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REGION OF EXISTENCE OF MULTIPLE SOLUTIONS FOR A CLASS OF ROBIN TYPE FOUR-POINT BVPS

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Abstract. This article aims to prove the existence of a solution and compute the region of existence for a class of four-point nonlinear boundary value problems (NLBVPs) defined as

$$-u''(x) = \psi(x, u, u'), \quad x \in (0, 1),$$
$$u'(0) = \lambda_1 u(\xi), \quad u'(1) = \lambda_2 u(\eta),$$

where $I = [0, 1]$, $0 < \xi \leq \eta < 1$ and $\lambda_1, \lambda_2 > 0$. The nonlinear source term $\psi \in C(I \times \mathbb{R}^2, \mathbb{R})$ is one sided Lipschitz in u with Lipschitz constant L_1 and Lipschitz in u', such that $\left|\psi(x, u, u') - \psi(x, u, v')\right| \leq L_2(x)|u' - v'|$. We develop monotone iterative technique (MI-technique) in both well ordered and reverse ordered cases. We prove maximum, anti-maximum principle under certain assumptions and use it to show the monotonic behaviour of the sequences of upper-lower solutions. The sufficient conditions are derived for the existence of solution and verified for two examples. The above NLBVPs is linearised using Newton's quasilinearization method which involves a parameter k equivalent to $\max_u \frac{\partial \psi}{\partial u}$. We compute the range of k for which iterative sequences are convergent.

Keywords: Green's function, monotone iterative technique, maximum principle, multi-point problem.

Mathematics Subject Classification: 34B05, 34B15, 34B10, 65L10, 47J25.

1. INTRODUCTION

In the field of differential equations (DEs), the concept of NLBVPs have great importance. Second and higher-order multi-point (m-point) NLBVPs are studied in several areas to describe many real life problems [17, 18, 20, 54]. Various methods are introduced to study the existence, multiplicity, and positivity of solutions of four-point NLBVPs, where the boundary conditions (BCs) may be Neuman, Dirichlet, or mixed type.

1.1. LITERATURE REVIEW ON FOUR-POINT BVPS

In 2010, Yang *et al.* [50] used the Krasnoselskii fixed point theorem and triple fixed point theorem to show the existence and multiplicity of positive solutions for the following class of four-point Dirichlet NLBVPs:

$$\begin{aligned} &u''(x) + \psi(x, u, u') = 0, \quad x \in (0, 1), \\ &u(0) = \alpha u(\xi), \quad u(1) = \beta u(\eta). \end{aligned} \tag{1.1}$$

Using Leggett–Williams norm-type theorem Shen *et al.* [35] established the existence of positive solution for the above class of second-order four-point NLBVPs (1.1) under different resonance conditions for α and β. Here $\xi, \eta \in (0, 1)$ and ψ is independent of u'. Chen *et al.* [8] provided existence of positive solutions for the following four-point NLBVPs:

$$u^{(4)}(x) + \psi(x, u) = 0, \quad u(0) = u(1) = 0, \quad x \in (0, 1), \tag{1.2}$$

$$au''(\xi) - bu'''(\xi) = 0, \quad cu''(\eta) + du'''(\eta) = 0, \tag{1.3}$$

where $0 \leq \xi < \eta \leq 1$ and $a, b, c, d \geq 0$ are constants. This equation is known as linear beam equation for $\psi(x, u) = a(x)g(u(x))$. Here they have used method of upper-lower solutions and fixed point theorem to study the existence results. Zhai *et al.* [51] established the existence-uniqueness (EU) of positive solutions and used fixed point theorem of concave operators in partial ordering Banach spaces for a class of four-point BVPs of Caputo fractional DEs for any given parameter.

Sun *et al.* [39] obtained various results to the existence of positive symmetric solutions for a class of second-order four-point NLBVPs with p-Laplacian. Bai *et al.* [2] studied second-order four-point BVPs:

$$\begin{aligned} &u''(x) + \lambda h(x)\psi(x, u) = 0 \quad x \in (0, 1), \\ &u'(0) = au(\xi), \quad u(1) = bu(\eta), \end{aligned}$$

where $0 < \xi < \eta < 1$, $0 \leq a, b < 1$, and $h : [0, 1] \to [0, \infty)$, $\psi : [0, 1] \times [0, \infty) \to [0, \infty)$ are continuous functions. Here the fixed-point index theory, Leray–Schauder degree, and upper-lower solution method are used to ensure the existence, non existence, and multiplicity of positive solutions in a given range. Chinni *et al.* [10] studied existence, localization, and multiplicity of positive solutions by using Schauder and Krasnoselskii's fixed point theorems, combined with a Harnack-type inequality. Here authors have considered the following class of four-point NLBVPs with singular ϕ-Laplacian:

$$-[\phi(u')]' = \psi(x, u, u'), \quad u(0) = \alpha u(\xi), \quad u(T) = \beta u(\eta), \tag{1.4}$$

where $\alpha, \beta \in [0, 1)$, $0 < \xi < \eta < T$, $\psi : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $\phi : (-a, a) \to \mathbb{R}$ $(0 < a < \infty)$ is an increasing homeomorphism, and (1.4) is always solvable. More works on nonlocal problems can be found in [1–4, 13, 25, 48].

Palamides *et al.* [27] used continuum property of the solutions funnel (Kneser's Theorem) which is combined with the corresponding vector field for fourth-order four-point BVPs. They investigated the existence of positive or a negative solution, although these problems do not always admit positive Green's function. Some more works are done by using various methods such as Leray–Schauder degree theory [2, 15, 26], Shooting method [41], Coincidence degree theory [42], Topological degree method [1, 3, 21, 25, 26], Upper-lower solution method [21, 22] and the list is not exhaustive. The afore mentioned methods come with its own set of advantages and disadvantages.

1.2. LITERATURE REVIEW ON MI-TECHNIQUE

The MI-technique is an inspiring method [9, 53] which gives ground for the theoretical and numerical EU of the solution of nonlinear IVPs and m-point BVPs. Higher-order m-point DEs are also studied by this technique.

MI-technique was first introduced by Picard [33] in 1890. He has studied the existence of a solution for Dirichlet BVPs. This method is related to the method of upper-lower solutions. In this method, for a class of linear problems, the monotone sequences of the upper-lower solution with initial guesses are constructed. Then by using the initial approximations, the convergence of these monotone sequences are shown. Further it is shown that the solution of NLBVPs lies between the convergent sequences of upper-lower solutions. The reader is also suggested to refer the book by Chaplygin [6] in which the MI-technique was proposed and its ideas gave a basis of many investigations of several other mathematicians. For a comprehensive and detail study, we suggest referring [11, 23].

There are some special types of problems in which MI-technique is well grounded such as impulsive integro DEs. For this problem Zhimin [19] investigated the EU of solution, where BCs are periodic. Wei *et al.* [49] studied the EU of slanted cantilever beam:

$$u^{(4)}(x) = \psi(x, u, u'), \quad u(0) = u'(0) = u''(1) = u'''(1) = 0, \quad 0 \le x \le 1,$$

where $\psi \in (I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is continuous and $u'(x)$ is slope, reflecting the curving degree of the elastic beam. Above problem illustrates the static deformation of an elastic beam which has its right extreme freed and left extreme fixed.

