max}^* = 1 - \frac{u_i}{n_{i,\max}^*} \quad (8)$$

where $n_{i,\max}^* = n_{i,\max}$ or $n_{i,\max}^* = n_{i+1,\max}$ for the case a) and b) respectively.

The graphs for $h_{i,\max}^*$ in individual u -intervals are shown in Fig. 2c.

It is necessary to note that the reasoning analogous to that above could be applied for leveling networks, assuming the lower-bound $h \gtrsim 0.3$.

3.3. Specification of the guidelines

We shall consider the individual u -intervals:

- I. $u \leq 55$; $k_h = k_1 = 1$.

Taking into account the inherent surplus of network reliability (see property 1), the increasing of the number of observations over $2u$ is not necessary, especially if it might considerably increase the cost of the survey. It is, however, possible to raise the reliability according to formula (7). Assuming $n = 220$, i.e. entering into the region of higher k_1 , yields $\bar{h} > 0.75$, which although brings us to an acceptable situation $k_h = k_1 = 2$, but at higher number of gross errors than the initial one.

- II. $55 < u \leq 110$; $k_h = 1$, $k_1 = 2$, and hence, $k_h < k_1$.

The surplus of network reliability due to the property (1) may compensate for this difference. The difference can additionally be diminished by raising the number of

observations to 220, i.e. not entering into the region of $k_1 = 3$, or to $n = 330$, but not reaching the reliability level necessary for the region $k_1 = 3$. The former option, i.e. $n = 220$, seems to be satisfactory.

III. $110 < u \leq 165$; $k_h = 1$, $k_1 = 3$, and hence, $k_h < k_1$.

The surplus of network reliability due to the property (1) may only slightly diminish this difference. To make the effect of the difference smaller we can raise the reliability by increasing the number of observations to 330, i.e. not entering into the region of $k_1 = 4$. Increasing the number of observations to 440 may only worsen the situation.

IV. $165 < u \leq 220$; $k_h = 1$, $k_1 = 4$, and hence, $k_h \ll k_1$.

The surplus of network robustness due to the property (1) may only slightly diminish this difference. It is possible to raise the reliability of the network not leaving the region $k_1 = 4$, i.e. assuming $n = 440$. Further increase in the number of observations may only worsen the situation.

Since in the above analysis the lower-bound $h \gtrsim 0.5$ which corresponds to $u \lesssim 0.5n$, was treated as $h = 0.5$, the resulting boundary values of u should, for practical purposes, be slightly lowered, e.g. $u = 0.48n$. For $n = 220$, we would get slightly reduced number of parameters, i.e. $u = 105$.

4. Conclusions

The law of gross errors was not a main topic of the present research, but a basis for getting orientation as regards realistic upper limits for a number of observations in a network.

- From the analysis based on the Adamczewski modified formula it follows, that
- for networks with up to 50 parameters and not more than 100 observations, the increase in the number of observations without surpassing this limit raises the network internal reliability without increasing the number of potential gross errors;
- for networks with more than 100 parameters, by increasing the number of observations we cannot practically secure a proper level of internal reliability.

Hence, in the light of this law, we may conclude that the increasing of the number of observations is not as advantageous as one might expect when disregarding accumulation of gross errors with the increase in the number of observations.

With a focus on engineering surveys it would be highly recommended to verify both the original and the modified Adamczewski formula by carrying out the empirical tests and theoretical analyses for currently applied setting-out and monitoring networks

of different types and sizes. The formulas when adapted to modern measuring techniques may as well display slower rate of gross error accumulation and yield longer ranges of the n -intervals and u -intervals.

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Poszukiwanie realistycznych ograniczeń górnych dla niezawodności wewnętrznej układów z obserwacjami nieskorelowanymi

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Streszczenie

Z teorii niezawodności wynika, że im większy jest nadmiar obserwacyjny w sieci, tym wyższy poziom jej niezawodności wewnętrznej. Biorąc jednakże pod uwagę fizykalną naturę procesu pomiaru można zauważyć, że projektowane dodatkowe obserwacje mogą zwiększyć liczbę potencjalnych błędów grubych w sieci, nie podnosząc niezawodności wewnętrznej do oczekiwanego według teorii poziomu. Niezbędne jest zatem ustalenie realistycznych poziomów górnych dla liczby obserwacji w sieci. W niniejszym artykule podjęta jest próba sformułowania zasad ustalania takich poziomów. Jako punkt wyjściowy w analizie przyjęto uzyskaną na drodze empirycznej formułę (Adamczewski 2003), nazwaną prawem błędów grubych, pozwalającą wyznaczyć prawdopodobieństwo wystąpienia w sieci określonej liczby błędów

grubych. Przy użyciu pomocniczej zależności wyprowadzonej na podstawie gaussowskiego rozkładu błędów dokonano modyfikacji formuły Adamczewskiego, przekształcając ją w jawną funkcję liczby obserwacji w sieci. Umożliwiło to skonstruowanie narzędzi niezbędnych do analizy, i ostatecznie sformułowanie wskazań co do wyznaczania górnych limitów niezawodności wewnętrznej sieci. Ponieważ formuła Adamczewskiego uzyskana została dla sieci klasycznych, wskazania niniejsze powinny być potraktowane jako wstępna propozycja wymagająca sprawdzenia w odniesieniu do nowoczesnych technik pomiarowych.