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On the quasi-efficient frontier of the set of optimal portfolios under hybrid uncertainty with short sales allowed*

by

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Abstract: The paper describes the methods for constructing a quasi-efficient frontier of minimum risk portfolio under conditions of hybrid uncertainty with allowed short sales. Investor's acceptable level of expected return is defined in crisp and fuzzy forms. Obtained results are illustrated on a model example. The dependence of the quasi-efficient frontier on the value of α-level is investigated.

Keywords: portfolio selection model, hybrid uncertainty, possibility, necessity, quasi-efficient solutions, minimum risk portfolio frontier, fuzzy random variable

1. Introduction

Classical mean–variance portfolio selection model was proposed by Markowitz (1952, 1959) and since then played an important role in the formation of modern portfolio theory. Its basic idea is to characterize a financial asset as a random variable with a probability distribution over its return and to quantify the expected return of portfolio as the investment gain and consider its variance as the investment risk.

However, in real life, it is usually impossible for investors to get the precise probability distributions of the assets' returns. In real world, there are many non-probabilistic factors that affect the markets. With the introduction of fuzzy sets and possibility theory by Zadeh (1965, 1978), Nahmias (1978) and Dubois and Prade (1988) many scholars began to employ this theory to manage portfolios in a fuzzy environment, because fuzzy approaches are, in general, more appropriate than probabilistic approaches for taking human subjective opinions

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into consideration. Thus, for example, Tanaka et al. (2000) and Inuiguchi and Tanino (2000) modeled security returns with fuzzy variables with possibility distributions and proposed the possibilistic portfolio selection models, respectively, while Parra et al. (2001) proposed a fuzzy goal programming approach for portfolio selection. Ammar and Khalifa (2003) proposed a quadratic programming approach for fuzzy portfolio selection problem. Zhang and Nie (2004) proposed the admissible efficient portfolio model. Bilbao-Terol et al. (2006) employed a fuzzy compromise programming technique to deal with the task. Giove et al. (2006) constructed a regret function to solve the interval portfolio problem. Vercher (2008) employed semi-infinite programming technique to solve the portfolio selection problem with fuzzy returns. And many more authors introduced fuzziness into portfolio theory.

The abovementioned portfolio selection models are mainly based on either probability theory or fuzzy/possibility theory, therefore only one kind of uncertainty is considered. In reality, randomness and fuzziness are often combined together and that leads to a hybrid uncertainty. Fuzzy random variables that were introduced by Kwakernaak (1978) and then further developed by Nahmias (1979) and other authors are the appropriate way to describe the hybrid uncertainty. Some studies on the fuzzy random programming can be found, for example, in Yazenin (1992), Liu (2002), Luhandjula (2004), Li, Xu and Gen (2006). Yazenin (2004, 2007) first introduced the formulation of portfolio selection problem under hybrid uncertainty of possibilistic-probabilistic type. Huang (2007) employed the random fuzzy theory by Liu (2002, 2004) to study portfolio selection in a random fuzzy environment in which the security returns are assumed to be stochastic variables with fuzzy information.

In the current paper we continue our previous works: Yazenin (2007), Yazenin and Soldatenko (2018, 2020, 2021) – where we also immersed Markowitz portfolio model in the context of hybrid uncertainty of the possibilistic-probabilistic type. However, in these studies only individual quasi-efficient solutions to the portfolio optimization problem were built. The question of constructing the whole set of quasi-efficient solutions in an analytical form remained open.

One of the classical methods for constructing an efficient minimum risk portfolio frontier is considered by Barbaumov (2003). In the present work this method is extended to a number of minimal risk portfolio models under conditions of hybrid uncertainty with allowed short sales and with fuzzy level of expected return acceptable to an investor. The obtained results are demonstrated on a numerical model example for a three-dimensional portfolio.

2. Minimal risk portfolio with allowed short sales

The model constructed according to the theory of Markowitz (1952) does not allow the possibility of opening short positions on securities. In turn, there is no such restriction in the Black approach, that is, the values of the shares of financial assets of the portfolio being formed can be both positive and negative (see Barbaumov, 2003). In this paper, we will consider investment portfolio model with allowed short sales.

Let us assume that there are n different assets on the market that an investor is interested in. The investor forms a portfolio by setting the values of the weight vector $w = (w_1, w_2, \dots, w_n)$, where w_i is the share of capital invested in securities of the i-th type. At the same time, obviously, the normalization condition $\sum_{i=1}^{n} w_i = 1$ must be met.

Let $R_i(\omega)$ be the profitability of the i-th asset, which in classical portfolio analysis is modeled using a random variable. Then the return of the entire portfolio is the weighted sum of its assets: $R(w, \omega) = \sum_{i=1}^{n} R_i(\omega) w_i$. Since the yield is a random variable, the investment portfolio can be described by two numerical characteristics: its mathematical expectation and variance, which are named, respectively, the expected return $\mathbf{E}[R(w, \omega)]$ and risk $\mathbf{D}[R(w, \omega)]$ of the portfolio. Then, in accordance with the classical Markowitz-Black theory, the minimal risk portfolio model with allowed short sales in the most general form can be written down as follows:

$$\mathbf{D}[R(w, \omega)] \to \min_{w}, \tag{1}$$

$$\begin{cases} \mathbf{E}[R(w, \omega)] \geq r, \\ \displaystyle\sum_{i=1}^{n} w_i = 1, \end{cases} \tag{2}$$

where r is the level of profitability acceptable to the investor.

Sometimes in the model of acceptable portfolios (2) the inequality is replaced by equality, which, firstly, somewhat simplifies the solution of the optimization problem itself, and secondly, does not change its essence, because, as is known from the classical portfolio optimization, if r is on the efficient frontier of the portfolio, then the minimum value of risk is achieved with equality in (2). We will replace the inequality with equality in the constraint later, because for now it will only complicate the immersion of the model into the possibility-necessity context.

