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ON SOME ADDITION FORMULAS FOR HOMOGRAPHIC TYPE FUNCTIONS

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Abstract

We deal with the functional equation (so called addition formula) of the form

$$f(x+y) = F(f(x), f(y)),$$

where F is an associative rational function. The class of associative rational functions was described by A. Chéritat [1] and his work was followed by a paper of the author. For function F defined by

$$F(x,y) = \varphi^{-1}(\varphi(x) + \varphi(y)),$$

where φ is a homographic function, the addition formula is fulfilled by homographic type functions. We consider the class of the associative rational functions defined by formula

$$F(u,v) = \frac{uv}{\alpha uv + u + v},$$

where α is a fixed real number.

1. Introduction

For the rational two-place real-valued function F given by

$$F(x,y) = \varphi^{-1}(\varphi(x) + \varphi(y)), \tag{H}$$

where φ is a homographic function, the addition formula has the form

$$\varphi(f(x+y)) = \varphi(f(x)) + \varphi(f(y))$$

and it is a conditional functional equation if the domain of φ is not equal to $\mathbb{R}^2$. Solutions of the above conditional equation are homographic type functions.

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If F is associative and rational then the form (H) states that it belongs to one of the following classes (all of them are considered in their natural domains):

$$F(u,v) = \frac{uv}{\alpha uv + u + v}$$

$$F(u,v) = \frac{u + v + 2\lambda uv}{1 - \lambda^2 uv}$$

$$F(u,v) = \frac{uv - \lambda^2}{u + v + 2\lambda}$$

$$F(u,v) = \frac{(1-2\lambda)uv - \frac{\lambda}{\mu}(u+v) - \frac{1}{\mu^2}}{\lambda\mu uv + u + v + \frac{2-\lambda}{\mu}}$$

where $\alpha \in \mathbb{R}$, $\lambda, \mu \in \mathbb{R} \setminus \{0\}$ (it is a consequence of the associativity).

Let $\alpha \in \mathbb{R}$ be arbitrary fixed. We consider the rational function

$$F : \{(x,y) \in \mathbb{R}^2 : \alpha xy + x + y \neq 0\} \longrightarrow \mathbb{R}$$

of the form

$$F(u,v) = \frac{uv}{\alpha uv + u + v}.$$

It is a rational two-place real-valued function defined on a disconnected subset of the real plane $\mathbb{R}^2$, which satisfies the equation

$$F(F(x,y),z) = F(x,F(y,z))$$

for all $(x,y,z) \in \mathbb{R}^3$ such that

$$\alpha xy + x + y, \alpha yz + y + z, \alpha F(x,y)z + F(x,y) + z, \alpha x F(y,z) + x + F(y,z)$$

are not equal to 0. We shall determine all functions $f : G \longrightarrow \mathbb{R}$, where $(G, \star)$ is a group, which satisfy the functional equation

$$f(x \star y) = \frac{f(x)f(y)}{\alpha f(x)f(y) + f(x) + f(y)}. \tag{1}$$

A neutral element of a group $(G, \star)$ will be written as 0.

By a solution of the functional equation (1) we understand any function $f : G \longrightarrow \mathbb{R}$ which satisfies equality (1) for every pair $(x,y) \in G^2$ such that $\alpha f(x)f(y) + f(x) + f(y) \neq 0$. Thus we deal with the following conditional functional equation:

$$\alpha f(x)f(y) + f(x) + f(y) \neq 0$$

implies

$$f(x \star y) = \frac{f(x)f(y)}{\alpha f(x)f(y) + f(x) + f(y)} \tag{E}$$

for all $x, y \in G$.

Some results on addition formulas can be found for example in the work of K. Domańska and R. Ger [2].

The following lemma will be useful in the sequel (see R. Ger [4]).

Lemma. (On characterization of subgroups). *Let $(G, \star)$ be a group. Then $(H, \star)$ is a subgroup of the group $(G, \star)$ if and only if $G \supset H \neq \emptyset$ and*

$$H \star H' \subset H',$$

where $H' := G \setminus H$.

2. Main results

We proceed with a description of solutions of (E) if $\alpha = 1$.

Theorem 1. *Let $(G, \star)$ be a group. A function $f : G \longrightarrow \mathbb{R}$ yields a nonconstant solution to the functional equation*

$$f(x)f(y) + f(x) + f(y) \neq 0$$

implies

$$f(x \star y) = \frac{f(x)f(y)}{f(x)f(y) + f(x) + f(y)} \tag{E1}$$

for all $x, y \in G$, if and only if either

$$f(x) := \begin{cases} -2 & \text{if } x \in H, \\ 0 & \text{if } x \in G \setminus H \end{cases}$$

or

$$f(x) := \begin{cases} -1 & \text{if } x \in \Gamma \\ -2 & \text{if } x \in G \setminus \Gamma \end{cases}$$

or

$$f(x) = \frac{1}{A(x) - 1}, \quad x \in G$$

where $(H, \star), (\Gamma, \star)$ are subgroups of the group $(G, \star)$, and $A : G \longrightarrow \mathbb{R}$ is a homomorphism such that $1 \notin A(G)$.

