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SOLUTION OF SOME PROBLEM ON THE DETERMINANT OF A SINE-TYPE MATRIX

Summary. In this paper we present the independent solution of problem E3178 posed in 1988 by G.A. Hively in American Mathematical Monthly. We also give a generalization of this problem together with the proof.

ROZWIĄZANIE PEWNEGO PROBLEMU DOTYCZĄCEGO WYZNACZNIKA MACIERZY SINUSÓW

Streszczenie. W artykule prezentujemy niezależne rozwiązanie problemu E3178 postawionego w 1988 r. przez G.A. Hively'ego w czasopiśmie American Mathematical Monthly. Podajemy również uogólnienie tego problemu wraz z dowodem.

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During some research conducted with my supervisor, Professor Roman Wituła, a problem published in American Mathematical Monthly has fallen into my hands. This problem concerns the value of determinant of the sine-type matrix and has been posed in 1988 by G.A. Hively. Author of this problem formulated it in the following way:

Problem 1. *Let x_i, y_i be any complex numbers and let S_n be a matrix of form $S_n = \big(\sin(x_i + y_j)\big)_{n\times n}$. It should be shown that for $n \geqslant 3$ the matrix S_n is singular.*

Proof. At first we will prove that for $n = 2$ the thesis of the above formulated problem is not satisfied. Let us take, for instance, $x_1 = \frac{\pi}{4},\ x_2 = y_1 = 0,\ y_2 = \frac{\pi}{4}$. Then we get

$$\det(S_2) = \begin{vmatrix} \sin(x_1+y_1) & \sin(x_1+y_2) \\ \sin(x_2+y_1) & \sin(x_2+y_2) \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{2}}{2} & 1 \\ 0 & \frac{\sqrt{2}}{2} \end{vmatrix} = \frac{1}{2} \neq 0.$$

Let us consider now the case when $n \geqslant 3$. Then, by using the formula for the sine of sum we can present the matrix S_n as follows

$$S_n = \begin{bmatrix} \sin(x_1+y_1) & \sin(x_1+y_2) & \sin(x_1+y_3) & \dots & \sin(x_1+y_n) \\ \sin(x_2+y_1) & \sin(x_2+y_2) & \sin(x_2+y_3) & \dots & \sin(x_2+y_n) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \sin(x_n+y_1) & \sin(x_n+y_2) & \sin(x_n+y_3) & \dots & \sin(x_n+y_n) \end{bmatrix} =$$

$$= \begin{bmatrix} \sin x_1 \cos y_1 + \sin y_1 \cos x_1 & \dots & \sin x_1 \cos y_n + \sin y_n \cos x_1 \\ \sin x_2 \cos y_1 + \sin y_1 \cos x_2 & \dots & \sin x_2 \cos y_n + \sin y_n \cos x_2 \\ \vdots & \dots & \vdots \\ \sin x_n \cos y_1 + \sin y_1 \cos x_n & \dots & \sin x_n \cos y_n + \sin y_n \cos x_n \end{bmatrix} =$$

$$= [\mathbf{k_1} \cos y_j + \mathbf{k_2} \sin y_j]_{j=1,\dots,n},$$

where $\mathbf{k_1} = [\sin x_1, \sin x_2, \dots, \sin x_n]^T$, $\mathbf{k_2} = [\cos x_1, \cos x_2, \dots, \cos x_n]^T$.

One can easily notice that $\det(S_n) = 0$ exactly when the columns of matrix S_n are linearly dependent which is, in turn, equivalent to the fact that the system of equations

$$\sum_{j=1}^{n} \alpha_j (\mathbf{k_1} \cos y_j + \mathbf{k_2} \sin y_j) = \mathbf{0}$$

possesses the non-zero solution. That is the system

$$\left(\sum_{j=1}^{n}\alpha_j\cos y_j\right)\mathbf{k_1}+\left(\sum_{j=1}^{n}\alpha_j\sin y_j\right)\mathbf{k_2}=\mathbf{0}$$

possesses the non-zero solution. And so it is for $n \geqslant 3$, because the system created from the coefficients placed by $\mathbf{k_1}$ and $\mathbf{k_2}$:

$$\begin{cases}\sum\limits_{j=1}^{n}\alpha_j\cos y_j=0\\ \sum\limits_{j=1}^{n}\alpha_j\sin y_j=0\end{cases}$$

for $n \geqslant 3$ has more variables than equations. □

Remark 2. The above fact can be also justified by using the Principle of Mathematical Induction. The proof can be then conducted in the following way:

