NONLINEAR HOMOGENIZATION OF HETEROGENEOUS PERIODIC PLATES OF REISSNER-MINDLIN TYPE

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In this paper, we propose a multiscale finite-strain plate theory for a composite nonlinear plate described by a repetitive periodic heterogeneity. We consider two scales, the macroscopic scale is linked to the entire plate and the microscopic scale is linked to the size of the heterogeneity. At the macroscopic scale, we approximate the displacement field by the Reissner-Mindlin model. By considering the equivalence between variations of the macroscopic elastic energy at each point of the mid surface and the microscopic one, we deduce that the macroscopic stress resultants can be expressed in terms of the microscopic stress.

Keywords: homogenization, plate, nonlinear hypothesis

1. Introduction

Composite plates are widely used in aeronautics applications because they offer the excellent ratio between stiffness or strength performance and weight. The size of fine scale details in such heterogeneous plates is typically much smaller compared to dimensions of the structure. Thus the making of direct numerical analyses is prohibitively expensive. To avoid large-scale computations, it is preferable to model the plates at the macroscale as a homogeneous continuum with effective properties obtained through a homogenization procedure. Based on asymptotic homogenization concepts, Caillerie (1984), Kohn and Vogelius (1984) discussed homogenization of heterogeneous periodic linear elastic plates. Their models are mathematically elegant and rigorous but only related to a simple engineering model (the Kirchhoff plate model). The Kirchhoff-Love plate model is the simplest and the most widely-used theory. Nevertheless, this model neglects contribution of out-of-plane stress components to stress energy. However, when the plate slenderness ratio L/h (h is plate thickness and L is characteristic dimension of its mid-plane) decreases, out-of-plane stresses have an increasing influence on the plate deflection. Exactly as Cecchi and Sab (2007) did for Reissner-Mindlin homogenization of periodic plates, Lebée and Sab (2012) proposed a homogenization theory for their bending gradient theory (Lebée and Sab, 2011). That approach corrected the homogenization theory of Lewiński (1991), Caillerie (1984), Kohn and Vogelius (1984) in order to take into account of out-of-plane stress components (transverse shearing and transverse normal stress). So, they used implicitly the superposition principle and then limited their theories to linear elasticity.

The study of Petracca *et al.* (2017) focused on periodic brick-masonry walls. The macro-scale behavior obeys the Reissner-Mindlin model and the local heterogeneous structures is assumed to be transverse isotropic. For the macroscopic Reissner-Mindlin plate model, Terada *et al.* (2017) proposed a new numerical plate testing (NPT) by adding a specific microscopic displacement terms such that the out-of-plane microscopic shear strain components contained the macroscopic curvature associated with torsional deformation.

The method of homogenization proposed by Lee *et al.* (2014) is deduced from the introduction of the double scale asympytotic expansion method into a new double scale variational

formulation. The developments given by this method are valid for the macro-scale Reissner-
-Mindlin behavior and restricted to small deformations and large rotations and displacements.
As a consequence, this approach seems to be intractable in the fully nonlinear setting. This is
due to fact that simplification linked to the application of the variational-asymptotic method
(Berdichevskii, 1979; Sutyrin, 1997) runs only under the small deformation assumption. More-
over, the recent extension of the asymptotic expansion for homogenization of the plate made of
a nonlinear Saint Venant-Kirchoff material proposed by Kalamkarov *et al.* (2017) seems to be
restricted to bending and stretching.

This paper concerns the modeling of the mechanical response of heterogeneous plate struc-
tures. The macroscopic displacement field is assumed to be of the Reissner-Mindlin type. So, we
avoid the Saint Venant-Kirchoff asumption considerd in Coenen *et al.* (2010), Cong *et al.* (2015).
The mechanical behaviour of the constituents of the plate is of a nonlinear hyperelastic type.
Then, it is necessary to define the relation between the definitions of the macroscopic generalized
strains and stresses for a plate continuum in terms of the microscopic ones. This macro-to-micro
scale transition is performed by imposing the macroscopic generalized deformation gradient on
the RVE (representative volume element) through essential boundary conditions that may be
periodic conditions. Upon solution of the microstructural boundary value problem, the macro-
scopic generalized stress resultants are expressed by averaging the computed RVE stress field
through the use of a generalised Hill-Mandel condition for shells (i.e. an energy condition where
the energy in the macroscale is equal to the one in the microscale). In our work, the through
thickness dimension is directly combined with the in-plane homogenization.

