

On the optimal control problem for two regions' macroeconomic model

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In this paper we consider a model of joint economic growth of two regions. This model bases on the classical Kobb-Douglas function and is described by a nonlinear system of differential equations. The interaction between regions is carried out by changing the balance of trade. The optimal control problem for this system is posed and the Pontryagin maximum principle is used for analysis the problem. The maximized functional represents the global welfare of regions. The numeric solution of the optimal control problem for particular regions is found. The used parameters was obtained from the basic scenario of the MERGE.

Key words: integrated assessment model for evaluating greenhouse gases reduction policies, optimal control, Pontryagin's maximum principle.

1. Introduction

Economic growth in the world economy has a rather accurate assessment, but the situation significantly changes when considering different regions of the world where growth rates can greatly vary. In modern economic theory there are many models of economic growth (see, for example [4, 13]). The interest of researchers is associated with finding major factors and analysis of their impact on a performance of national economies.

Dependencies of the country's GDP on the existing in country fixed assets (capital), a labor, determined by the number of working population, investment, etc. are studied, among others, in macroeconomics. A problem of modeling and investigation such dependencies become more complicated when considering the relationship of economies of the different countries in the global context. By viewing a group of national economies, the global climate change should also be considered in some sense. It is quite a complex and contradictory concept, but many related issues continue to provoke vigorous debates. Of course, human economic activities have an impact on environmental

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and social systems of regions, but the assessment of this impact seems to be quite a difficult problem in conjunction with the climate changes. The integrated assessment models are used for research and forecasting of changes of various region characteristics; one of them is the MERGE proposed in [5, 6] and modified in the works [2, 7]. This model is a framework for research of climate changes and an assessment of their impact on the development of social-economic systems of regions. Also it provides to build different scenarios of their dynamics.

The present work is closely related to the economic-energy submodel of the MERGE, which is intended for the simulation of region macroeconomic indicators on a large time interval. It is a fully integrated applied model of the general equilibrium. At each time point supply and demand are equalized by commodity prices in the energy sector, as well as the national common product, uniting all goods produced outside the sector. Each region is represented by as a single producer-consumer. Decisions about the investment are modeled by such choice of the consumption level sequence maximizing the sum of discounted utilities of consumption. The module presented in the MERGE is a discrete model with a possible non-uniform mesh of the considered time period. Optimal trajectories of components of the region economic-energy systems are found with a help of the intertemporal optimization by maximizing the sum of discounted utilities of the regional consumption in a whole time interval. The optimization problem in this case is a problem of nonlinear programming and for its solution the iterative joint maximization method is used [10].

2. Model

In the present paper we consider a model of joint growth of two regions similar to those used in the MERGE. Dynamics of main characteristics of each region are described by the system obtained using a combination of Cobb–Douglas type classical production functions which nested in the constant elasticity of substitution production function:

$$\begin{aligned}
 \dot{Y}_r &= -\mu Y_r + (a u_r^{\alpha\rho} l_r^{(1-\alpha)\rho} + b v_r^{\beta\rho} w_r^{(1-\beta)\rho})^{1/\rho}, \\
 \dot{K}_r &= -\mu K_r + u_r, \\
 \dot{E}_r &= -\mu E_r + v_r, \\
 \dot{N}_r &= -\mu N_r + w_r, \quad t \in [t_0, T],
 \end{aligned} \tag{1}$$

where $Y_r(t)$ is the economic output in every period t (GDP); $K_r(t)$ is the capital (fixed assets); $E_r(t)$ is the electricity; $N_r(t)$ is the non-electric energy; $l_r(t)$ is a continuous function described the labour and depending on the number of working population; α is an elasticity of substitution between capital and labour; β is an elasticity of substitution between electricity and non-electric energy; μ is the coefficient of depreciation; ρ is an elasticity of substitution between capital-labour and energy bundle; a and b are scale productivity factors. Here and further we assume that $r = 1, 2$.

We impose the restrictions on control parameters of the system (1) of the form:

$$0 < a_u^r \leq u_r \leq b_u^r, \quad 0 < a_v^r \leq v_r \leq b_v^r, \quad 0 < a_w^r \leq w_r \leq b_w^r. \tag{2}$$

The functions $u_r(\cdot)$, $v_r(\cdot)$, and $w_r(\cdot)$ satisfying the relations (2) will be called admissible controls. Let us introduce a vector-function $c_r(\cdot) = (u_r(\cdot), v_r(\cdot), w_r(\cdot))^T$ and denote a set of the admissible controls by the symbol $U \subset L_2([t_0, T], \mathbb{R}^3)$.