3. Minimal risk portfolio in conditions of hybrid uncertainty

All the concepts and definitions from the theory of possibility that are used in this paper can be found, for example, in the works of Yazenin (2007, 2016), Yazenin and Wagenknecht (1996), Yazenin and Soldatenko (2018, 2020, 2021).

We will model the profitability of the i-th financial asset with a fuzzy random variable $R_i(\omega, \gamma)$. For a better intuitive understanding of this model, let us imagine a situation, in which a financial expert is asked to evaluate the profitability of a certain financial asset. Both the profitability and its assessment by the expert are uncertain values. We assume that the uncertainty of market conditions is probabilistic. On the other hand, the uncertainty of an expert's assessment is more naturally described by some distribution of possibilities. For example, one way to model these uncertainties would be the following. The expert gives his estimate of the spread for the profitability of the asset in the form of, for example, a triangular or trapezoidal fuzzy value. At the same time, the boundaries of the spread, its width and relative position can be modified by random components of a fuzzy random variable of the asset. The shift-scale representation of a fuzzy random variable described here in Section 5 is a convenient tool for constructing such a model.

The exact representation of a fuzzy random variable is not essential at the moment. A specific example will be discussed in Section 4.

Let us consider a model of minimizing the risk of a portfolio with restrictions on possibility (necessity) on its expected return. According to the classical model (1)-(2) it is necessary to build a portfolio risk function, and its return should be included in the restrictions system. First, we will identify all the components of the model.

In the conditions of hybrid uncertainty of possibilistic-probabilistic type the return of the investment portfolio will be a fuzzy random function:

$$R(w, \omega, \gamma) = \sum_{i=1}^{n} R_i(\omega, \gamma) w_i, \tag{3}$$

which is a linear function of the equity shares $w = (w_1, w_2, \dots, w_n)$ in the investment portfolio.

We will use the so-called indirect method to solve the optimization problem. The essence of the method is to construct an equivalent deterministic analogue of the possibilistic-probabilistic problem, in this case there will be no more randomness and fuzziness, and therefore it can further be solved by "ordinary" methods. This can be done by the stepwise removal of uncertainty. The uncertainty of the probabilistic type will be removed based on the principle of expected return. To do this, we identify the possibility distribution of the mathematical expectation of the function $R(w, \omega, \gamma)$. Based on the properties of mathematical expectation, we obtain the following formula for the expected return of the portfolio:

$$\mathbf{E}[R(w, \omega, \gamma)] = \mathbf{E}\Big[\sum_{i=1}^{n} R_i(\omega, \gamma) w_i\Big] = \sum_{i=1}^{n} \hat{R}_i(\gamma) w_i, \tag{4}$$

where $\hat{R}_i(\gamma)$ is the expected value of a fuzzy random variable $R_i(\omega, \gamma)$.

Next, we need a formula to calculate the covariance of two fuzzy random variables $R_i(\omega, \gamma)$ and $R_j(\omega, \gamma)$. According to Feng's (Feng, Hu and Shu, 2001) formula, it will be determined through the covariance of their α-level sets:

$$\operatorname{cov}(R_i, R_j) = \frac{1}{2}\int_0^1 \Big(\operatorname{cov}\big(R_i^-(\omega, \alpha), R_j^-(\omega, \alpha)\big) + \operatorname{cov}\big(R_i^+(\omega, \alpha), R_j^+(\omega, \alpha)\big)\Big) d\alpha,$$

where $R_k^-(\omega, \alpha)$, $R_k^+(\omega, \alpha)$ are boundaries of an α-level set of the fuzzy value $R_k(\omega, \gamma)$. Note that the covariance, and, accordingly, the variance of a fuzzy random variable, according to this definition, will be crisp quantities.

Given that the variance of a (fuzzy) random variable X is

$$\mathbf{D}[X] = \operatorname{cov}(X, X),$$

with the above-stated assumptions and notations the variance of the i-th asset will have the following form:

$$\mathbf{D}[R_i(\omega, \gamma)] = \frac{1}{2}\int_0^1 (\mathbf{D}\left[R_i^-(\omega, \alpha)\right] + \mathbf{D}\left[R_i^+(\omega, \alpha)\right]) d\alpha, \tag{5}$$

where $R_i^-(\omega, \alpha)$ and $R_i^+(\omega, \alpha)$ are the left and right boundaries of the α-level set of fuzzy random variable $R_i(\omega, \gamma)$.

Let us introduce the following notation:

$$\sigma_{ij} = \operatorname{cov}(R_i(\omega, \gamma), R_j(\omega, \gamma)), \qquad \sigma_i^2 = \mathbf{D}\left[R_i(\omega, \gamma)\right].$$

Using the introduced notation and properties of the dispersion determined by Feng (Feng, Hu and Shu, 2001), we write the formula for $\mathbf{D}[R(w, \omega, \gamma)]$:

$$\mathbf{D}[R(w, \omega, \gamma)] = \sum_{i=1}^{n} \mathbf{D}[R_i(\omega, \gamma) w_i] + 2\sum_{i<j}^{n} \operatorname{cov}(R_i(\omega, \gamma) w_i, R_j(\omega, \gamma) w_j) = \\ = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2\sum_{i<j}^{n} w_i w_j \sigma_{ij}. \tag{6}$$