As the exact solution for the fractional DEs can not be obtained easily, so we look for approximate solutions. For the approximation of the solution, various methods can be used but MI-technique is an effective mechanism for both IVPs and BVPs related to fractional type DEs. Cui [12] used this technique to approximate maximal and minimal solutions and derived uniqueness result for nonlinear Riemann–Liouville fractional DEs:

$$D^p u(x) + \psi(x, u) = 0, \quad u(0) = u'(1) = 0, \quad u(1) = 0, \quad x \in (0, 1),$$

where D^p is the standard Riemann–Liouville derivative and $p \in (2, 3]$. We can see use of MI-technique for Riemann–Liouville fractional DEs also in [34, 46]. Other works which may be referred are such as Impulsive DEs in Banach space [7], Casual DEs [47], Stieltjes integral BCs [40], etc.

There are various methods to deal with singular NLBVPs for example, shooting method, the topological degree method, and method of upper-lower solutions. The method of upper-lower solution is a very promising method as mentioned in [53]. In [28–32] the EU of solutions for a class of singular and doubly singular two-point BVPs have been established. Also, the region of multiple solutions has been determined. Singh *et al.* [38] developed MI-technique for three-point singular BVPs and studied the existence of a solution. Zhang [53] proved necessary and sufficient condition for existence of positive solution for the following Dirichlet singular BVPs

$$-u''(x) = \psi(x, u), \quad u(0) = u(1) = 0, \quad x \in (0, 1).$$

There are a lot of works on two, three, and m-point BVPs using MI-technique. In 1931, Dragoni [14] introduced MI-technique for two-point Dirichlet BVPs where the nonlinear term is derivative dependent. Cabada *et al.* [5], Cherpion *et al.* [9] also developed this technique for two-point second-order BVPs and studied the existence and approximation. Singh *et al.* [36] considered three-point BVPs (1.1), where $u'(0) = 0$ and $u'(1) = \beta u(\eta)$. They developed this technique and derived some new results for the existence of a solution. For several other three-point BVPs, the existence of solution can be found in [24, 36, 37, 44].

MI-Technique has also been done for four point BVPs. Ge *et al.* [16] studied multiplicity of solution for four-point BVPs via the variational approach and MI-technique. Zhang *et al.* [52] developed the method of upper-lower solutions with MI-technique and obtained some new existence results for fourth order four-point BVPs (1.2)–(1.3), where ψ is dependent on the derivative of solution u. Recently, Verma *et al.* [45] proved the existence of solution for the BVPs (1.1) with BCs $u'(0) = 0$, $u(1) = \lambda_1 u(\xi) + \lambda_2 u(\eta)$, where $\xi \le \eta \in (0, 1)$ and $\lambda_1, \lambda_2 > 0$. They proposed the method of upper-lower solutions in both reverse and well ordered cases. Urus *et al.* [43] explored this technique for the above BVPs where ψ is independent of u'.

In this paper, we investigate the existence of a solution for the following NLBVPs:

$$-u''(x) = \psi(x, u, u'), \quad x \in (0, 1), \tag{1.5}$$

$$u'(0) = \lambda_1 u(\xi), \quad u'(1) = \lambda_2 u(\eta), \tag{1.6}$$

where $0 < \xi \le \eta < 1$, $\lambda_1, \lambda_2 > 0$ and $\psi \in C(I \times \mathbb{R}^2, \mathbb{R})$. We prove maximum, anti-maximum principle, and develop MI-technique with upper-lower solution. Existence and approximation of solution are proved for reverse and well ordered cases. We obtain that the nonlinear source term $\psi(x, u, u')$ is Lipschitz in u' and one sided Lipschitz in u. To this end, we compute the region for the existence of solution.

To study the NLBVPs (1.5)–(1.6), define an iterative scheme which is given as follows:

$$\begin{gathered}-u''_{n+1}(x) - ku_{n+1}(x) = \psi(x, u_n(x), u'_n(x)) - ku_n(x),\\ u'_{n+1}(0) = \lambda_1 u_{n+1}(\xi), \quad u'_{n+1}(1) = \lambda_2 u_{n+1}(\eta),\end{gathered}$$

where $\psi \in C(I \times \mathbb{R}^2, \mathbb{R})$, $0 < \xi \le \eta < 1$, $\lambda_1, \lambda_2 \ge 0$, $n \in \mathbb{N}$ and $k \in \mathbb{R} \setminus \{0\}$ is constant. The linear BVPs corresponding to above iterative scheme is

$$-u''(x) - ku(x) = g(x), \quad x \in (0, 1), \tag{1.7}$$

$$u'(0) = \lambda_1 u(\xi), \quad u'(1) = \lambda_2 u(\eta) + c, \tag{1.8}$$

where $g(x) = \psi(x, u, u') - ku$ is continuous in I and c has a constant value.

This paper is divided into four sections. In the second section, we study the existence of a NLBVPs for the case $0 < k < \pi^2/4$, and numerically we verify it. Similarly, in the third section, we study the case where $k < 0$, and finally in the last section, we have concluded our results.

2. WHEN $0 < k < \frac{\pi^2}{4}$

This section is divided into five subsections. In the first subsection, we derived Green's function, its sign, solution of BVPs (1.7)–(1.8), and anti-maximum principle. In the second subsection, the existence of some differential inequalities are proved, to determine the monotonicity of sequences of upper-lower solutions. The third subsection is devoted to develop the MI-technique in reverse ordered cases. Some lemmas and propositions are obtained, which are used to prove the existence of solutions. In the fourth subsection, we obtain bounds for the derivative of the solution and we establish the existence theorem which is used to proves the existence of solutions between upper-lower solutions. In the last subsection, through an example we have shown that all the sufficient conditions are true in the specific region of k and monotonic sequences converge to the solution of the NLBPVs.

2.1. DEDUCTION OF GREEN'S FUNCTION

Consider linear BVPs (1.7)–(1.8) in the following manner:

$$-u''(x) - ku(x) = g(x), \quad x \in (0, 1), \tag{2.1}$$

$$u'(0) = \lambda_1 u(\xi), \quad u'(1) = \lambda_2 u(\eta). \tag{2.2}$$

Here $c = 0$ and $g(x) \in C(I)$.