We suppose that the parameters of regions are known at the initial moment t_0 , i.e. it is given an initial state

$$Y_r(t_0) = Y_r^0, \quad E_r(t_0) = E_r^0, \quad K_r(t_0) = K_r^0, \quad N_r(t_0) = N_r^0, \quad Y_r^0, E_r^0, K_r^0, N_r^0 > 0. \tag{3}$$

3. Statement of the optimal control problem

Further, an optimal control problem of the system (1)–(3) is considered.

Problem P1. It is required to determine the functions $Y_r^*(\cdot)$, $K_r^*(\cdot)$, $E_r^*(\cdot)$, $N_r^*(\cdot)$, $c_r^*(\cdot)$, and $f_r^*(\cdot)$ solving the extremal problem

$$\max_{Y_r, K_r, E_r, N_r, c_r, f_r} J(Y_r, K_r, E_r, N_r, c_r, f_r), \tag{4}$$

$$J(Y_r, K_r, E_r, N_r, c_r, f_r) = \int_{t_0}^T \sum_{r=1}^2 d_r(t) \ln C_r(t) dt, \tag{5}$$

satisfying the system (1) and ensuring the fulfillment of restrictions (2). Here $d_r(t)$ is a coefficient which represents the social discount factor and the economic loss factor due to the impact of climate change, the functions $d_r(t)$ are assumed to be given, and $d_r(t) > 0$ for $t \in [t_0, T]$, $d_1(t) + d_2(t) \leq 1$; $f_r(t)$ is the balance of trade (the difference between regional export and import of consumer goods), this functions are also the control parameters satisfying the conditions

$$f_1(t) = f(t), \quad f_2(t) = -f(t), \quad |f(t)| \leq b_f, \quad t \in [t_0, T]. \tag{6}$$

Consumption $C_r(t)$ at the moment t is determined by the classical formula [5]:

$$C_r(t) = Y_r(t) - I_r(t) - f_r(t) - G_r(E_r(t), N_r(t)), \tag{7}$$

where $I_r(t)$ is an investment used to built capital stock, we assume $I_r(t) = u_r(t)$; $G_r(E_r(t), N_r(t))$ is the energy cost function that represents the total costs of extracting resources and supplying electric and non-electric energy and is determined by the equality

$$G_r(E_r(t), N_r(t)) = g_r E_r(t) + h_r N_r(t), \tag{8}$$

where the positive coefficients g_r and h_r characterize the production cost of electricity and non-electric energy, respectively.

The variables K_r are not present explicitly in the definition of maximized functional (5) and in the equations for characteristics $Y_r(t)$, $E_r(t)$, and $N_r(t)$, therefore the optimal trajectories $K_r(t)$ are determined only by the initial conditions K_r^0 and the optimal controls u_r^* . Thus, we can ignore the corresponding equation of the system (1) when solving the extremal problem (4).

Because of the economic sense of parameters of the system (1) let us impose restrictions in the following form:

$$0 < \alpha, \beta < 1, \quad \rho < 0, \mu > 0, \quad a_u^r, b_u^r, a_v^r, b_v^r, a_w^r, b_w^r, b_f > 0. \tag{9}$$

Lemma 1 *For the functions $Y_r(t)$, $E_r(t)$ and $N_r(t)$ the following estimates are valid:*

$$Y_r(t) \geq Y_m^r(t_Y^r), \quad E_r(t) \leq E_M^r(t_E^r), \quad N_r(t) \leq N_M^r(t_N^r), \quad t \in [0, T], \tag{10}$$

where

$$\begin{aligned} Y_m^r(t) &= e^{-\mu t} Y_r^0 + \mu^{-1} \xi_r (1 - e^{-\mu t}), \\ E_M^r(t) &= e^{-\mu t} E_r^0 + \mu^{-1} b_v^r (1 - e^{-\mu t}), \\ N_M^r(t) &= e^{-\mu t} N_r^0 + \mu^{-1} b_w^r (1 - e^{-\mu t}), \\ \xi_r &= (a_u^r)^{\alpha\rho} \min_{\tau \in [t_0, T]} I_r^{(1-\alpha)\rho}(\tau) + b(a_v^r)^{\beta\rho} (a_w^r)^{(1-\beta)\rho})^{1/\rho}, \end{aligned} \tag{11}$$