Since the expected return (4) of a portfolio in conditions of hybrid uncertainty is a fuzzy value, in order to remove the uncertainty of possibilistic type, we will introduce a restriction on possibility (necessity) regarding the level of expected return acceptable to the investor (which, in general, may be fuzzy) into the system of restrictions that defines the set of acceptable portfolios. Then, the model of the minimum risk portfolio by Black will take the following form with a crisp level of expected return r:

$$\mathbf{D}[R(w,\omega,\gamma)] \to \min_{w}, \tag{7}$$

$$\begin{cases} \tau\big\{\mathbf{E}[R(w,\omega,\gamma)] \geq r\big\} \geq \alpha, \\ \sum_{i=1}^{n} w_i = 1, \end{cases} \tag{8}$$

and the following form with a fuzzy level of expected return $r(\gamma)$:

$$\mathbf{D}[R(w,\omega,\gamma)] \to \min_{w}, \tag{9}$$

$$\begin{cases} \tau\big\{\mathbf{E}[R(w,\omega,\gamma)] \geq r(\gamma)\big\} \geq \alpha, \\ \sum_{i=1}^{n} w_i = 1, \end{cases} \tag{10}$$

where $\tau \in \{\pi, \nu\}$, π is a measure of possibility, ν is a measure of necessity, α is the level of possibility/necessity.

After that, we will remove the uncertainty of the possibilistic type using an indirect method, namely, we will build an equivalent deterministic analogue of models of acceptable portfolios (8) and (10). Let $\tau =$ 'π'. Based on the results of Yazenin and Wagenknekht (1996), we obtain the following equivalent deterministic analogue for a crisp level of expected return (8):

$$\begin{cases} \sum_{i=1}^{n} \hat{R}_i^{+}(\alpha) w_i \geq r, \\ \sum_{i=1}^{n} w_i = 1, \end{cases} \tag{11}$$

and in the case of a fuzzy level of expected return (10):

$$\begin{cases} \sum_{i=1}^{n} \hat{R}_i^{+}(\alpha) w_i \geq r^{-}(\alpha), \\ \sum_{i=1}^{n} w_i = 1, \end{cases} \tag{12}$$

where $\hat{R}_i^+(\alpha)$ is the right boundary of α-the level set of the expected return of the i-th financial asset, and $r^-(\alpha)$ - the left boundary of α-the level set of the fuzzy level of the expected return of the portfolio.

Note that if we replace inequality with equality in (8) or (10) under the sign of the measure of possibility, we get the following systems – for (8):

$$\begin{cases} \sum_{i=1}^{n} \hat{R}_i^+(\alpha) w_i \geq r, \\ \sum_{i=1}^{n} \hat{R}_i^-(\alpha) w_i \leq r, \\ \sum_{i=1}^{n} w_i = 1, \end{cases}$$

and for (10):

$$\begin{cases} \sum_{i=1}^{n} \hat{R}_i^+(\alpha) w_i \geq r^-(\alpha), \\ \sum_{i=1}^{n} \hat{R}_i^-(\alpha) w_i \leq r^+(\alpha), \\ \sum_{i=1}^{n} w_i = 1. \end{cases}$$

However, the resulting upper bounds of expected return in the system of restrictions do not play any role, because at the efficient frontier of the portfolio, the risk reaches its minimum value at the lower boundary of expected return, that is, with the equality in (11) and (12).

We consider now the necessity context. Let now in models (8) and (10) assume τ = 'ν'. Then, the equivalent deterministic analogue will take the following form for a crisp level of expected return (8):

$$\begin{cases} \sum_{i=1}^{n} \hat{R}_i^-(1-\alpha) w_i \geq r, \\ \sum_{i=1}^{n} w_i = 1, \end{cases} \tag{13}$$

and the following one in the case of a fuzzy level of expected return (10):

$$\begin{cases} \sum_{i=1}^{n} \hat{R}_i^-(1-\alpha) w_i \geq r^+(1-\alpha), \\ \sum_{i=1}^{n} w_i = 1, \end{cases} \tag{14}$$

where $\hat{R}_i^-(1-\alpha)$ and $r^+(1-\alpha)$ are left and right borders of $(1-\alpha)$–level sets of corresponding fuzzy values.

Note that it is for this case that we require in (8) and (10) the inequality to hold under the sign of the uncertainty measure, since there is no deterministic equivalent for equality in the necessity context (due to the properties of the measure ν, see Yazenin, 2016).

As one can see, when replacing a crisp level of expected return with a fuzzy one in equivalent analogues of the model of acceptable portfolios (12) and (14), only the right part of the restriction changes: the crisp level of r changes to the corresponding boundary of the α-level set $r(\gamma)$. Therefore, for brevity, we will further denote the level as $\bar{r}$, which, depending on the context, will be equal to:

- $\bar{r} = r$ – in case of a crisp level of expected return,
- $\bar{r} = r^-(\alpha)$ – in the case of a fuzzy level of expected return in a possibilistic context,
- $\bar{r} = r^+(1-\alpha)$ – in the case of an fuzzy level of expected return in the necessity context.

Also, to simplify things, we will introduce the notation $\bar{R}_i$, which will be equal to:

- $\bar{R}_i = \hat{R}_i^+(\alpha)$ in the context of possibility measure,
- $\bar{R}_i = \hat{R}_i^-(1-\alpha)$ in the context of necessity measure.

Then, the equivalent deterministic analogues of minimum risk portfolios in the various contexts discussed above can be compactly written as:

$$\mathbf{D}[R(w, \omega, \gamma)] \to \min_{w}, \tag{15}$$

$$\begin{cases} \sum_{i=1}^{n} \bar{R}_i w_i \geq \bar{r}, \\ \sum_{i=1}^{n} w_i = 1. \end{cases} \tag{16}$$

Once again, we recall that for the criterion (15) there is no such variability of models, since we have determined the variance in a crisp form.