Define

$$G(x,s)=\frac{1}{\sqrt{k}D_k}\begin{cases}\sqrt{k}\cos\sqrt{k}x\big[\sqrt{k}\cos\sqrt{k}\,(s-1)\\ +\lambda_2\sin\sqrt{k}\,(s-\eta)\big]+\lambda_1\sin\sqrt{k}\,(s-x)\\ \cdot\big[\lambda_2\sin\sqrt{k}\,(\eta-\xi)-\sqrt{k}\cos\sqrt{k}\,(\xi-1)\big], & 0\le x\le s\le\xi,\\ \sqrt{k}\cos\sqrt{k}s\big[\sqrt{k}\cos\sqrt{k}\,(x-1)\\ +\lambda_2\sin\sqrt{k}\,(x-\eta)\big], & 0\le x,s\le\xi,\\ \Big(\sqrt{k}\cos\sqrt{k}x+\lambda_1\sin\sqrt{k}\,(x-\xi)\Big)\\ \cdot\big[\sqrt{k}\cos\sqrt{k}\,(s-1)+\lambda_2\sin\sqrt{k}\,(s-\eta)\big], & \xi\le x\le s\le\eta,\\ \Big(\sqrt{k}\cos\sqrt{k}s+\lambda_1\sin\sqrt{k}\,(s-\xi)\Big)\\ \cdot\big[\sqrt{k}\cos\sqrt{k}\,(x-1)+\lambda_2\sin\sqrt{k}\,(x-\eta)\big], & s\le x,s\le\eta,\\ \sqrt{k}\cos\sqrt{k}\,(s-1)\ \big[\sqrt{k}\cos\sqrt{k}x\\ +\lambda_1\sin\sqrt{k}\,(x-\xi)\big], & \eta\le x\le s\le 1,\\ \sqrt{k}\cos\sqrt{k}\,(x-1)\ \big[\sqrt{k}\cos\sqrt{k}s\\ +\lambda_1\sin\sqrt{k}\,(s-\xi)\big]+\lambda_2\sin\sqrt{k}\,(x-s)\\ \cdot\big[\sqrt{k}\cos\sqrt{k}\eta+\lambda_1\sin\sqrt{k}\,(\eta-\xi)\big], & s\le x,s\le 1,\end{cases}\tag{2.3}$$

where

$$D_k=k\sin\sqrt{k}+\lambda_2\sqrt{k}\cos\sqrt{k}\eta+\lambda_1\big[\lambda_2\sin\sqrt{k}\,(\eta-\xi)-\sqrt{k}\cos\sqrt{k}\,(\xi-1)\big].$$

Lemma 2.1. *$G(x,s)$ defined by* (2.3) *is the Green's function of the BVPs* (2.1)–(2.2).

Proof. From equations (2.1)–(2.2) we obtain

$$G(x,s)=\begin{cases}a_1\cos\sqrt{k}x+b_1\sin\sqrt{k}x, & 0\le x\le s\le\xi,\\ a_2\cos\sqrt{k}x+b_2\sin\sqrt{k}x, & 0\le x,s\le\xi,\\ a_3\cos\sqrt{k}x+b_3\sin\sqrt{k}x, & \xi\le x\le s\le\eta,\\ a_4\cos\sqrt{k}x+b_4\sin\sqrt{k}x, & s\le x,s\le\eta,\\ a_5\cos\sqrt{k}x+b_5\sin\sqrt{k}x, & \eta\le x\le s\le 1,\\ a_6\cos\sqrt{k}x+b_6\sin\sqrt{k}x, & s\le x,s\le 1.\end{cases}$$

Applying properties of Green's function we can get values of a_i and b_i, where $i=1,2,\dots,6$. The proof is similar to the proof described in [45]. □

Let us assume that

$[A_1]$ $D_k > 0,\ \sqrt{k}\cos\sqrt{k} - \lambda_2 \sin\sqrt{k}\eta \geq 0,\ \sqrt{k} - \lambda_1 \sin\sqrt{k}\xi > 0.$

In Section 2.5, we have shown graphically that above inequalities are true for $k \in (\alpha, \beta) \subset (0, \pi^2/4)$, where (α, β) is the range of k for which monotone sequences converge to the solution.

Lemma 2.2. *If* $[A_1]$ *holds, then* $G(x, s) \geq 0$.

Proof. Since $[A_1]$ holds for all $\xi, \eta \in I$. Let us prove for the interval $0 \leq x \leq s \leq \xi$. From Lemma 2.1 we have

$$\begin{aligned} a_1 = \frac{1}{\sqrt{k}D_k}\big[&\lambda_2\sqrt{k}\sin\sqrt{k}(s-\eta) + k\cos\sqrt{k}(s-1) \\ &+ \lambda_1 \sin\sqrt{k}s\{\lambda_2 \sin\sqrt{k}(\eta-\xi) - \sqrt{k}\cos\sqrt{k}(\xi-1)\}\big] \end{aligned}$$

which implies that

$$\begin{aligned} a_1 = \frac{1}{\sqrt{k}D_k}\big[&(\sqrt{k}\sin\sqrt{k} + \lambda_2\cos\sqrt{k}\eta)(\sqrt{k} - \lambda_1 \sin\sqrt{k}\xi)\sin\sqrt{k}s \\ &+ (\sqrt{k}\cos\sqrt{k} - \lambda_2\sin\sqrt{k}\eta)(\sqrt{k}\cos\sqrt{k}s - \lambda_1\sin\sqrt{k}s\cos\sqrt{k}\xi)\big]. \end{aligned}$$

As $\cos\sqrt{k}\xi \leq \cos\sqrt{k}s$ and $\sin\sqrt{k}s \leq \sin\sqrt{k}\xi$, we have

$$\begin{aligned} \cos\sqrt{k}\xi(\sqrt{k} - \lambda_1\sin\sqrt{k}\xi) &\leq \cos\sqrt{k}\xi(\sqrt{k} - \lambda_1 \sin\sqrt{k}s) \\ &\leq \sqrt{k}\cos\sqrt{k}s - \lambda_1\sin\sqrt{k}s\cos\sqrt{k}\xi. \end{aligned}$$

Hence,

$$\begin{aligned} a_1 \geq \frac{(\sqrt{k} - \lambda_1\sin\sqrt{k}\xi)}{\sqrt{k}D_k}&[(\sqrt{k}\sin\sqrt{k} + \lambda_2\cos\sqrt{k}\eta)\sin\sqrt{k}s \\ &+ (\sqrt{k}\cos\sqrt{k} - \lambda_2\sin\sqrt{k}\eta)\cos\sqrt{k}\xi]. \end{aligned}$$

Now we have

$$\begin{aligned} b_1 &= \frac{1}{\sqrt{k}D_k}\lambda_1\cos\sqrt{k}s[\sqrt{k}\cos\sqrt{k}(\xi-1) - \lambda_2\sin\sqrt{k}(\eta-\xi)], \\ &= \frac{1}{\sqrt{k}D_k}\lambda_1\cos\sqrt{k}s[\cos\sqrt{k}\xi(\sqrt{k}\cos\sqrt{k} - \lambda_2\sin\sqrt{k}\eta) \\ &\quad + \sin\sqrt{k}\xi(\sqrt{k}\sin\sqrt{k} + \lambda_2\cos\sqrt{k}\eta)]. \end{aligned}$$

By applying $[A_1]$ it can be easily seen that $a_1,\ b_1 \geq 0$. Hence $G(x, s) \geq 0$, for $0 \leq x \leq s \leq \xi$. In similar fashion we can prove for other intervals. □

Lemma 2.3. *If* $g(x) \in C(I)$ *and* c *is any constant, then the solution* $u(x) \in C^2(I)$ *of BVPs* (1.7)–(1.8) *is given by*

$$u(x) = -\frac{c}{D_k}\big(\sqrt{k}\cos\sqrt{k}x + \lambda_1\sin\sqrt{k}(x-\xi)\big) - \int_0^1 G(x,s)g(s)ds. \qquad (2.4)$$

Proof. It is easy to deduce by using the concept of CF (Complimentary Function) and PI (Particular Integral). □

Proposition 2.4 (Anti-maximum principle). *Let* $[A_1]$ *be satisfied,* $c \geq 0$, $g(x) \geq 0$ *and* $g(x) \in C(I)$, *then the solution* $u(x)$ *of BVPs* (1.7)–(1.8) *is non-positive.*

Proof. Given that $g(x) \geq 0$, $c \geq 0$ and $[A_1]$ is satisfied. Now (2.4) can be written as

$$u(x) = -\frac{c}{D_k}\big(\cos\sqrt{k}x(\sqrt{k} - \lambda_1 \sin\sqrt{k}\xi) + \lambda_1 \cos\sqrt{k}\xi \sin\sqrt{k}x\big) - \int_0^1 G(x,s)g(s)ds.$$

Applying $[A_1]$ and Lemma 2.2 in above equation, we obtain the required result. □

2.2. EXISTENCE OF SOME DIFFERENTIAL INEQUALITIES

Lemma 2.5. *Suppose* $L_1 \in \mathbb{R}^+$ *and* $L_2 : I \to \mathbb{R}^+$ *are such that*

(i) $L_1 - k \leq 0$,
(ii) $L_2(0) = 0$, $L_2'(x) \geq 0$.