$$t_Y^r = \begin{cases} t_0, & \xi_r \geq \mu Y_r^0, \\ T, & \xi_r < \mu Y_r^0, \end{cases} \quad t_E^r = \begin{cases} T, & b_v^r \geq \mu E_r^0, \\ t_0, & b_v^r < \mu E_r^0, \end{cases} \quad t_N^r = \begin{cases} T, & b_w^r \geq \mu N_r^0, \\ t_0, & b_w^r < \mu N_r^0. \end{cases} \tag{12}$$

Proof for each region can be found in [12].

Theorem 1 *Let the parameters of the system (1) satisfy the conditions (9), the restrictions on the controls (2), (6) and the initial values (3) satisfy the inequalities*

$$Y_m^r(t_Y^r) - b_u^r - b_f - g_j E_M^r(t_E^r) - h_r N_M^r(t_N^r) > 0,$$

where Y_m^r , E_M^r , and N_M^r are defined by the formulas (11), t_Y^r , t_E^r , and t_N^r – by the formulas (12); then the functions of consumption take only positive values

$$C_r(t) > 0, \quad t \in [t_0, T].$$

Proof By taking into account the functions of energy expenditures (8) the following estimate for the functions of consumption (7) is valid

$$C_r(t) = Y_r(t) - u_r(t) - f_r(t) - g_r E_r(t) - h_r N_r(t) \geq Y_r(t) - b_u^r - b_f - g_r E_r(t) - h_r N_r(t).$$

By using the inequalities of Lemma we obtain that

$$C_r(t) \geq Y_m^r(t_Y^r) - b_u^r - b_f - g_r E_M^r(t_E^r) - h_r N_M^r(t_N^r).$$

We finished the proof of the Theorem. □

Further, we shall assume that the statement of Theorem is valid. Let us reduce the system (1) by introducing the notation $Z_r(t) = Y_r(t) - g_r E_r(t) - h_r N_r(t)$. Then we re-write the system (1) to the form

$$\dot{Z}_r(t) = -\mu Z_r(t) + (a u_r^{\alpha\rho}(t) l_r^{(1-\alpha)\rho}(t) + b v_r^{\beta\rho}(t) w_r^{(1-\beta)\rho}(t))^{1/\rho} - g_r v_r(t) - h_r w_r(t) \quad (13)$$

with the corresponding boundary conditions

$$Z_r(0) = Z_r^0 = Y_r^0 - g_r E_r^0 - h_r N_r^0. \quad (14)$$

As a result we obtain the following optimal control problem.

Problem P2 It is required to define functions $Z_r^*(\cdot)$, $c_r^*(\cdot)$, and $f_r^*(\cdot)$, solving the extremal problem

$$\max_{Z_r, c_r, f_r} J(Z_r, c_r, f_r),$$

$$J(Z_r, c_r, f_r) = \int_{t_0}^T \sum_{r=1}^2 d_r(t) \ln(Z_r(t) - u_r(t) - f_r(t)) dt,$$

satisfying the equations (13) with the boundary conditions (14) and ensuring the implementation of restrictions (2) and (6).

4. Solution of the problem P2

We use the Pontryagin maximum principle [8] for investigation of properties of the vector functions $c_r(t)$ and $f_r(t)$, $t_0 \leq t \leq T$, which are the control optimal programs for the Problem P2.

Let us write the Hamilton–Pontryagin function for the problem assuming $\psi_0 = -1$:

$$H(t, Z_r, \psi_r, c_r, f_r) = \sum_{r=1}^2 (\psi_r (-\mu Z_r + (a u_r^{\alpha\rho} l_r^{(1-\alpha)\rho}(t) + b v_r^{\beta\rho} w_r^{(1-\beta)\rho}(t))^{1/\rho} - g_r v_r - h_r w_r) + d_r(t) \ln(Z_r - u_r - f_r(t))). \quad (15)$$

Then we find the partial derivatives of the hamiltonian corresponding to the coordinates Z_r :

$$\frac{\partial H}{\partial Z_r} = -\mu\psi_r + \frac{d_r(t)}{Z_r - u_r - f_r(t)}.$$