4. Quasi-efficient minimum risk portfolio frontier

As it is known from the classical portfolio theory, the result of solving the problem of a minimum risk portfolio is its efficient frontier, which is the dependence of the minimum risk of the portfolio upon its expected return. In our case, we call this curve a *quasi-efficient* frontier, because, due to the uncertainty of possibilistic type that we introduced to the model, it now depends on the level of α, which is set by the expert.

In the work of Barbaumov (2003), a method for constructing an efficient frontier of the classical minimal risk portfolio (1)-(2), based on the Lagrange multiplier method is considered. We generalize it to the case of a minimal risk portfolio in conditions of hybrid uncertainty of possibilistic-probabilistic type (15)-(16). To do this, first we replace the inequality in (16) with equality. As mentioned above, if $\bar{r}$ is on the efficient portfolio frontier, then the minimum risk value is achieved when the equality in (16) is met. As a result, the Lagrange function will look like this:

$$\mathcal{L}(w,\omega,\gamma) = \mathbf{D}[R(w,\omega,\gamma)] + \lambda_1\left(\sum_{i=1}^{n}\overline{R}_i w_i - \bar{r}\right) + \lambda_2\left(\sum_{i=1}^{n} w_i - 1\right).$$

After substituting the formula for the variance (6) we get:

$$\mathcal{L}(w,\omega,\gamma) = \sum_{i=1}^{n} w_i^2\sigma_i^2 + 2\sum_{i<j}^{n} w_i w_j \sigma_{ij} + \lambda_1\left(\sum_{i=1}^{n}\overline{R}_i w_i - \bar{r}\right) + \lambda_2\left(\sum_{i=1}^{n} w_i - 1\right). \quad (17)$$

Next we differentiate (17) by variables w_i, $i = 1,\ldots,n$, λ_1, λ_2, the results of differentiation we equate to zero and obtain the following system of linear equations:

$$\begin{cases} 2\sigma_1^2 w_1 + 2\sigma_{12}w_2 + \cdots + 2\sigma_{1n}w_n + \lambda_1\overline{R}_1 + \lambda_2 = 0, \\ 2\sigma_{21}w_1 + 2\sigma_2^2 w_2 + \cdots + 2\sigma_{2n}w_n + \lambda_1\overline{R}_2 + \lambda_2 = 0, \\ \cdots \\ 2\sigma_{n1}w_1 + 2\sigma_{n2}w_2 + \cdots + 2\sigma_n^2 w_n + \lambda_1\overline{R}_n + \lambda_2 = 0, \\ w_1 + w_2 + \cdots + w_n = 1, \\ \overline{R}_1 w_1 + \cdots + \overline{R}_n w_n = \bar{r}. \end{cases} \quad (18)$$

The augmented matrix of this system of equations is given in Table 1.

Having solved the system (18) using the Gauss-Jordan method, we get the result schematically shown in Table 2.

Table 1. Augmented matrix of a system of linear equations (18)

w_1	w_2	$\cdots$	w_n	λ_1	λ_2	
$2\sigma_1^2$	$2\sigma_{12}$	$\cdots$	$2\sigma_{1n}$	$\overline{R}_1$	1	0
$2\sigma_{21}$	$2\sigma_2^2$	$\cdots$	$2\sigma_{2n}$	$\overline{R}_2$	1	0
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$2\sigma_{n1}$	$2\sigma_{n2}$	$\cdots$	$2\sigma_n^2$	$\overline{R}_n$	1	0
1	1	$\cdots$	1	0	0	1
$\overline{R}_1$	$\overline{R}_2$	$\cdots$	$\overline{R}_n$	0	0	$\bar{r}$

Table 2: Solution for a system of linear equations (18)

w_1	w_2	$\cdots$	w_n	λ_1	λ_2	
1	0	$\cdots$	0	0	0	$a_1 + b_1\bar{r}$
0	1	$\cdots$	0	0	0	$a_2 + b_2\bar{r}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
0	0	$\cdots$	1	0	0	$a_n + b_n\bar{r}$
0	0	$\cdots$	0	1	0	$a_{n+1} + b_{n+1}\bar{r}$
0	0	$\cdots$	0	0	1	$a_{n+2} + b_{n+2}\bar{r}$

The numbers a_i and b_i ($i = 1, 2, \dots, n$) are the coefficients that were found by the Gauss-Jordan method. They depend only on mathematical expectations, variances and covariances between the returns of various assets and do not depend on the level of expected return $\bar{r}$. Therefore, the optimal solution of the problem has the form:

$$w^*(\bar{r}) = \{a_1 + b_1\bar{r},\, a_2 + b_2\bar{r},\, \dots,\, a_n + b_n\bar{r}\}. \tag{19}$$

Substituting (19) into (6), we find the quasi-efficient frontier of the set of investment portfolios in the context of hybrid uncertainty with allowed short sales:

$$D^*(\bar{r}) = \sqrt{A\bar{r}^2 + B\bar{r} + C}, \text{ where } \bar{r} \geq -\frac{B}{2A}, \tag{20}$$

$$A = \sum_{i=1}^{n} \sigma_i^2 b_i^2 + 2\sum_{i<j} \sigma_{ij} b_i b_j,$$

$$B = 2\sum_{i=1}^{n} \sigma_i^2 a_i b_i + 2\sum_{i<j} \sigma_{ij}(a_i b_j + b_i a_j),$$

$$C = \sum_{i=1}^{n} \sigma_i^2 a_i^2 + 2\sum_{i<j} \sigma_{ij} a_i a_j.$$

A visualization of the resulting frontier shall be presented in the next section of the paper, i.e. in Section 5.