Then we have the following:

(a) *if* $(L_1 - k)\cos\sqrt{k} + \sup\{L_2(x)\sqrt{k}\sin\sqrt{k}\} \leq 0$, *then*

$$Y_1(x) = (L_1 - k)\cos\sqrt{k}x + L_2(x)\sqrt{k}\sin\sqrt{k}x \leq 0, \quad \forall x \in I,$$

(b) *if* $(L_1 - k) + \sup L_2'(x) \leq 0$, *then*

$$Y_2(x) = (L_1 - k)\sin\sqrt{k}x + L_2(x)\sqrt{k}\cos\sqrt{k}x \leq 0, \quad \forall x \in I.$$

Proof. (a) Since $\cos x$ is decreasing and $\sin x$ is increasing function in $(0, \frac{\pi}{2})$. Using these properties for all $x \in I$, we have

$$\begin{aligned} Y_1(x) &= (L_1 - k)\cos\sqrt{k}x + L_2(x)\sqrt{k}\sin\sqrt{k}x \\ &\leq (L_1 - k)\cos\sqrt{k} + \sup\{L_2(x)\sqrt{k}\sin\sqrt{k}\} \end{aligned}$$

The desired result follows from the assumption.

(b) We have

$$Y_2'(x) = \sqrt{k}\big((L_1 - k) + L_2'(x)\big)\cos\sqrt{k}x - kL_2(x)\sin\sqrt{k}x \leq 0,$$

whenever $(L_1 - k) + \sup L_2'(x) \leq 0$. Therefore, $Y_2(x)$ is decreasing for all $x \in I$, and $Y_2(0) = 0$. Hence, the result follows. □

Lemma 2.6. *Suppose* $[A_1]$ *and conditions of Lemma 2.5 are satisfied. Then the following inequalities hold:*

(a) $(L_1-k)(\sqrt{k}\cos\sqrt{k}x+\lambda_1\sin\sqrt{k}(x-\xi))$
$\pm L_2(x)\sqrt{k}(\sqrt{k}\sin\sqrt{k}x-\lambda_1\cos\sqrt{k}(x-\xi))\le 0,\quad \forall x\in I.$

(b) $(L_1-k)G(x,s)\pm L_2(x)\frac{\partial G(x,s)}{\partial x}\le 0,\quad \forall x,s\in I, x\neq s.$

Proof. (a) Consider the positive sign, i.e.,

$$\begin{aligned}&(L_1-k)(\sqrt{k}\cos\sqrt{k}x+\lambda_1\sin\sqrt{k}(x-\xi))\\&+L_2(x)\sqrt{k}(\sqrt{k}\sin\sqrt{k}x-\lambda_1\cos\sqrt{k}(x-\xi))\\&=(\sqrt{k}-\lambda_1\sin\sqrt{k}\xi)((L_1-k)\cos\sqrt{k}x+L_2(x)\sqrt{k}\sin\sqrt{k}x)\\&\quad+\lambda_1\cos\sqrt{k}\xi((L_1-k)\sin\sqrt{k}x-L_2(x)\sqrt{k}\cos\sqrt{k}x).\end{aligned}$$

By using inequality (a) of Lemma 2.5 we conclude the result. A similar proof follows for a negative sign.

(b) Consider the positive sign, i.e.,

$$(L_1-k)G(x,s)+L_2(x)\frac{\partial G(x,s)}{\partial x},\quad x,s\in I,\quad x\neq s. \tag{2.5}$$

To evaluate sign of (2.5) we first evaluate $\frac{\partial G}{\partial x}$, $x\neq s$, from Lemma 2.1 for each interval individually. Then we substitute the values of $G(x,s)$ and $\frac{\partial G}{\partial x}$ for each subinterval of I in equation (2.5).

For brevity, let us define

$$\begin{aligned}Z_1&=\sqrt{k}\cos\sqrt{k}(s-1)+\lambda_2\sin\sqrt{k}(s-\eta),\\Z_2&=\sqrt{k}\cos\sqrt{k}(\xi-1)+\lambda_2\sin\sqrt{k}(\xi-\eta),\\Z_3&=\sqrt{k}\cos\sqrt{k}s+\lambda_1\sin\sqrt{k}(s-\xi),\\Y_3&=(L_1-k)\cos\sqrt{k}x-L_2(x)\sqrt{k}\sin\sqrt{k}x.\end{aligned}$$

By simple calculations, it can be seen that $Z_1,\ Z_2,\ Z_3\ge 0$ and $Y_3(x)\le 0$.

(i) When $0\le x\le s\le\xi$, then $G(x,s)$ and $\frac{\partial G}{\partial x}$ can be written as

$$\begin{aligned}G(x,s)&=Z_1\sqrt{k}\cos\sqrt{k}x-Z_2\lambda_1\sin\sqrt{k}(s-x),\\\frac{\partial G(x,s)}{\partial x}&=-Z_1k\sin\sqrt{k}x+Z_2\lambda_1\sqrt{k}\cos\sqrt{k}(s-x),\quad x\neq s.\end{aligned}$$

Now expression (2.5) becomes

$$\begin{aligned}(L_1-k)G(x,s)+L_2(x)\frac{\partial G}{\partial x}&=Y_1(x)(Z_1\sqrt{k}-Z_2\lambda_1\sin\sqrt{k}s)\\&\quad+Y_2(x)Z_2\lambda_1\cos\sqrt{k}s,\quad x\neq s,\end{aligned}$$

where $Y_1(x)$ and $Y_2(x)$ are given in Lemma 2.5.

Substituting values of Z_1 and Z_2 in $(Z_1\sqrt{k} - Z_2\lambda_1 \sin\sqrt{k}s)$ and simplifying we get

$$\begin{aligned}
&Z_1\sqrt{k} - Z_2\lambda_1 \sin\sqrt{k}s \\
&= (\sqrt{k}\cos\sqrt{k} - \lambda_2 \sin\sqrt{k}\eta)(\sqrt{k}\cos\sqrt{k}s - \lambda_1 \sin\sqrt{k}s \cos\sqrt{k}\xi) \\
&\quad + (\sqrt{k}\sin\sqrt{k} + \lambda_2 \cos\sqrt{k}\eta)(\sqrt{k}\sin\sqrt{k}s - \lambda_1 \sin\sqrt{k}s \sin\sqrt{k}\xi), \\
&\geq \cos\sqrt{k}\xi(\sqrt{k}\cos\sqrt{k} - \lambda_2 \sin\sqrt{k}\eta)(\sqrt{k} - \lambda_1 \sin\sqrt{k}s) \\
&\quad + \sin\sqrt{k}s(\sqrt{k}\sin\sqrt{k} + \lambda_2 \cos\sqrt{k}\eta)(\sqrt{k} - \lambda_1 \sin\sqrt{k}\xi).
\end{aligned}$$

Applying inequality $[A_1]$ and Lemma 2.5 we obtain that expression (2.5) is non-positive.