Proof. Assume that f is a nonconstant solution of the equation (E1). First we show that $f(0) \in \{-2,-1,0\}$. Indeed, setting $x=y=0$ in (E1) we obtain

$$f(0)^2+2f(0)=0 \quad \text{or} \quad f(0)=\frac{f(0)^2}{f(0)^2+2f(0)},$$

hence $f(0) \in \{-2,-1,0\}$.

First assume that $f(0)=-2$. We show that $f(G) \subset \{-2,0\}$. In fact, putting $y=0$ in (E1) we obtain

$$-2f(x)+f(x)-2=0 \quad \text{or} \quad f(x)=\frac{-2f(x)}{-2f(x)+f(x)-2}$$

for all $x \in G$. Consequently

$$f(x)=-2 \quad \text{or} \quad f(x)=\frac{2f(x)}{f(x)+2}$$

for all $x \in G$ and since the equality

$$c=\frac{2c}{c+2}$$

forces c to vanish, we infer that

$$f(x)=-2 \quad \text{or} \quad f(x)=0$$

for all $x \in G$. Since f is assumed to be nonconstant, both the complementary sets

$$H := \{x \in G: \quad f(x)=-2\} \quad \text{and} \quad H'=\{x \in G: \quad f(x)=0\}$$

are nonempty.

We shall show that $H \star H' \subset H'$, which implies that H is a subgroup of the group G (see Lemma). Fix arbitrarily elements $x \in H$ and $y \in H'$. Since $f(x)f(y)+f(x)+f(y)=-2$, we get $f(x \star y)=0$ by (E1) i.e. $x \star y \in H'$, which was to be shown. So, in this case we have

$$f(x) := \begin{cases} -2 & \text{if } x \in H, \\ \ \ 0 & \text{if } x \in G \setminus H. \end{cases}$$

Let now $f(0)=-1$. Assume that $f(a)=0$ for some $a \neq 0$. Putting $x=0$ and $y=a$ in (E1) we get $f(0)=0$ which leads to a contradiction. Consequently, in this case we have $f(x) \neq 0$ for all $x \in G$. We define a function $A: G \longrightarrow \mathbb{R}$ by the formula

$$A(x)=\frac{1}{f(x)}+1, \quad x \in G.$$

Observe that $1 \notin A(G)$. A straightforward verification shows that

$$f(x)f(y) + f(x) + f(y) = 0 \quad \text{if and only if} \quad A(x) + A(y) = 1$$

for all $x, y \in G$. Thus jointly with (E1) we infer that

$$A(x) + A(y) \neq 1$$

implies

$$A(x \star y) = 1 + \frac{f(x)f(y) + f(x) + f(y)}{f(x)f(y)} = 2 + \frac{1}{f(x)} + \frac{1}{f(y)} = A(x) + A(y)$$

which states that the function A yields a solution of the equation

$$g(x) + g(y) \neq 1 \quad \text{implies} \quad g(x \star y) = g(x) + g(y)$$

for all $x, y \in G$.

Since $f(0) = -1$, evidently $A(0) = 0$. From the theorem proved by R. Ger [3] (since $A(0) = 0$) we conclude that A yields a homomorphism of groups G and $\mathbb{R}$ or there exists a subgroup Γ of the group G such that A is of the form

$$A(x) := \begin{cases} 0 & \text{if } x \in \Gamma, \\ \frac{1}{2} & \text{if } x \in G \setminus \Gamma. \end{cases}$$

Accordingly,

$$f(x) = \frac{1}{A(x) - 1}, \quad x \in G$$

or

$$f(x) := \begin{cases} -2 & \text{if } x \in \Gamma, \\ 0 & \text{if } x \in G \setminus \Gamma. \end{cases}$$

Now let $f(0) = 0$. Putting $y = 0$ in (E1) we obtain that $f(x) = 0$ for all $x \in G$, contradicting the assumption that f is nonconstant. It is easy to check that each of those functions yields a solution to the equation (E1). Thus the proof has been completed. □

Now we proceed with a description of solutions of (E).

Theorem 2. *Let $(G, \star)$ be a group and $\alpha \in \mathbb{R} \setminus \{0\}$ be fixed. A function $f : G \longrightarrow \mathbb{R}$ yields a nonconstant solution to the functional equation*

$$\alpha f(x)f(y) + f(x) + f(y) \neq 0$$

implies

$$f(x\star y)=\frac{f(x)f(y)}{\alpha f(x)f(y)+f(x)+f(y)} \tag{E}$$

for all $x,y\in G$ *if and only if either*

$$f(x):=\begin{cases}-\frac{2}{\alpha} & \text{if } x\in H,\\ 0 & \text{if } x\in G\setminus H\end{cases}$$

or

$$f(x):=\begin{cases}-\frac{1}{\alpha} & \text{if } x\in \Gamma\\ -\frac{2}{\alpha} & \text{if } x\in G\setminus \Gamma\end{cases}$$

or

$$f(x)=\frac{1}{\alpha(A(x)-1)},\quad x\in G$$

where $(H,\star)$, $(\Gamma,\star)$ *are subgroups of the group* $(G,\star)$, *and* $A:G\longrightarrow\mathbb{R}$ *is a homomorphism such that* $1\notin A(G)$.