5. Specification of a model for one class of fuzzy random variables

We specify a minimal risk portfolio model under conditions of hybrid uncertainty and a method for constructing its quasi-efficient frontier for one particular class of fuzzy random variables $R_i(\omega, \gamma)$, which we will model using a shift-scale representation:

$$R_i(\omega, \gamma) = a_i(\omega) + \delta_i(\omega) X_i(\gamma).$$

Here the fuzzy values $X_i(\gamma) \in Tr(c_i, d_i)$ are mutually min-related, and the shift and scale coefficients $a_i(\omega)$, $\delta_i(\omega)$ are independent random variables uniformly distributed over the segments $[l_i^a, r_i^a]$ and $[l_i^\delta, r_i^\delta]$, respectively. For convenience, let us put $l_i^\delta \geq 0$, in order not to introduce the absolute value operation into the fuzzy coefficient of the value $\delta_i(\omega) X_i(\gamma)$.

Recall that $Tr(a, d)$ is a class of triangular fuzzy values with a modal value of a, a fuzziness coefficient of d and a distribution function of possibilities shown in Fig. 1.

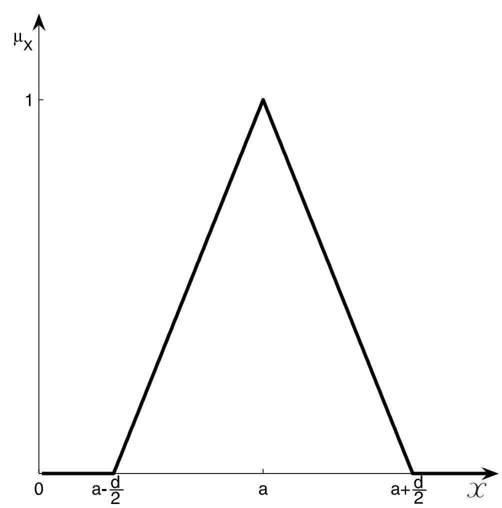

Figure 1. The possibility distribution function for a triangular class of fuzzy values $Tr(a, d)$

To simplify the writing, we introduce the following notations:

$$\hat{a}_i = \mathbf{E}[a_i(\omega)] = \frac{l_i^a + r_i^a}{2}, \qquad \hat{\delta}_i = \mathbf{E}[\delta_i(\omega)] = \frac{l_i^\delta + r_i^\delta}{2},$$
$$\tilde{a}_i = \mathbf{D}[a_i(\omega)] = \frac{(r_i^a - l_i^a)^2}{12}, \qquad \tilde{\delta}_i = \mathbf{D}[\delta_i(\omega)] = \frac{(r_i^\delta - l_i^\delta)^2}{12}.$$

Let us put a fuzzy level of expected return $r(\gamma) \in Tr(c_r, d_r)$.

In accordance with the calculus of the possibilities of fuzzy values (see Yazenin and Wagenknekht, 1996; or Yazenin, 2016), we have the following parameterized representation for $R_i(\omega, \gamma)$:

$$R_i(\omega, \gamma) \in Tr\big(a_i(\omega) + \delta_i(\omega)c_i, \delta_i(\omega)d_i\big) \tag{21}$$

and based on the linearity of the mathematical expectation from (21), we get a parameterized representation of the expected return of the ith asset:

$$\hat{R}_i(\gamma) = \mathbf{E}[R_i(\omega, \gamma)] \in Tr(\hat{a}_i + \hat{\delta}_i c_i, \hat{\delta}_i d_i). \tag{22}$$

In order to construct a model of acceptable portfolios (16), we define the boundaries of the α-level set of fuzzy random variables (21):

$$R_i^-(\omega, \alpha) = a_i(\omega) + \delta_i(\omega)c_i - \frac{\delta_i(\omega)d_i}{2}(1-\alpha),$$
$$R_i^+(\omega, \alpha) = a_i(\omega) + \delta_i(\omega)c_i + \frac{\delta_i(\omega)d_i}{2}(1-\alpha).$$

Replacing in the above formulas the random variables $a_i(\omega)$ and $\delta_i(\omega)$ with their expected values $\hat{a}_i$ and $\hat{\delta}_i$, we get the boundaries of α-level sets of fuzzy values (22).

We will also write out the boundaries of the α-level set of the fuzzy level of expected return:

$$r^-(\alpha) = c_r - \frac{d_r}{2}(1-\alpha), \qquad r^+(1-\alpha) = c_r + \frac{d_r}{2}\alpha.$$

Now we specify the portfolio risk function (15). With the assumptions we made earlier and based on the formulas (5) and (6), it takes the following form:

$$\mathbf{D}[R(w,\omega,\gamma)] = \sum_{i=1}^{n} w_i^2 \mathbf{D}[R_i(\omega,\gamma)] + 2\sum_{i<j}^{n} w_i w_j \cdot 0 =$$
$$= \frac{1}{2}\sum_{i=1}^{n} w_i^2 \int_0^1 \Big(\mathbf{D}\left[R_i^-(\omega,\alpha)\right] + \mathbf{D}\left[R_i^+(\omega,\alpha)\right]\Big) d\alpha =$$
$$= \frac{1}{2}\sum_{i=1}^{n} w_i^2 \int_0^1 \left(\mathbf{D}\Big[a_i(\omega) + \delta_i(\omega)c_i - \frac{\delta_i(\omega)d_i}{2}(1-\alpha)\Big] + \right.$$
$$\left. +\mathbf{D}\Big[a_i(\omega) + \delta_i(\omega)c_i + \frac{\delta_i(\omega)d_i}{2}(1-\alpha)\Big]\right) d\alpha =$$
$$= \frac{1}{2}\sum_{i=1}^{n} w_i^2 \int_0^1 \left(\tilde{a}_i + \tilde{\delta}_i\Big(c_i - \frac{d_i}{2}(1-\alpha)\Big)^2 + \tilde{a}_i + \tilde{\delta}_i\Big(c_i + \frac{d_i}{2}(1-\alpha)\Big)^2\right) d\alpha =$$
$$= \sum_{i=1}^{n} w_i^2 \left(\tilde{a}_i + \tilde{\delta}_i \left(c_i^2 + \frac{d_i^2}{4}\int_0^1 (1-\alpha)^2 d\alpha\right)\right) =$$
$$= \sum_{i=1}^{n} w_i^2 \left(\tilde{a}_i + \tilde{\delta}_i \left(c_i^2 + \frac{d_i^2}{12}\right)\right).$$