(ii) When $0 \leq s \leq x \leq \xi$, from Lemma 2.1

$$\frac{\partial G(x,s)}{\partial x} = \sqrt{k}\cos\sqrt{k}s(-k\sin\sqrt{k}(x-1) + \lambda_2\sqrt{k}\cos\sqrt{k}(x-\eta)), \quad x \neq s.$$

Now expression (2.5) becomes

$$\begin{aligned}
&(L_1 - k)G(x,s) + L_2(x)\frac{\partial G}{\partial x} \\
&= \sqrt{k}\cos\sqrt{k}s\big(\sqrt{k}\{(L_1 - k)\cos\sqrt{k}(x-1) - L_2(x)\sqrt{k}\sin\sqrt{k}(x-1)\} \\
&\quad + \lambda_2\{(L_1 - k)\sin\sqrt{k}(x-\eta) + L_2(x)\sqrt{k}\cos\sqrt{k}(x-\eta)\}\big), \\
&= \sqrt{k}\cos\sqrt{k}s\big(Y_3(x)(\sqrt{k}\cos\sqrt{k} - \lambda_2\sin\sqrt{k}\eta) \\
&\quad + Y_2(x)(\sqrt{k}\sin\sqrt{k} + \lambda_2\cos\sqrt{k}\eta)\big).
\end{aligned}$$

Applying inequalities $[A_1]$ and (b) of Lemma 2.5 we get the required result.

(iii) When $\xi \leq x \leq s \leq \eta$,

$$\begin{aligned}
G(x,s) &= Z_1(\sqrt{k}\cos\sqrt{k}x + \lambda_1\sin\sqrt{k}(x-\xi)), \\
\frac{\partial G(x,s)}{\partial x} &= Z_1(-k\sin\sqrt{k}x + \lambda_1\sqrt{k}\cos\sqrt{k}(x-\xi)), \quad x \neq s.
\end{aligned}$$

Now for $x \neq s$,

$$\begin{aligned}
&(L_1 - k)G(x,s) + L_2(x)\frac{\partial G}{\partial x} \\
&= Z_1\big((L_1 - k)(\sqrt{k}\cos\sqrt{k}x + \lambda_1\sin\sqrt{k}(x-\xi)) \\
&\quad + L_2(x)\sqrt{k}(-\sqrt{k}\sin\sqrt{k}x + \lambda_1\cos\sqrt{k}(x-\xi))\big) \\
&= Z_1\big(\sqrt{k}Y_3(x) + \lambda_1\big(Y_2(x)\cos\sqrt{k}\xi - Y_3(x)\sin\sqrt{k}\xi\big)\big).
\end{aligned}$$

Applying inequalities $[A_1]$ and (b) of Lemma 2.5 we obtain that expression (2.5) is non-positive.

(iv) When $\xi \leq s \leq x \leq \eta$,

$$G(x,s) = Z_3\big(\sqrt{k}\cos\sqrt{k}(x-1) + \lambda_2 \sin\sqrt{k}(x-\eta)\big),$$
$$\frac{\partial G(x,s)}{\partial x} = Z_3\big(-k\sin\sqrt{k}(x-1) + \sqrt{k}\lambda_2\cos\sqrt{k}(x-\eta)\big), \quad x \neq s.$$

$$\begin{aligned}
&(L_1-k)G(x,s) + L_2(x)\frac{\partial G}{\partial x}\\
&= Z_3[\sqrt{k}\{(L_1-k)\cos\sqrt{k}(x-1) - L_2(x)\sqrt{k}\sin\sqrt{k}(x-1)\}\\
&\quad + \lambda_2\{(L_1-k)\sin\sqrt{k}(x-\eta) + L_2(x)\sqrt{k}\cos\sqrt{k}(x-\eta)\}]\\
&= Z_3[\sqrt{k}\big(Y_3(x)\cos\sqrt{k} + Y_2(x)\sin\sqrt{k}\big) + \lambda_2\big(Y_2(x)\cos\sqrt{k}\eta - Y_3(x)\sin\sqrt{k}\eta\big)]\\
&= Z_3\big(Y_3(x)(\sqrt{k}\cos\sqrt{k} - \lambda_2\sin\sqrt{k}\eta) + Y_2(x)(\sqrt{k}\sin\sqrt{k} + \lambda_2\cos\sqrt{k}\eta)\big) \leq 0.
\end{aligned}$$

(v) When $\eta \leq x \leq s \leq 1$,

$$G(x,s) = \sqrt{k}\cos\sqrt{k}(s-1)(\sqrt{k}\cos\sqrt{k}x + \lambda_1\sin\sqrt{k}(x-\xi)),$$
$$\frac{\partial G(x,s)}{\partial x} = k\cos\sqrt{k}(s-1)\big(-\sqrt{k}\sin\sqrt{k}x + \lambda_1\cos\sqrt{k}(x-\xi)\big), \quad x \neq s.$$

We have

$$\begin{aligned}
&(L_1-k)G(x,s) + L_2(x)\frac{\partial G}{\partial x}\\
&= \sqrt{k}\cos\sqrt{k}(s-1)[\sqrt{k}Y_3(x) + \lambda_1\big(Y_2(x)\cos\sqrt{k}\xi - Y_3(x)\sin\sqrt{k}\xi\big)]\\
&= \sqrt{k}\cos\sqrt{k}(s-1)[Y_3(x)(\sqrt{k} - \sin\sqrt{k}\xi) + \lambda_1 Y_2(x)\cos\sqrt{k}\xi] \leq 0,
\end{aligned}$$

(vi) When $\eta \leq s \leq x \leq 1$,

$$\begin{aligned}
G(x,s) &= Z_3\sqrt{k}\cos\sqrt{k}(x-1)\\
&\qquad + \lambda_2\sin\sqrt{k}(x-s)(\sqrt{k}\cos\sqrt{k}\eta + \lambda_1\sin\sqrt{k}(\eta-\xi)),\\
\frac{\partial G(x,s)}{\partial x} &= -Z_3 k\sin\sqrt{k}(x-1)\\
&\qquad + \sqrt{k}\lambda_2\cos\sqrt{k}(x-s)(\sqrt{k}\cos\sqrt{k}\eta + \lambda_1\sin\sqrt{k}(\eta-\xi)), \quad x \neq s.
\end{aligned}$$