So the conjugate system $\dot{\psi}_r = -H'_{Z_r}$ takes the form

$$\dot{\psi}_r = \mu\psi_r - \frac{d_r(t)}{Z_r - u_r - f_r(t)}. \tag{16}$$

The right ends of trajectories are free, therefore the conjugate variables satisfy the transversality conditions

$$\psi_r(T) = 0. \tag{17}$$

By using the equations (13) with the conditions (14), as well as (16) and (17), we obtain the maximum principle boundary value problem for the Problem P2 of the form

$$\begin{aligned} \dot{Z}_r &= -\mu Z_r + (a u_r^{\alpha\rho} l_r^{(1-\alpha)\rho}(t) + b v_r^{\beta\rho} w_r^{(1-\beta)\rho})^{1/\rho} - g v_r - h w_r, & Z_r(t_0) &= Z_r^0, \\ \dot{\psi}_r &= \mu\psi_r - \frac{d_r(t)}{Z_r - u_r - f_r(t)}, & \psi_r(T) &= 0. \end{aligned} \tag{18}$$

Proposition 1 *Let the assumptions of the Theorem hold. Then the partial derivatives of the hamiltonian (15) corresponding to the controls u_r are not equal to zero.*

The validity of the proposition is established by analogy with the proof in [12].

Two cases follows from this assertion: either $u_r^*(t) = a_r^r$ or $u_r^*(t) = b_u^r$, $t \in [t_0, T]$. Let us find derivatives of the hamiltonian corresponding to the controls f , v_r , and w_r :

$$\begin{aligned} \frac{\partial H}{\partial f} &= \frac{d_2(Z_1(t) - u_1(t) - f(t)) - d_1(t)(Z_2(t) - u_2(t) + f(t))}{(Z_1(t) - u_1(t) - f(t))(Z_2(t) - u_2(t) + f(t))}, \\ \frac{\partial H}{\partial v_r} &= \psi_r (b\beta(a(u_r^*)^{\alpha\rho} l_r^{(1-\alpha)\rho}(t) + b v_r^{\beta\rho} w_r^{(1-\beta)\rho})^{1/\rho-1} v_r^{\beta\rho-1} w_r^{(1-\beta)\rho} - g_r), \\ \frac{\partial H}{\partial w_r} &= \psi_r (b(1-\beta)(a(u_r^*)^{\alpha\rho} l_r^{(1-\alpha)\rho}(t) + b v_r^{\beta\rho} w_r^{(1-\beta)\rho})^{1/\rho-1} v_r^{\beta\rho} w_r^{(1-\beta)\rho-1} - h_r). \end{aligned}$$

By equating the derivatives to zero, we have that

$$\begin{aligned} f^*(t) &= \frac{d_2(t)(Z_1(t) - u_1(t)) - d_1(t)(Z_2(t) - u_2(t))}{d_1(t) + d_2(t)}, \\ v_r^*(t) &= a^{1/\rho} u_r^*(t) l_r^{1-\alpha}(t) (p_r^\gamma q_r^{(1-\beta)\gamma\rho} - b q_r^{\beta-1}), \\ w_r^*(t) &= a^{1/\rho} u_r^*(t) l_r^{1-\alpha}(t) q_r^{-1} (p_r^\gamma q_r^{(1-\beta)\gamma\rho} - b q_r^{\beta-1}), \quad t \in [0, T], \end{aligned} \tag{19}$$

where $p_r = g_r / (b\beta)$, $q_r = \beta h_r / ((1-\beta)g_r)$, and $\gamma = 1 / (1-\rho)$.

5. An algorithm of solving the Problem P1

An analytical solution of the problem P1 as an optimal control problem is complicated because of the nonlinearity of the system (1), as well as concavity of the functional (5). For the solution of this problem methods of concave optimization may be used [3,11]. In this case we shall construct an optimal control by investigating the restrictions (2).

Let $\{\tau_i\}_0^n$, $\tau_0 = t_0$, $\tau_n = T$, is a time mesh of the interval $[t_0, T]$ with the step $\delta = (T - t_0) / n$. We assume $f_r^{(0)} = 0$, $u_r^{(0)} = 0$, $v_r^{(0)} = 0$, and $w_r^{(0)} = 0$. In the i -th iteration the optimal controls $u_r^{(i)}$ according to the Lemma can be selected as a_r or as b_r . For the optimal controls $f_r^{(i)}$, $v_r^{(i)}$, and $w_r^{(i)}$ we assume that they can be selected as the boundary values of intervals (6), (2) or as values defined by the formulas (19) respectively.