Proof. Let α be arbitrarily fixed nonzero number. Assume that f is a nonconstant solution of the equation (E) i.e.

$$\alpha f(x)f(y)+f(x)+f(y)\neq 0 \quad \text{implies} \quad f(x\star y)=\frac{f(x)f(y)}{\alpha f(x)f(y)+f(x)+f(y)}$$

for all $x,y\in G$. Hence

$$\alpha^2 f(x)f(y)+\alpha f(x)+\alpha f(y)\neq 0 \text{ implies } \alpha f(x\star y)=\frac{\alpha f(x)f(y)}{\alpha f(x)f(y)+f(x)+f(y)}$$

for all $x,y\in G$ i.e.

$$\alpha f(x)\alpha f(y)+\alpha f(x)+\alpha f(y)\neq 0 \quad \text{implies} \quad \alpha f(x\star y)=$$

$$=\frac{\alpha^2 f(x)f(y)}{\alpha^2 f(x)f(y)+\alpha f(x)+\alpha f(y)}=\frac{\alpha f(x)\alpha f(y)}{\alpha f(x)\alpha f(y)+\alpha f(x)+\alpha f(y)}$$

for all $x,y\in G$. Thus it is easy to observe that (E) states that the function $g:=\alpha f$ satisfies the following functional equation:

$$g(x)g(y)+g(x)+g(y)\neq 0 \quad \text{implies} \quad g(x\star y)=\frac{g(x)g(y)}{g(x)g(y)+g(x)+g(y)}$$

for all $x,y\in G$. From the Theorem 1 we conclude that g is of the form

$$g(x):=\begin{cases}-2 & \text{if } x\in H,\\ 0 & \text{if } x\in G\setminus H\end{cases}$$

or

$$g(x) := \begin{cases} -1 & \text{if } x \in \Gamma \\ -2 & \text{if } x \in G \setminus \Gamma \end{cases}$$

or

$$g(x) = \frac{1}{A(x) - 1}, \quad x \in G$$

where $(H, \star), (\Gamma, \star)$ are subgroups of the group $(G, \star)$, and $A : \Gamma \longrightarrow \mathbb{R}$ is a homomorphism such that $1 \notin A(\Gamma)$. This states that f is of the above form.

It is easy to check that each of those functions yields a solution to the equation (E). Thus the proof has been completed. □

The following remark gives the forms of constant solutions of the equation (E).

Remark. *Let* $(G, \star)$ *be a groupoid and* α *be a fixed nonzero number. The only constant solutions of* (E) *are as follows:* $f = -\frac{2}{\alpha}$, $f = 0$ *and* $f = -\frac{1}{\alpha}$.

To check this, assume that $f = c$ fulfils (E). Then

$$\alpha c^2 + 2c \neq 0 \quad \text{implies} \quad c = \frac{c^2}{\alpha c^2 + 2c}$$

i.e.

$$c = 0 \quad \text{or} \quad \alpha c + 2 = 0 \quad \text{or} \quad 1 = \frac{1}{\alpha c + 2}.$$

Hence

$$c \in \left\{ -\frac{2}{\alpha}, -\frac{1}{\alpha}, 0 \right\},$$

which was to be shown. □

The following theorem gives the form of solutions of the equation (E) if $\alpha = 0$, i.e. the form of solutions of the following equation

$$f(x) + f(y) \neq 0 \quad \text{implies} \quad f(x \star y) = \frac{f(x)f(y)}{f(x) + f(y)} \tag{E0}$$

Theorem 3. *Let* $(G, \star)$ *be a monoid. The only solution* $f : G \longrightarrow \mathbb{R}$ *of the equation* (E0) *is* $f = 0$.

Proof. Assume that f is a solution of the equation (E0). First we show that $f(0) = 0$. In fact, setting $x = y = 0$ in (E1) we obtain

$$f(0) = 0 \quad \text{or} \quad f(0) = \frac{f(0)^2}{2f(0)} = \frac{1}{2} f(0)$$

whence $f(0) = 0$. Fix arbitrarily an $x \in G$ and take $y = 0$. Then, by (E0), we get

$$f(x) \neq 0 \Longrightarrow f(x) = 0$$

which implies $f = 0$ and which was to be shown. It is easy to check that $f = 0$ yields a solution to the equation (E0). Thus the proof has been completed. $\square$

References

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