- by using the formula for the sine of sum we compute directly the value of determinant $\det(S_3)$ receiving $\det(S_3)=0$,
- we pose the inductive assumption that for some $n \geqslant 3$ we have $\det(S_n)=0$,
- we expand the determinant $\det(S_{n+1})$ (for example, with respect to the first row) and we get

  $$\det(S_{n+1})=\sum_{j=1}^{n+1}(-1)^{1+j}\sin(x_1+y_j)\det(S_n(\{x_i\}_{i=2}^{n+1},\{y_i\}_{i=1,\ i\neq j}^{n+1})).^{1}$$

  For each $j\in\{1,2,3,\dots,n+1\}$ from the inductive assumption we have

  $$\det\left(S_n(\{x_i\}_{i=2}^{n+1},\{y_i\}_{i=1,\ i\neq j}^{n+1})\right)=0,$$

  hence $\det(S_{n+1})=0$. Thus, in view of the the Principle of Mathematical Induction we obtain that for $n \geqslant 3$ we have $\det(S_n)=0$. □

[1] Notation $S_n(\{x_i\}_{i=2}^{n+1},\{y_i\}_{i=1,\ i\neq j}^{n+1})$ means that for creating the matrix S_n we use successively the numbers $(x_2,x_3,\dots,x_{n+1})$, $(y_1,y_2,\dots,y_{j-1},y_{j+1},\dots,y_{n+1})$.

Problem, discussed in this paper, can be generalized in the following way:

Problem 3. *Let x_i, y_i be any complex numbers and let A_n be the matrix of dimension $n \times n$ of the form*

$$A_n = \begin{bmatrix} x_1y_1 + x_2y_2 & x_1y_3 + x_2y_4 & \dots & x_1y_{2n-1} + x_2y_{2n} \\ x_3y_1 + x_4y_2 & x_3y_3 + x_4y_4 & \dots & x_3y_{2n-1} + x_4y_{2n} \\ x_5y_1 + x_6y_2 & x_5y_3 + x_6y_4 & \dots & x_5y_{2n-1} + x_6y_{2n} \\ \vdots & \vdots & \dots & \vdots \\ x_{2n-1}y_1 + x_{2n}y_2 & x_{2n-1}y_3 + x_{2n}y_4 & \dots & x_{2n-1}y_{2n-1} + x_{2n}y_{2n} \end{bmatrix} =$$

$$= [\mathbf{k_1}y_{2j-1} + \mathbf{k_2}y_{2j}]_{j=1,2,\dots,n},$$

where $\mathbf{k_1} = [x_1, x_3, x_5, \dots, x_{2n-1}]^T$, $\mathbf{k_2} = [x_2, x_4, x_6, \dots, x_{2n}]^T$. *It should be proven that* $\det(A_n) = 0$ *for any* $n \geqslant 3$.

Proof. The only thing needed to be done is to repeat the reasoning from the proof of Problem 1. Value $\det(A_n)$ is equal to zero when the columns of matrix A_n are linearly dependent which is equivalent to the fact that the system of equations

$$\sum_{j=1}^{n} \alpha_j \left(\mathbf{k_1}y_{2j-1} + \mathbf{k_2}y_{2j}\right) = \mathbf{0}$$

possesses the non-zero solution. That is the system:

$$\left(\sum_{j=1}^{n} \alpha_j y_{2j-1}\right)\mathbf{k_1} + \left(\sum_{j=1}^{n} \alpha_j y_{2j}\right)\mathbf{k_2} = \mathbf{0}$$

possesses the non-zero solution. And so it is for $n \geqslant 3$, because the system created from the coefficients placed by $\mathbf{k_1}$ and $\mathbf{k_2}$:

$$\begin{cases} \sum_{j=1}^{n} \alpha_j y_{2j-1} = 0 \\ \sum_{j=1}^{n} \alpha_j y_{2j} = 0 \end{cases}$$

for $n \geqslant 3$ has more variables than equations. □

Remark 4. Although the solution presented in this paper, concerning the used idea, is consistent with Solution I in [1] and, probably, many mathematicians investigating this problem come to this idea in a quite natural way, I want to emphasize that I found this solution without any help, and only later my supervisor showed me the mentioned above solutions presented in American Mathematical Montly [1]. Moreover, I want to notice that I have presented my solution in a full form with the proper generalization.

References

1. Hively G.A., Tomsky J., Dearden B.: *E3178.* Amer. Math. Monthly **95**, no. 7 (1988), 664–665.
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