2. Two scales description of a heterogeneous plate

Let us denote by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ the canonical orthonormal bases of $\mathbb{R}^3$. In the plane $(\mathbf{O}, \mathbf{e}_1, \mathbf{e}_2)$,
the domain defines the mid-plane of the plate. Then, we denote by $\mathbf{x}' = (x_1, x_2)$ the cartesian
coordinate of a point of the mid-plane which also defines the macro-scale of the plate. So, the size
of its mid-plane defines the global length-scale. This plate is heterogeneous and we assume that
the microstructure of the plate is repetitive periodic in the mid-plane. This microstructure is
defined at the microscopic scale $\mathbf{y}$ and it is sufficient to define the distribution of the constituents
on the smallest period or unit cell as follows

$$\mathbf{Y} = \left\{ \mathbf{y} = (y_1, y_2, y_3); \; \mathbf{y}' = (y_1, y_2) \in \mathbf{Y}' = [0, a] \times [0, b], -\frac{h}{2} \leqslant y_3 \leqslant \frac{h}{2} \right\} \tag{2.1}$$

The size of heterogeneities, which is assumed to be of the same order of magnitude of thick-
ness h, is very small with regard to the global length-scale l. The plate is formed by an integer
number of the unit cell. Both the upper and lower boundaries of the plate can be defined in
microscopic coordinates $\mathbf{y}$: $\partial \mathbf{Y}^{\pm} = \{(\mathbf{y}', \pm h/2)\}$. The lateral boundary $\partial \boldsymbol{\omega} \times \mathbf{Y}$ is split into two
parts, the first one $\partial \boldsymbol{\omega}_{\bar{\mathbf{t}}} \times [-h/2, h/2]$ is subjected to the surface force $\bar{\mathbf{t}}(\mathbf{x}', y_3)$ and on the second
one $\partial \boldsymbol{\omega}_{\bar{\mathbf{x}}} \times [-h/2, h/2]$, the current position vector $\widetilde{\mathbf{x}}(\mathbf{x}', y_3)$ is prescribed by $\overline{\mathbf{x}}(\mathbf{x}', y_3)$.

The load acting on both the upper and lower boundary of the plate $(\overline{\mathbf{t}}^{\pm}(\mathbf{x}', \mathbf{y}'))$ and the body
forces $(\mathbf{f}(\mathbf{x}', \mathbf{y}))$ are periodic in $\mathbf{y}'$.

3. Nonlinear homogenization by using the Hill-Mandel macro-micro concept

3.1. Homogeneous plate model

We consider a Reissner-Mindlin model, then the current macroscopic position is approxima-
ted in the form

$$\widetilde{x}_\alpha(\mathbf{x}', y_3) = \widetilde{x}_\alpha^{(0)}(\mathbf{x}') + y_3 \widetilde{x}_\alpha^{(1)}(\mathbf{x}') \qquad \widetilde{x}_3(\mathbf{x}', y_3) = y_3 + \widetilde{x}_3^{(0)}(\mathbf{x}') \tag{3.1}$$

Accordingly, the deformation gradient $\mathbf{F}$ is defined by

$$
\begin{aligned}
F_{\alpha\beta}(\mathbf{x}', y_3) &= \widetilde{x}^{(0)}_{\alpha,\beta}(\mathbf{x}') + y_3 \widetilde{x}^{(1)}_{\alpha,\beta}(\mathbf{x}') \\
F_{\alpha 3}(\mathbf{x}', y_3) &= \widetilde{x}^{(1)}_\alpha(\mathbf{x}') \qquad F_{3\alpha}(\mathbf{x}', y_3) = \widetilde{x}^{(0)}_{3,\alpha}(\mathbf{x}') \qquad F_{33}(\mathbf{x}', y_3) = 1
\end{aligned}
\tag{3.2}
$$

Greek indices take value 1 or 2, and we use the notation $u_{,\alpha} = \partial u / \partial x_\alpha$.