For each collection of the control parameters $c_r^{(1)}, \dots, c_r^{(n)}$, $f_r^{(1)}, \dots, f_r^{(n)}$ we solve the system of differential equations (1) with a help of the Euler method [1]. Then in $(i + 1)$ -th step of the iteration procedure we obtain that

$$\begin{aligned}
 Y_r^{(i+1)} &= Y_r^{(i)} + \delta(-\mu Y_r^{(i)} + (au_r^{(i+1)\alpha\rho} l_r^{(i)(1-\alpha)\rho} + bv_r^{(i+1)\beta\rho} w_r^{(i+1)(1-\beta)\rho})^{1/\rho}), \\
 K_r^{(i+1)} &= K_r^{(i)} + \delta(-\mu K_r^{(i)} + u_r^{(i+1)}), \\
 E_r^{(i+1)} &= E_r^{(i)} + \delta(-\mu E_r^{(i)} + v_r^{(i+1)}), \\
 N_r^{(i+1)} &= N_r^{(i)} + \delta(-\mu N_r^{(i)} + w_r^{(i+1)}), \quad i = 0, \dots, n-1.
 \end{aligned}$$

We take into account the found values of functions in the mesh points to obtain the functional J (5) by calculating the integral using the rectangle method [9]:

$$J = \delta \sum_{i=1}^n \sum_{r=1}^2 d_r^{(i)} \ln C_r^{(i)},$$

where

$$C_r^{(i)} = Y_r^{(i)} - u_r^{(i)} - f_r^{(i)} - G^{(i)}(E_r^{(i)}, N_r^{(i)}).$$

Finally, we choose the largest value of the functional among obtained and emphasize the corresponding set of controls, which will be optimal.

6. Results of the numerical modeling

By using the algorithm described in the previous section let us construct a solution of the problem P1 for specific values of parameters of the system (1) and restrictions (2). We consider two cooperated regions (R1 and R2, numbers $r = 1$ and $r = 2$ respectively), region R1 has macroeconomic characteristics belonging to Russian Fed-

eration, and R2 has characteristics of Ukraine. By using results of the work [5] we set up the following values of parameters: $\alpha = 0.3$, $\beta = 0.45$, $\mu = 0.05$, $\rho = -1.5$. We investigate the dynamics of main indexes on the time interval from 2010 to 2018 years; the initial state is following: $Y_1^0 = \$2.242$ trillion, $K_1^0 = \$6.061$ trillion, $E_1^0 = 1.038$ TkWh, $N_1^0 = 20.3344$ EJ [14], $Y_2^0 = \$0.306$ trillion, $K_2^0 = \$0.857$ trillion, $E_2^0 = 0.2$ TkWh, $N_2^0 = 5.258$ EJ [16]. Hereinafter the symbol \$ is USD2005.

Parameters of the energy expenditures function (8) are defined by the formulas $g_1 = \$0.09$ trillion/TkWh, $h_1 = \$0.0054$ trillion/EJ, $g_2 = \$0.0563$ trillion/TkWh, $h_2 = \$0.0025$ trillion/EJ [5].

We assume that the differences between exports and imports $f_r(t)$ do not exceed \$0.03 trillion. The functions $l_r(t)$ describing a labor productivity and being measured in efficiency units are chosen as constants: $l_1(t) = l_2(t) = 1$. The discount factors of utility are defined by the formulas $d_1(t) = 0.032$ and $d_2(t) = 0.005$. By using the relations for the regions' basic characteristics introduced in [5], we find out $a = 5.44$ and $b = 0.64$.

As boundary values of the controls u_1 we choose the investment volumes of 2008 and 2010 years, i.e. $a_u^1 = \$0.005$ trillion, $b_u^1 = \$0.5$ trillion [14, 17]. As boundary values of the control u_2 we choose the investment volumes of 1996 and 2010 years, i.e. $a_u^2 = \$0.005$ trillion, $b_u^2 = \$0.03$ trillion [16].