After determining all the components, the equivalent deterministic analogue of the minimum risk portfolio model (15)-(16) can be written down in the following form in a possibilistic context with a fuzzy level of expected return:

$$\sum_{i=1}^{n} w_i^2 \left(\tilde{a}_i + \tilde{\delta}_i \Big(c_i^2 + \frac{d_i^2}{12}\Big)\right) \to \min_w,$$
$$\begin{cases} \sum_{i=1}^{n} \Big(\hat{a}_i + \hat{\delta}_i c_i + \frac{\hat{\delta}_i d_i}{2}(1-\alpha)\Big) w_i \geq c_r - \frac{d_r}{2}(1-\alpha), \\ \sum_{i=1}^{n} w_i = 1, \end{cases} \tag{23}$$

and in the necessity context:

$$\sum_{i=1}^{n} w_i^2 \left(\tilde{a}_i + \tilde{\delta}_i \Big(c_i^2 + \frac{d_i^2}{12}\Big)\right) \to \min_w,$$
$$\begin{cases} \sum_{i=1}^{n} \Big(\hat{a}_i + \hat{\delta}_i c_i - \frac{\hat{\delta}_i d_i}{2}\alpha\Big) w_i \geq c_r + \frac{d_r}{2}\alpha, \\ \sum_{i=1}^{n} w_i = 1. \end{cases} \tag{24}$$

By replacing the right-hand sides of the inequalities in (23) and (24) with r, we get models with a crisp level of expected return.

Using Python packages for symbolic mathematics, we have derived formulas for the optimal solution and the quasi-efficient frontier in a closed analytical form. The optimal vector of weights is equal to:

$$w_i = \eta_i \left(\mathcal{A} + \frac{\mathcal{B} - s_i}{\mathcal{C}} \mathcal{D} \right),$$

and the dependence of risk on the level of expected return in a fuzzy form:

$$\mathbf{D}[R(w, \omega, \gamma)] = \sum_{i=1}^{n} w_i^2 v_i = \frac{1}{\mathcal{C}^2} \left(\mathcal{D}^2 \sum_{i=1}^{n} \eta_i s_i^2 - 2\mathcal{D}\mathcal{X} \sum_{i=1}^{n} \eta_i s_i + \mathcal{X}^2 \sum_{i=1}^{n} \eta_i \right),$$

where

- $\eta_i = \frac{1}{v_i}$,
- $v_i = \frac{1}{48}\big(4(r_i^a - l_i^a)^2 + \frac{1}{3}(12c_i^2 + d_i^2)(r_i^\delta - l_i^\delta)^2\big)$,
- $\mathcal{A} = \frac{1}{\sum_{i=1}^{n} \eta_i}$,
- $\mathcal{B} = \mathcal{A} \sum_{i=1}^{n} \eta_i s_i$,
- $\mathcal{C} = \sum_{i=1}^{n} \eta_i \cdot \frac{\sum_{i=1}^{n} \eta_i s_i^2}{\sum_{i=1}^{n} \eta_i s_i} - \sum_{i=1}^{n} \eta_i s_i$,
- $\mathcal{D} = 1 - \frac{(c_r - \beta d_r) \sum_i \eta_i}{\sum_{i=1}^{n} \eta_i s_i}$,
- $\mathcal{X} = \mathcal{A}\mathcal{B} + \mathcal{C}\mathcal{D}$,
- $s_i = p_i + \beta q_i$,
- $\beta = \frac{1}{2}(1 - \alpha)$,
- $p_i = \hat{a}_i + \hat{\delta}_i c_i$,
- $q_i = \hat{\delta}_i d_i$.

6. Model example

As an example, let us take a three-dimensional portfolio ($n = 3$) and the model specified in the previous section. Assume the following values:

$$\begin{aligned} &X_1(\gamma) \in Tr(-0.5,\ 1), && a_1(\omega) \in \mathcal{U}(1.065,\ 1.137), && \delta_1(\omega) \in \mathcal{U}(0.399,\ 0.401),\\ &X_2(\gamma) \in Tr(-0.5,\ 1), && a_2(\omega) \in \mathcal{U}(1.172,\ 1.232), && \delta_2(\omega) \in \mathcal{U}(0.34,\ 0.46),\\ &X_3(\gamma) \in Tr(0.5,\ 1), && a_3(\omega) \in \mathcal{U}(0.9,\ 0.91), && \delta_3(\omega) \in \mathcal{U}(0.3,\ 0.5), \end{aligned}$$

$$r(\gamma) \in Tr(1.07,\ 0.1), \quad \alpha = 0.6.$$

We will consider the portfolio model in a possibilistic context. For the specified input data, the vector of optimal portfolio shares will be equal to:

$$w^* = (0.501,\ 0.307,\ 0.192).$$

Figure 2 shows a graphical interpretation of fuzzy random variables that determine the profitability of individual assets (on the left), and the structure of the optimal portfolio (on the right). For each distribution, the dotted line shows one of the possible triangular fuzzy values characterizing the asset return spread (initially set by an expert). In turn, the solid colored sides of the triangle show a random uniform fluctuation of the left and right shoulders of the distribution, as well as its modal value (due to market's stochastic uncertainty).