We have

$$\begin{aligned}
&(L_1-k)G(x,s) + L_2(x)\frac{\partial G}{\partial x}\\
&= Z_3\sqrt{k}\big((L_1-k)\cos\sqrt{k}(x-1) - L_2(x)\sqrt{k}\sin\sqrt{k}(x-1)\big) + \lambda_2\big(\sqrt{k}\cos\sqrt{k}\eta,\\
&\quad + \lambda_1\sin\sqrt{k}(\eta-\xi)\big)\big((L_1-k)\sin\sqrt{k}(x-s) + L_2(x)\sqrt{k}\cos\sqrt{k}(x-s)\big)\\
&= Z_3\sqrt{k}(Y_2(x)\cos\sqrt{k} + Y_3(x)\sin\sqrt{k}) + \lambda_2(\sqrt{k}\cos\sqrt{k}\eta + \lambda_1\sin\sqrt{k}(\eta-\xi))\\
&\quad \cdot (Y_3(x)\cos\sqrt{k} - Y_2(x)\sin\sqrt{k}s).
\end{aligned}$$

Applying inequality $[A_1]$ and Lemma 2.5, it is easy to show that above equation is non-positive. This completes the proof. Similarly (b) can be proved for the negative sign also. □

2.3. NON-WELL ORDERED CASE: CONSTRUCTION OF UPPER-LOWER SOLUTIONS

In this section, upper-lower solutions are defined and some conditions on $c(x)$, $d(x)$ and $\psi(x,u,u')$ are assumed. Then we define the sequences of functions $\{c_n(x)\}_n$ and $\{d_n(x)\}_n$, and develop MI-technique based on these sequences. We prove some lemmas which show that sequences of upper solutions and lower solutions are respectively monotonically non-decreasing and non-increasing. Also we develop a theorem which gives that the sequence of functions $\{c_n(x)\}_n$ and $\{d_n(x)\}_n$ are uniformly convergent and converge to the solution of NLBVPs (1.5)–(1.6).

Definition 2.7. A function $c(x) \in C^2(I)$ is called lower solution of NLBVPs (1.5)–(1.6) if it satisfies the following inequalities:

$$-c''(x) \leq \psi(x, c(x), c'(x)), \quad c'(0) = \lambda_1 c(\xi), \quad c'(1) \leq \lambda_2 c(\eta), \quad x \in (0,1).$$

Definition 2.8. A function $d(x) \in C^2(I)$ is called an upper solution of NLBVPs (1.5)–(1.6), if it satisfy the following inequalities,

$$-d''(x) \geq \psi(x, d(x), d'(x)), \quad d'(0) = \lambda_1 d(\xi), \quad d'(1) \geq \lambda_2 d(\eta), \quad x \in (0,1).$$

Let us assume some conditions as follows:

[A_2] there exist $c(x)$, $d(x) \in C^2(I)$ such that $c(x) \geq d(x)$ for all $x \in I$, where $c(x)$, $d(x)$ are respective lower-upper solutions of NLBVPs (1.5)–(1.6),

[A_3] $\psi(x,v,w) : E \to \mathbb{R}$ is continuous function on

$$E := \{(x,v,w) \in I \times \mathbb{R}^2 : d(x) \leq v \leq c(x)\},$$

[A_4] there exists $L_1 \geq 0$ such that for all $(x,v_1,w), (x,v_2,w) \in E$

$$v_1 \leq v_2 \Rightarrow \psi(x,v_2,w) - \psi(x,v_1,w) \leq L_1(v_2 - v_1),$$

[A_5] there exists $L_2 : I \to \mathbb{R}^+$ such that $L_2(0) = 0$, $L_2'(x) \geq 0$, and

$$|\psi(x,v,w_1) - \psi(x,v,w_2)| \leq L_2|(w_1 - w_2)|$$

for all (x,v,w_1), $(x,v,w_2) \in E$.

Also we propose sequences of functions $\{c_n(x)\}_n$ and $\{d_n(x)\}_n$ such that $c_0(x) = c(x)$, $d_0(x) = d(x)$,

$$-c''_{n+1}(x) - kc_{n+1}(x) = \psi(x, c_n(x), c'_n(x)) - kc_n(x), \tag{2.6}$$

$$c'_{n+1}(0) = \lambda_1 c_{n+1}(\xi), \quad c'_{n+1}(1) = \lambda_2 c_{n+1}(\eta), \tag{2.7}$$

$$-d''_{n+1}(x) - kd_{n+1}(x) = \psi(x, d_n(x), d'_n(x)) - kd_n(x), \tag{2.8}$$

$$d'_{n+1}(0) = \lambda_1 d_{n+1}(\xi), \quad d'_{n+1}(1) = \lambda_2 d_{n+1}(\eta). \tag{2.9}$$

Lemma 2.9. *If $c_n(x)$ is a lower solution of* (1.5)–(1.6). *Then $c_n(x) \geq c_{n+1}(x)$ for all $x \in I$, where $c_{n+1}(x)$ is given by equation* (2.6)–(2.7).

Proof. Given that $c_n(x)$ is a lower solution of (1.5)–(1.6). From (2.6)–(2.7) we have

$$\begin{aligned} &-(c''_{n+1}(x) - c''_n(x)) - k(c_{n+1}(x) - c_n(x)) \geq 0, \quad n \in \mathbb{N}, \\ &(c_{n+1} - c_n)'(0) = \lambda_1(c_{n+1} - c_n)(\xi), \\ &(c_{n+1} - c_n)'(1) \geq \lambda_2(c_{n+1} - c_n)(\eta). \end{aligned}$$

This is in the form of equations (1.7)–(1.8) with solution $u(x)$, where

$$\begin{aligned} u(x) &= c_{n+1}(x) - c_n(x), \\ g(x) &= -(c''_{n+1}(x) - c''_n(x)) - k(c_{n+1}(x) - c_n(x)) \geq 0, \end{aligned}$$

and $c \geq 0$. Hence, the result can be concluded from Proposition 2.4. □

Proposition 2.10. *Assume* $[A_1]$–$[A_5]$ *are true, and* $L_1 \in \mathbb{R}^+$, $L_2 : I \to \mathbb{R}^+$ *are such that conditions of Lemma 2.5 hold. If* $c_n(x)$ *is a lower solution of* (1.5)–(1.6), *then*

$$(k - L_1)u(x) + L_2 \operatorname{sign}(u'(x))u'(x) \leq 0, \quad \text{for all } x \in I,$$

where $u(x)$ *is any solution of BVPs* (1.7)–(1.8).

Proof. Let $u(x) = (c_{n+1} - c_n)(x)$. We have

$$\begin{aligned} -u''(x) - ku(x) &= -c''_{n+1}(x) + c''_n(x) - kc_{n+1}(x) + kc_n(x) \\ &= c''_n(x) + \psi(x, c_n, c'_n) \geq 0, \\ (c_{n+1} - c_n)'(0) &= \lambda_1(c_{n+1} - c_n)(\xi), \quad (c_{n+1} - c_n)'(1) \geq \lambda_2(c_{n+1} - c_n)(\eta). \end{aligned}$$

This is in the form of equations (1.7)–(1.8) with the solution $u(x)$. Therefore $u(x)$ can be written in the form of (2.4) with $g(x) = c''_n(x) + \psi(x, c_n, c'_n)$. Using $u(x)$ and $u'(x)$ from (2.4), we obtain

$$\begin{aligned} &(k - L_1)u(x) + L_2 \operatorname{sign}(u'(x))u'(x) \\ &= \frac{b}{D_k}[(L_1 - k)(\sqrt{k}\cos\sqrt{k}x + \lambda_1 \sin\sqrt{k}(x - \xi)) \pm L_2(x)\sqrt{k}(\sqrt{k}\sin\sqrt{k}x \\ &\quad - \lambda_1 \cos\sqrt{k}(x - \xi))] + \int_0^1 \Big[(L_1 - k)G(x, s) \pm L_2(x)\frac{\partial G(x, s)}{\partial x}\Big] g(s)ds. \end{aligned}$$

By inequalities of Lemma 2.6, the result can be concluded. □

Lemma 2.11. *Assume* $[A_1]$–$[A_5]$ *are true, and* $L_1 \in \mathbb{R}^+$, $L_2 : I \to \mathbb{R}^+$ *are such that conditions of Lemma 2.5 hold. Then the function* $c_n(x)$ *given by equation* (2.6)–(2.7) *satisfy:*

(a) $c_n(x) \geq c_{n+1}(x)$,
(b) $c_n(x)$ *is a lower solution of* (1.5)–(1.6).