We choose bounds of the control v_1 as $a_v^1 = 0.01$ TkWh and $b_v^1 = 0.08$ TkWh which base on the production growth of electricity in 2008–2009 and 2009–2010 years respectively [14]. For the second region we choose bounds of the control v_2 as the growth rate of electricity production in 2005–2010 and 1987–1988 years [16]. As a result we have that $a_v^2 = 0.005$ TkWh and $b_v^2 = 0.012$ TkWh. For the controls w_r we define the following restrictions: $a_w^1 = 0.01$ EJ, $b_w^1 = 0.6$, $a_w^2 = 0.01$ EJ and $b_w^2 = 0.15$ EJ. Then we choose a step of the time interval mesh equal to one year, i.e. $n = 8$.

Graphics of the GDP and the electricity production of regions are demonstrated in Fig. 1–4. The black lines illustrate results obtained with a help of the algorithm of solving problem P1. Data of International Monetary Fund [15] are presented by the grey line in Fig. 1 and Fig. 2. Data of the MERGE' basic scenario is shown also by the grey line in Fig. 3 and Fig. 4.

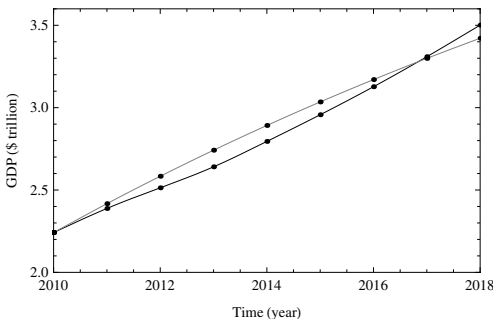


Figure 1: Realized GDP of region R1.

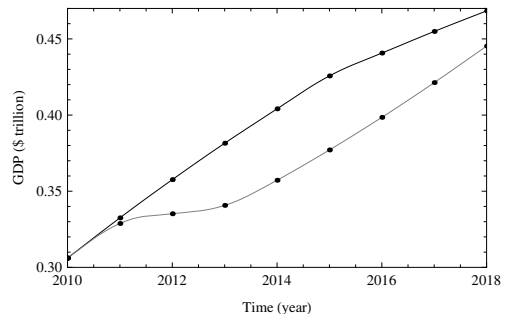


Figure 2: Realized GDP of region R2.

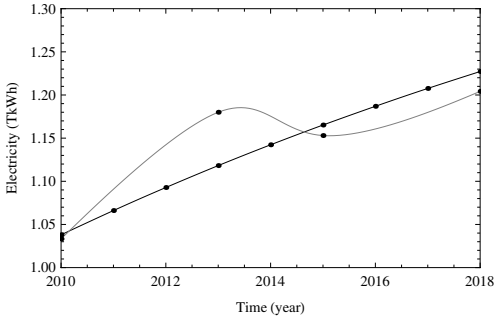


Figure 3: Electricity production of region R1.

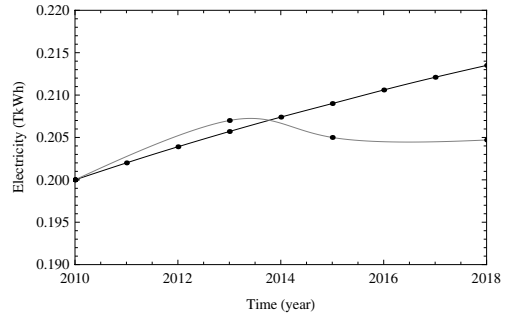


Figure 4: Electricity production of region R2.

Investment of the region R1 taken as the control parameter u_1 reach the maximum b_u^1 from 2010 to 2018 years, but investment of the region R2 (u_2) take b_u^2 from 2010 to 2015 years, then take the value a_u^2 up to 2018 year.

In the considered problem for the given parameters set, the optimal control u_2^* has a single switch point. The optimal controls of energy production are defined by the formulas

$$v_r^*(t) = b_v^r, \quad w_r^*(t) = b_w^r.$$

The optimal control $f^*(t)$ conforming to the difference between imports and exports for each region is shown in Fig. 5 and Fig. 6 respectively.

Graphics demonstrate a similar behavior in Fig. 1, whereas essential distinctions are observed at the GDP graph of Ukraine (Fig. 2). Discrepancies between obtained values and the IMF forecast can be explained by that at first the data of 2011–2013 years is used by the IMF for prediction, but we use only data at the initial moment (2010 year). Secondly there are differences in the forecasts of electricity production (Fig. 4).