For a hyperelastic material with strain energy function $W(\mathbf{F})$, the first Piola-Kirchoff strain tensor is defined by

$$
\boldsymbol{\pi} = \frac{\partial W}{\partial \mathbf{F}}
\tag{3.3}
$$

In the case of dead loading, the potential energy is given by

$$
E = \Psi - V
\tag{3.4}
$$

in which Ψ is the internal energy

$$
\Psi = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \int\limits_\omega W(\mathbf{F}) \, d\mathbf{x}' \, dy_3
$$

V is the work of exterior forces

$$
V = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \int\limits_\omega \mathbf{f}^M \cdot \widetilde{\mathbf{x}} \, d\mathbf{x}' \, dy_3 + \int\limits_\omega \mathbf{t}^{M\pm}\Big(\mathbf{x}', \pm \frac{h}{2}\Big) \cdot \widetilde{\mathbf{x}}\Big(\mathbf{x}', \pm \frac{h}{2}\Big) \, d\mathbf{x}' + \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \int\limits_{\partial\omega_{\overline{\mathbf{t}}}} \overline{\mathbf{t}} \cdot \widetilde{\mathbf{x}} \, ds \, dy_3
$$

and s is the curvilinear coordinate along $\partial\boldsymbol{\omega}$ with

$$
\mathbf{f}^M(\mathbf{x}', y_3) = \frac{1}{|\mathbf{Y}'|} \int\limits_{\mathbf{Y}'} \mathbf{f}(\mathbf{x}', \mathbf{y}) \, d\mathbf{y}' \qquad \mathbf{t}^{M\pm}(\mathbf{x}') = \frac{1}{|\mathbf{Y}'|} \int\limits_{\mathbf{Y}'} \mathbf{t}^{M\pm}(\mathbf{x}', \mathbf{y}') \, d\mathbf{x}'
$$

Remark: On the global macroscale, we consider the effect of the load on both upper and lower faces of the plate and the body forces in similar fashion as in the asymptotic analysis proposed by Lewiński (1991). Nevertheless, since the result is formal, we can also consider a model for which the load acts on the unit cell $\mathbf{Y}$ linked to the local microscopic scale. And then the load acts indirectly on the macroscopic through the local scale (Pruchnicki, 2019b).

The principle of stationary potential energy requires that the variation of the potential energy vanishes

$$
\delta E = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \int\limits_\omega \boldsymbol{\pi} : \delta\mathbf{F} \, d\mathbf{x}' \, dy_3 - \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \int\limits_\omega \mathbf{f}^M \cdot \delta\widetilde{\mathbf{x}} \, d\mathbf{x}' \, dy_3 - \int\limits_\omega \mathbf{t}^{M\pm}(\mathbf{x}') \cdot \delta\widetilde{\mathbf{x}} \, d\mathbf{x}' - \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \int\limits_{\partial\omega_{\overline{\mathbf{t}}}} \overline{\mathbf{t}} \cdot \delta\widetilde{\mathbf{x}} \, ds \, dy_3 = 0
\tag{3.5}
$$

By inserting (3.1) and (3.2) the potential energy becomes

$$
\begin{aligned}
\delta E &= \int\limits_\omega \delta E_M \, d\mathbf{x}' - \int\limits_\omega \mathbf{f}^{M0} \cdot \delta\widetilde{\mathbf{x}}^{(0)} \, d\mathbf{x}' - \int\limits_\omega f^{M1}_\alpha \delta\widetilde{x}^{(1)}_\alpha \, d\mathbf{x}' - \int\limits_\omega \mathbf{t}^{M\pm} \cdot \delta\widetilde{\mathbf{x}}^{(0)} \, d\mathbf{x}' \\
&\quad - \frac{h}{2} \int\limits_\omega \mathbf{t}^{M\pm}_\alpha \delta\widetilde{x}^{(1)}_\alpha \, d\mathbf{x}' - \int\limits_{\partial\omega_{\overline{\mathbf{t}}}} \mathbf{n} \cdot \delta\widetilde{\mathbf{x}}^{(0)} \, ds - \int\limits_{\partial\omega_{\overline{\mathbf{t}}}} m_\alpha \delta\widetilde{x}^{(1)}_\alpha \, ds = 0
\end{aligned}
\tag{3.6}
$$

in which the variation of the elastic energy at each point of the plate is

$$\delta E_M = N_{\alpha\beta}\delta\widetilde{x}^{(0)}_{\alpha,\beta} + N_{3\alpha}\delta\widetilde{x}^{(0)}_{3,\alpha} + N_{\alpha3}\delta\widetilde{x}^{(1)}_{\alpha} + M_{\alpha\beta}\delta\widetilde{x}^{(1)}_{\alpha,\beta} \tag{3.7}$$

with

$$\mathbf{N} = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \boldsymbol{\pi}\, dy_3 \qquad\qquad M_{\alpha\beta} = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \pi_{\alpha\beta} y_3\, dy_3$$

$$\mathbf{f}^{Mi} = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \mathbf{f}^{M} y_3^i\, dy_3 \qquad \text{for} \qquad i = 0,1 \tag{3.8}$$

$$\mathbf{n} = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \overline{\mathbf{t}}\, dy_3 \qquad\qquad \mathbf{m} = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \overline{\mathbf{t}} y_3\, dy_3$$