Proof. We use the principle of mathematical induction to prove the monotonicity of $c_n(x)$.

Step 1. If $n = 0$, $c_0(x) = c(x)$, where $c(x)$ is a lower solution of (1.5)–(1.6), therefore by Lemma 2.9 we have $c_1 \leq c_0$.

Step 2. Suppose for $n-1$ that $c_{n-1}(x)$ is a lower solution of (1.5)–(1.6) and $c_n \leq c_{n-1}$. By definition of lower solutions,

$$c''_{n-1}(x) + \psi(x, c_{n-1}, c'_{n-1}) \geq 0, \quad x \in (0,1),$$
$$c'_{n-1}(0) = \lambda_1 c_{n-1}(\xi), \quad c'_{n-1}(1) \leq \lambda_2 c_{n-1}(\eta).$$

To show that c_n is a lower solution of (1.5)–(1.6) we have

$$\begin{aligned} -c''_n - \psi(x, c_n, c'_n) &= -\psi(x, c_n, c'_n) + \psi(x, c_n, c'_n) + kc_n - kc_{n-1}, \\ &\leq L_1(c_{n-1} - c_n) + L_2(x) \mid c'_{n-1} - c'_n \mid + k(c_n - c_{n-1}), \\ &\leq (k - L_1)(c_n - c_{n-1}) + L_2(x) \mid c'_{n-1} - c'_n \mid . \end{aligned}$$

Let $u = c_n - c_{n-1}$. By using Proposition 2.10 we arrive at $-c''_n - \psi(x, c_n, c'_n) \leq 0$. This proves that c_n is a lower solution of (1.5)–(1.6) and therefore by Lemma 2.9, $c_{n+1} \leq c_n$. □

Lemma 2.12. *If $d_n(x)$ is an upper solution of* (1.5)–(1.6)*, then $d_n(x) \leq d_{n+1}(x)$ for all $x \in I$, where $d_{n+1}(x)$ is given by equation* (2.8)–(2.9)*.*

The proof of Lemma 2.12 is similar to that of Lemma 2.9.

Lemma 2.13. *Assume $[A_1]$–$[A_5]$ are true, and $L_1 \in \mathbb{R}^+$, $L_2 : I \to \mathbb{R}^+$ are such that conditions of Lemma 2.5 hold. Then the function $d_n(x)$ given by equation* (2.8)–(2.9) *satisfy:*

(a) $d_n(x) \leq d_{n+1}(x)$,
(b) $d_n(x)$ *is an upper solution of* (1.5)–(1.6).

The proof of Lemma 2.13 is similar to that of Lemma 2.11.

Proposition 2.14. *Assume $[A_1]$–$[A_5]$ are true, $L_1 \in \mathbb{R}^+$, $L_2 : I \to \mathbb{R}^+$ are such that conditions of Lemma 2.5 hold, and*

$$\psi(x, d(x), d'(x)) - \psi(x, c(x), c'(x)) - k(d(x) - c(x)) \geq 0.$$

Then $c_n \geq d_n$ for all $x \in I$, where c_n and d_n are given by equation (2.6)–(2.7) *and* (2.8)–(2.9)*, respectively.*

Proof. Define

$$g_i(x) = \psi(x, d_i, b'_i) - \psi(x, c_i, c'_i) - k(d_i - c_i), \quad i \in \mathbb{N},$$

and let $u_i = d_i - c_i$ which satisfies

$$-u''_i - ku_i = \psi(x, d_{i-1}, d'_{i-1}) - \psi(x, c_{i-1}, c'_{i-1}) - k(d_{i-1} - c_{i-1}) = g_{i-1}(x),$$
$$(d_i - c_i)'(0) = \lambda_1 (d_i - c_i)(\xi), \quad (d_i - c_i)'(1) = \lambda_2 (d_i - c_i)(\eta).$$

Claim 1. $c_1 \geq d_1$. For $i = 1$, we have

$$-u_1'' - ku_1 = g_0(x) \geq 0,$$
$$u_1'(0) = \lambda_1 u_1(\xi), \quad u_1'(1) \geq \lambda_2 u_1(\eta).$$

Therefore u_1 is a solution of equations (1.7)–(1.8) with $g(x) = g_0(x)$. Hence by Proposition 2.4, $c_1 \geq d_1$.

Claim 2. $d_n \leq c_n$. Suppose $g_{n-2} \geq 0$ and $c_{n-1} \geq d_{n-1}$. Now,

$$\begin{aligned} g_{n-1}(x) &= \psi(x, d_{n-1}, d'_{n-1}) - \psi(x, c_{n-1}, c'_{n-1}) - k(c_{n-1} - c_{n-1}), \\ &\geq -(k - L_1)(d_{n-1} - c_{n-1}) - L_2(x)|c'_{n-1} - d'_{n-1}|, \\ &\geq -[(k - L_1)u_{n-1} + L_2(x)(\operatorname{sign} u'_{n-1})u'_{n-1}]. \end{aligned}$$

With the help of Proposition 2.10 we can prove that

$$(k - L_1)u_{n-1} + L_2(x)(\text{sign} \;\; u'_{n-1})u'_{n-1} \leq 0.$$

Therefore $g_{n-1} \geq 0$, also we have $u_n = d_n - c_n$ for $i = n$. Then u_n satisfies

$$-u_n'' - ku_n = g_{n-1}(x) \geq 0,$$
$$u_n'(0) = \lambda_1 u_n(\xi), \quad u_n'(1) \geq \lambda_2 u_n(\eta).$$

We deduce from Proposition 2.4 that $d_n \leq c_n$. □

2.4. BOUND ON DERIVATIVE OF SOLUTION

[A_6] Let $|\psi(x, v, w)| \leq \phi(|w|)$ for all $(x, v, w) \in E$, where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous which satisfies

$$\int_\gamma^\infty \frac{sds}{\phi(s)} \geq \max_{x \in I} c(x) - \min_{x \in I} d(x),$$

such that

$$\gamma = 2 \max \Big\{ \sup_{x \in [0,1]} |c(x)|, \sup_{x \in [0,1]} |d(x)| \Big\}.$$

Lemma 2.15. *If* [A_6] *holds, then there exists* $P > 0$ *such that* $\| u' \|_\infty \leq P$ *for all* $x \in I$, *where* u *is any solution of inequality*

$$-u''(x) \geq \psi(x, u, u'), \quad x \in (0, 1), \tag{2.10}$$
$$u'(0) = \lambda_1 u(\xi), \quad u'(1) \geq \lambda_2 u(\eta), \tag{2.11}$$

such that $d(x) \leq u(x) \leq c(x)$.