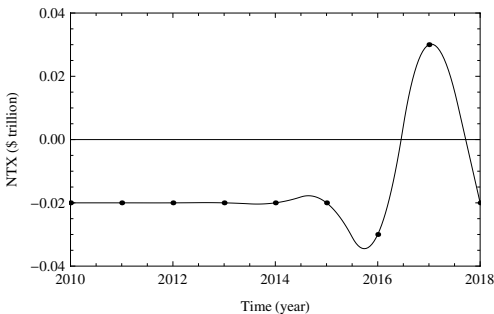


Figure 5: Difference between import and export of region R1.

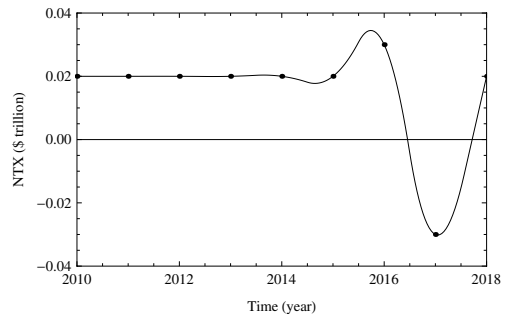


Figure 6: Difference between import and export of region R2.

7. Summary

In this paper we investigated one economic model of two regions' joint growth. The model described by nonlinear differential equations and the nonlinearity of this system is provided by the classical Cobb-Douglas function and the constant elasticity of substitution production function.

The optimal control problem was posed for this model and we applied the Pontryagin maximum principle to carry out the analysis. The problem is significantly complicated, so we tried to simplify it for simulation. We used the macroeconomic parameters belonging to real regions. They had been found out from the MERGE. The proposed approach was applied and we obtained good results of simulation. The deep analysis of some of them will be held in future investigations. One of the main directions of research is considering a several interacting regions.

References

- [1] J.C. BUTCHER: Numerical Methods for Ordinary Differential Equations. John Wiley & Sons, New York, 2008.
- [2] B.V. DIGAS and V.L. ROZENBERG: Application of an optimization model to studying some aspects of Russia's economic development. *Int. J. Environmental Policy and Decision Making*, **1**(1), (2010), 51-63.
- [3] D.Y. GAO: Duality Principles in Nonconvex Systems. Theory, Methods and Applications. Springer series on Nonconvex optimization and its application, **39**, 2000.
- [4] D.H. ROMER: Advanced Macroeconomics. Fourth ed. McGraw-Hill, New York, 2011.
- [5] A. MANNE, R. MENDELSON and R. RICHEL: MERGE — a model for evaluating regional and global effects of GHG reduction policies. *Energy Policy*, **23**(1), (1995), 17-34.
- [6] A. MANNE: Energy, the environment and the economy: hedging our bets. Global Climate Change, ed. by J.M. Griffin. Edward Elgar, Northampton(MA), 2000.
- [7] V. MAKSIMOV, L. SCHRATTENHOLZER and Y. MINULLIN: Computer analysis of the sensitivity of the integrated assessment model MERGE-5I. *Lecture Notes in Computer Science*, **3911**, (2006), 583-590.
- [8] L.S. PONTRYAGIN, V.G. BOLTYANSKII, R.V. GAMKRELIDZE and E.F. MISHCHENKO: The Mathematical Theory of Optimal Processes. John Wiley & Sons, New York. 1962.
- [9] A. RALSTON and P. RABINOWITZ: A First Course in Numerical Analysis. 2nd ed. Dover, New York. 2001.

- [10] T. RUTHERFORD: Sequential Joint Maximization. Energy and Environmental Policy Modeling. ed. by J.P. Weyant. Kluwer Academic Publishers, Norwell(MA). 1999, 139-175.
- [11] R.G. STRONGIN and Y.D. SERGEYEV: Global Optimization with Non-Convex Constraints. Springer series on Nonconvex optimization and its application, **45**, (2000).
- [12] P.G. SURKOV: On an optimal control problem for a nonlinear economic model. *Applied Mathematical Sciences*, **8**(171), (2014), 8517-8527.
- [13] W. WELFE: Macroeconometric Models. Springer-Verlag, Berlin Heidelberg, 2013.
- [14] Federal state statistics service. URL: <http://www.gks.ru/>.
- [15] International Monetary Fund. URL: <http://www.imf.org/>.
- [16] The state statistics service of Ukraine. URL: <http://www.ukrstat.gov.ua/>.
- [17] World Bank. URL: <http://data.worldbank.org>.