The strong form of variational formulation (3.6) is

$$\text{div}_{\mathbf{x}'}\mathbf{N} + \mathbf{f}'^{M0} + \mathbf{t}^{M\pm} = 0 \qquad\qquad M_{\alpha\beta,\beta} + N_{\alpha3} + f_{\alpha}^{M1} + t_{\alpha}^{M\pm} = 0 \qquad\qquad \text{in} \quad \omega$$

$$\mathbf{N}\cdot\boldsymbol{\nu} = \mathbf{n} \qquad M_{\alpha\beta}\nu_{\beta} = m_{\alpha} \qquad \text{on} \quad \partial\omega_{\overline{\mathbf{t}}}$$

$$\widetilde{\mathbf{x}}^{(0)}(\mathbf{x}') = \overline{\mathbf{x}}(\mathbf{x}',0) \qquad\qquad \widetilde{x}^{(1)}_{\alpha}(\mathbf{x}') = \overline{x}^{[1]}_{\alpha}(\mathbf{x}',0) \qquad\qquad \text{on} \quad \partial\omega_{\overline{\mathbf{x}}}$$

where ν is the exterior unit normal on $\partial\omega$.

We assume that $\overline{\mathbf{x}}(\mathbf{x}',y_3)$ is C^1 in y_3, then we consider a Taylor expansion of the prescribed current position up to the first order: $\overline{\mathbf{x}}(\mathbf{x}',y_3) = \overline{\mathbf{x}}(\mathbf{x}',0) + \overline{\mathbf{x}}^{[1]}(\mathbf{x}',0)y_3 + O(y_3^2)$ in which $\overline{\mathbf{x}}^{[1]} = \partial\overline{\mathbf{x}}/\partial y_3$.

3.2. The transition law between macroscopic and microscopic scales and the microscopic problem

We assume that the microscopic deformation gradient can be expressed in terms of the macroscopic displacement gradient $\mathbf{F}$ and the microscopic current position $\mathbf{x}^m$

$$\mathbf{F}^m(\mathbf{x}',\mathbf{y}) = \mathbf{F}(\mathbf{x}',y_3) + \mathbf{F}^{\mathbf{y}}(\mathbf{x}^m(\mathbf{x}',\mathbf{y})) \tag{3.9}$$

in which

$$\mathbf{F}^{\mathbf{y}}(\mathbf{x}^m(\mathbf{x}',\mathbf{y})) = \frac{\partial\mathbf{x}^m(\mathbf{x}',\mathbf{y})}{\partial y_i}\otimes\mathbf{e}_i = \mathbf{x}^m_{ji}(\mathbf{x}',\mathbf{y})\otimes\mathbf{e}_i$$

in which the symbol $\otimes$ denotes tensorial product.

Thus the microscopic first Piola-Kirchoff tensor is given by

$$\boldsymbol{\pi}^m = \frac{\partial W(\mathbf{F}^m)}{\partial\mathbf{F}^m} \tag{3.10}$$

The equilibrium equation on the unit cell $\mathbf{Y}$ is

$$\pi^m_{ij;j} = 0 \qquad \text{in} \quad \mathbf{Y} \tag{3.11}$$

The Neumann boundary conditions on both the upper and lower boundaries of the RVE are

$$\pi^m_{i3} = 0 \qquad \text{on} \quad \partial\mathbf{Y}^{\pm} \tag{3.12}$$

Due to mid-plane periodicity of the heterogeneity, we can impose the periodicity of $\boldsymbol{\pi}^m$ and $\mathbf{x}^m$ on $\partial\mathbf{Y}' \times [-h/2, h/2]$ (lateral boundary of the unit cell).

So the microscopic problem is defined by (3.11) and (3.12) and the periodicity condition of $\boldsymbol{\pi}^m$ and $\mathbf{x}^m$ on $\partial\mathbf{Y}' \times [-h/2, h/2]$. In addition to this periodicity condition, we can impose $\langle\mathbf{x}^m\rangle = 0$ in order to prevent rigid displacement.