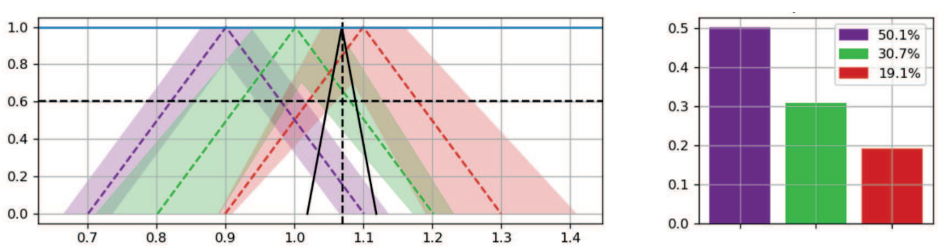

Figure 2. Fuzzy random values of individual assets and the level of expected return (left) and the structure of the found minimum risk portfolio (right)

The quasi-efficient frontier of the found portfolio is shown in Fig. 3 (the right part of the parabola). This boundary is constructed with $\alpha = 0.6$.

When the α-level changes, the efficient frontier of the solution will change. This is shown in Fig. 4.

Let us illustrate the difference in the resulting models with crisp and fuzzy levels of expected return. With a fuzzy level of expected return the efficient frontier can be built in two ways: as a curve of risk dependence on the modal value of the fuzzy level and as a curve of risk dependence on the left boundary (in the possibility context) of the α-level set of the fuzzy level of expected return. Figure 5 shows both options.

As expected, the quasi-efficient frontier that depends on the left boundary of the α-level set of the fuzzy level of expected return behaves identically to the quasi-efficient frontier that depends on the crisp level of expected return. If the dependence is built on the modal value, then we observe a shift of the parabola by the amount of displacement of the boundary of the α-level set from the modal value. This spread will become wider when the coefficient of fuzziness of the level of expected return increases. We see that lowering the α level reduces the riskiness of the model.

The final plot in Fig. 6 shows the quasi-efficient frontiers of the minimal risk portfolio found in the contexts of possibility and necessity. As can be seen from the plots, the necessity context is more "cautious", giving a greater risk compared to the possibility context for the same level of expected return (and, accordingly, a lower return for the same level of risk).

As one can see, changing the context of possibility/necessity, varying the α-level and the degree of fuzziness of the expected return level acceptable to the investor allow us to flexibly manage the quasi-efficient frontier of the minimum risk portfolio.

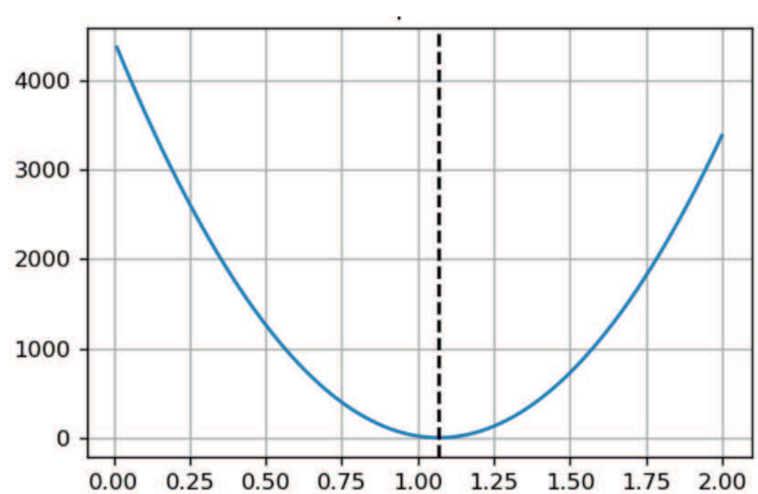

Figure 3. Quasi-efficient frontier of the minimum risk portfolio with $\alpha = 0.6$ (the righthand arm of the parabola)

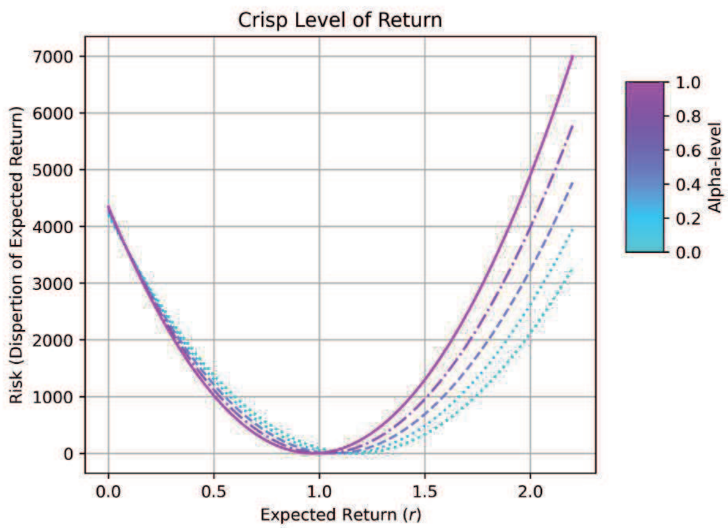

Figure 4. Changes of the quasi-efficient frontier of the minimum risk portfolio when changing the α-level

7. Summary

The article develops a method for constructing a quasi-efficient minimum risk portfolio frontier under conditions of hybrid uncertainty of the possibilistic-probabilistic type with allowed short sales and when an acceptable for the investor level of expected return is defined as crisp or fuzzy value. The formula for the dependence of the quasi-efficient frontier on the α-level is derived in analytical form. The results are illustrated by a numerical example.

The proposed method of analytical construction of a set of quasi-effective frontiers of a minimal risk portfolio simplifies the study of various mechanisms for managing the uncertainty of the model (various types of triangular norms, distributions of fuzzy and random components, etc.), which is an area for continuation of the presented work.