For the out-of-plane shear mode ($F_{13} \neq 0$ or $F_{31} \neq 0$ or $F_{23} \neq 0$ or $F_{32} \neq 0$), the solutions to microscopic equilibrium problems (3.11) and (3.12) are indeterminate because the constraint conditions provided earlier for the microscopic displacement are not sufficient to prevent rigid body rotations of an in-plane unit cell (Geers *et al.*, 2007; Petracca *et al.*, 2017). To avoid rigid body rotations in imposing out-of-plane shear deformations, the following constraint conditions seem to be effective (Petracca *et al.*, 2017)

$$\langle y_3 w_1\rangle = 0 \qquad \text{when} \qquad F_{13} \neq 0 \quad \text{or} \quad F_{31} \neq 0$$
$$\langle y_3 w_2\rangle = 0 \qquad \text{when} \qquad F_{23} \neq 0 \quad \text{or} \quad F_{32} \neq 0$$

Now we consider an equivalence between the variation of both the macroscopic and the microscopic elastic energy

$$\delta E_M = \frac{1}{|\mathbf{Y}'|} \int\limits_{\mathbf{Y}} \boldsymbol{\pi}^m : \delta\mathbf{F} \, d\mathbf{y} \tag{3.13}$$

Equivalence condition (3.13) implies that we can define $\mathbf{N}$, $M_{\alpha\beta}$, Eqs. (3.8), in terms of the microscopic stress field

$$\mathbf{N} = \frac{1}{|\mathbf{Y}'|} \int\limits_{\mathbf{Y}} \boldsymbol{\pi}^m \, d\mathbf{y} \qquad\qquad M_{\alpha\beta} = \frac{1}{|\mathbf{Y}'|} \int\limits_{\mathbf{Y}} \boldsymbol{\pi}^m y_3 \, d\mathbf{y} \tag{3.14}$$

We can observe that, as in Cong *et al.* (2015), we can define the relation between the macroscopic ($\boldsymbol{\pi}$) and the microscopic first Piola-Kirchhoff ($\boldsymbol{\pi}^m$) tensors as follows

$$\boldsymbol{\pi} = \frac{1}{|\mathbf{Y}'|} \int\limits_{\mathbf{Y}} \boldsymbol{\pi}^m \, d\mathbf{y}' \tag{3.15}$$

Then, from (3.15) and (3.8), we deduce (3.14).

4. Conclusion and discussion

We have presented in this work a multiscale theory for simulating the mechanical response of a highly heterogeneous plate based on the concept of computational homogenization. The concept has been described in terms of structural description of both microscopic and macroscopic scales and the resulting boundary value problems. A similar approach was presented and tested numerically with success in Cong *et al.* (2015) and Terada *et al.* (2017). The macroscopic generalized strain and the macroscopic generalized resultant stress are deduced so as to satisfy the macrohomogeneity or the Hill-Mandel condition in the sense that the local strain energy density in a homogenized thick plate must be the same as the volume average of the strain energy over an in-plane unit cell. Nevertheless, it is not a rigiourous mathematical argument, then this type of model can be only validated by numerical computation. For the homogeneous plate model (macroscopic behavior of the plate), we can also consider a more generalized shear deformable plate model recently introduced by Polizzotto (2018). This family of plate models spans from the Kirchhoff plate to the Reissner-Mindlin ones. For the sake of simplicity, we have considered in this work the Reissner-Mindlin plate model.

Another way for solving this type of problem is firstly to propose a two-scale expression of the potential energy (Lewiński, 1991; Lee *et al.*, 2014). But this problem is more complex that the initial scale problem, and so to the author's knowledge there exist two possibilities for solving it in a tractable way. The first one is the formal asymptotic expansion method used by Lewiński (1991). The second one is the variational asymptotic method proposed by Sutyrin (1997), Berdichevskii (1979) and Lee *et al.* (2014). Anyway it will seem to be difficult to extend this approach in nonlinear setting. Rigourous Γ-convergence is not established for every formal result, and when it can be established, additional assumptions are generally necessary.

The problem of homogenization of the heterogeneous plate couples two problems. The first one is reduction of the initial three dimensional problem to a two dimensional one, and the second one is the homogenization of the heterogeneous structure. So, we can simplify the problem by considering only the reduction problem without considering the concept of homogenization. For the homogeneous plate, this problem is successfully addressed by truncation of the elastic potential (Schneider *et al.*, 2014), and for the heterogeneous plate by Pruchnicki (2019b).

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