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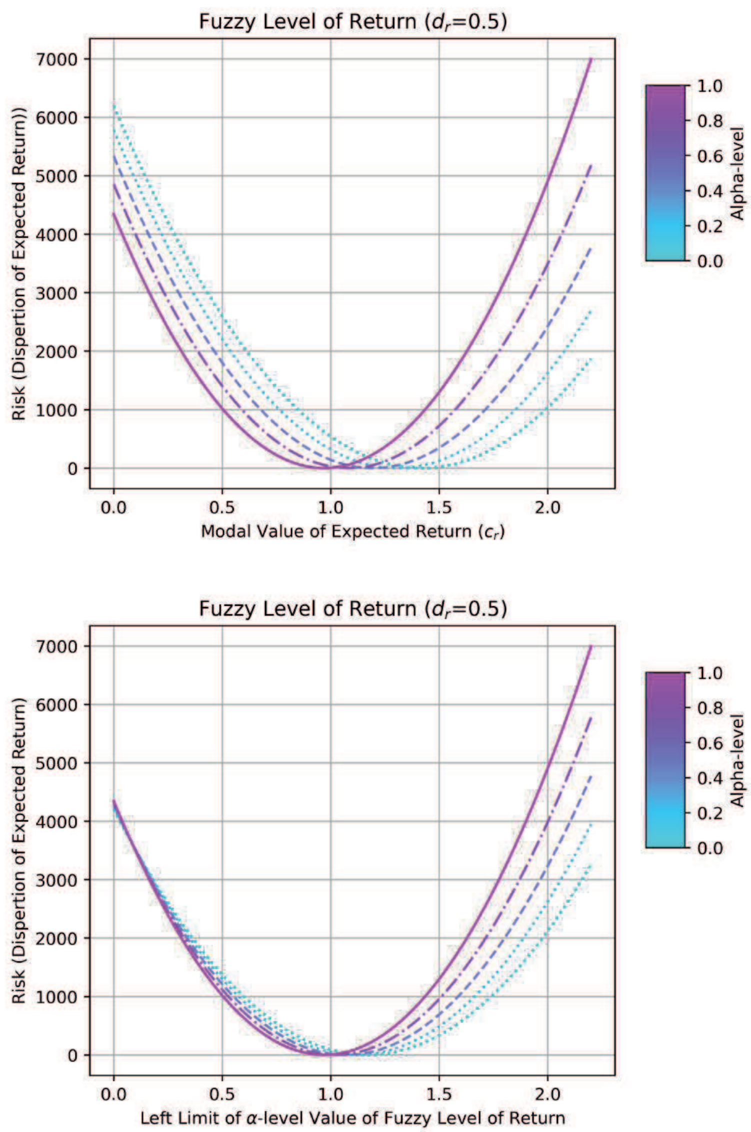

Figure 5. The curve of risk dependence on the modal value of the fuzzy return level (top) and the left boundary of the α-level set of the fuzzy return level (bottom)

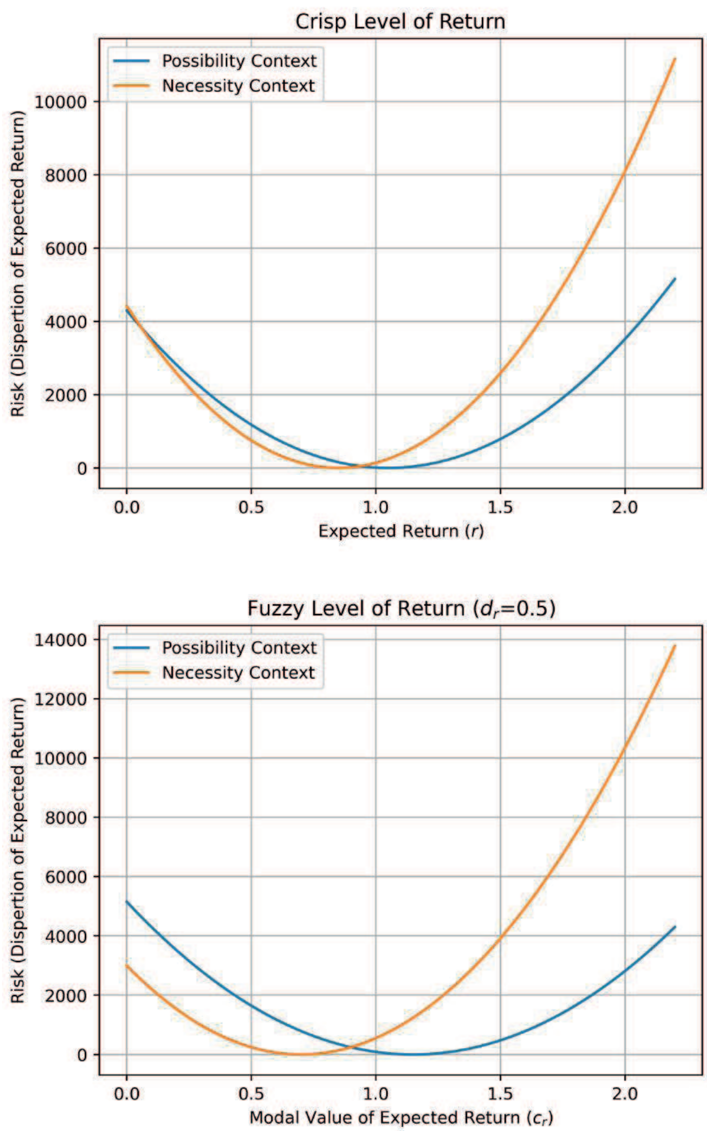

Figure 6. Quasi-efficient frontiers of the minimum risk portfolio in the contexts of possibility and necessity in cases where the investor's acceptable level of expected return is crisp (top) or fuzzy (bottom) value

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