Proof. We consider three cases.

Case 1. If $u(x)$ is monotonically increasing in $(0,1)$, by the mean value theorem there exists $\alpha \in (0,1)$ such that

$$u'(\alpha) = u(1) - u(0),$$

which gives

$$|u'(\alpha)| \leq \gamma,$$

where

$$\gamma = 2\max\Big\{ \sup_{x\in[0,1]} |c(x)|, \sup_{x\in[0,1]} |d(x)| \Big\}.$$

Using $|\ \psi(x,v,w)\ |\leq \phi(|\ w\ |)$ in equation (2.10) and integrating from the limit α to x the equation becomes

$$\int_{\alpha}^{x} \frac{u''(x)u'(x)}{\phi(|u'(x)|)} dx \leq \int_{\alpha}^{x} u'(x)dx \leq \max_{x\in I} c(x) - \min_{x\in I} d(x).$$

Let $u'(x) = s(>0)$. Since $|u'(\alpha)| \leq \gamma$, then

$$\int_{\gamma}^{u'(x)} \frac{sds}{\phi(s)} \leq \int_{u'(\alpha)}^{u'(x)} \frac{sds}{\phi(s)} \leq \max_{x\in I} c(x) - \min_{x\in I} d(x).$$

Adding $\int_0^{\gamma} \frac{sds}{\phi(s)}$ on both side of above equation and applying condition $[A_6]$ we obtain

$$\|\ u'\ \|_{\infty} \leq P, \quad \forall x \in I.$$

Case 2. If $u(x)$ is monotonically decreasing in $(0,1)$, the proof is similar to Case 1.

Case 3. If $u(x)$ is neither monotonically decreasing nor monotonically increasing in $(0,1)$. The proof of this case is divided into two subcases.

Subcase 1. Consider the interval $(x_0, x] \subset (0,1)$ such that $u'(x_0) = 0$ and $u'(x) > 0$ for $x > x_0$. Using $|\psi(x,v,w)| \leq \phi(|w|)$ in equation (2.10) and integrating from x_0 to x the equation becomes

$$\int_{x_0}^{x} \frac{u''(x)u'(x)}{\phi(|u'(x)|)} dx \leq \int_{x_0}^{x} u'(x)dx \leq \max_{x\in I} c(x) - \min_{x\in I} d(x).$$

Let $u'(x) = s(>0)$ and choose $P > 0$. Using condition $[A_6]$ we obtain

$$\int_{0}^{u'(x)} \frac{sds}{\phi(s)} \leq \max_{x\in I} c(x) - \min_{x\in I} d(x) \leq \int_{0}^{P} \frac{sds}{\phi(s)} \Rightarrow \|\ u'\ \|_{\infty} \leq P, \quad \forall x \in I.$$

Subcase 2. Consider the interval $[x, x_0) \subset (0,1)$ such that $u'(x) < 0$ for $x < x_0$. The proof of this subcase is similar to Subcase 1. □

Lemma 2.16. *If $[A_6]$ holds, then there exists $P > 0$ such that $\| u' \|_\infty \leq P$ for all $x \in I$, where u is any solution of inequality*

$$-u''(x) \leq \psi(x, u, u'), \quad x \in (0,1),$$
$$u'(0) = \lambda_1 u(\xi), \quad u'(1) \leq \lambda_1 u(\eta),$$

such that $d(x) \leq u(x) \leq c(x)$.

The proof of Lemma 2.16 is similar to Lemma 2.15.

Theorem 2.17. *Assume $[A_1]$–$[A_5]$ are true, $L_1 \in \mathbb{R}^+$, $L_2 : I \to \mathbb{R}^+$ are such that conditions of Lemma 2.5 hold, and*

$$\psi(x, d(x), d'(x) - \psi(x, c(x), c'(x)) - k(d(x) - c(x)) \geq 0, \quad \forall x \in I.$$

Then $(c_n)_n \to y$ and $(d_n)_n \to z$ uniformly in $C^1(I)$, and $d \leq y \leq z \leq c$, where $y(x)$ and $z(x)$ are solutions of (1.5) *and* (1.6)*.*

Proof. We have already proved that the sequences $(c_n)_n$ and $(d_n)_n$ are such that

$$c = c_0 \geq c_1 \ldots \geq c_n \ldots \geq d_n \geq \ldots \geq d_1 \geq d_0 = d. \tag{2.12}$$

Now we prove that the sequences $(c_n)_n$ and $(d_n)_n$ converges uniformly in $C^1(I)$ to solutions y and z of NLBVPs (1.5) and (1.6) such that $d \leq y \leq z \leq c$ for all $x \in I$.

Firstly, we prove that $(c_n)_n$ and $(d_n)_n$ converge in $C^1(I)$. Since $(c_n)_n$ and $(d_n)_n$ are bounded as well as monotonic, therefore by monotone convergence theorem $(c_n)_n$ and $(d_n)_n$ are convergent pointwise. Let $\lim_{n\to\infty} c_n(x) = y(x)$ and $\lim_{n\to\infty} d_n(x) = z(x)$. From (2.12) and Lemma 2.16 it can be deduced that $(c_n)_n$ is uniformly bounded and equicontinuous in $C^2(I)$, i.e., for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|(c_n)(x) - (c_n)(y)| < \epsilon, \quad \text{if} \quad |x - y| < \delta, \quad \forall n \in \mathbb{N}.$$

Therefore every subsequence $(c_{n_i})_i$ of $(c_n)_n$ is equibounded and equicontinuous in $C^2(I)$. We know by the Arzela–Ascoli theorem that there exist a sub-subsequence $(c_{n_{i_j}})_j$ of sub-sequence $(c_{n_i})_i$ which converges in $C^2(I)$. Since convergent sequences have unique limit point, hence $c_n(x) \to y(x)$ uniformly in $C^2(I)$. Similarly, it can also be shown that $(d_n)_n(x) \to z(x)$ uniformly in $C^2(I)$.

We finally prove that $y(x)$ and $z(x)$ are solutions of NLBVPs (1.5)–(1.6). Since equations (2.6)–(2.7) and (2.8)–(2.9) are in the form of equations (1.7)–(1.8), so the solution of these equations can be expressed as the form of equation (2.4) for $(c_n)_n$ and $(d_n)_n$. After taking limit $n \to \infty$ and using Lebesgue dominated convergence theorem we can conclude that $y(x)$ and $z(x)$ are the solutions of NLBVPs (1.5)–(1.6). Hence the theorem is proved. □

2.5. NUMERICAL ILLUSTRATION

In this section, we have considered an example for reverse order case. This example gives uniformly convergent sequences of upper-lower solutions which converge to the solution of our NLBVPs for the specific range of $k > 0$.

2.5.1. Example

Consider four point NLBVPs

$$-u''(x) = \frac{e^u - xe^{u'}}{195},$$
$$u'(0) = 2u(0.1), \quad u'(1) = 3u(0.2),$$

where $\psi(x,u,u') = \frac{e^u - xe^{u'}}{195}$, $\xi = 0.1$, $\eta = 0.2$, $\lambda_1 = 2$ and $\lambda_2 = 3$. We consider initial upper-lower solutions as $d(x) = -(1 + 2.525x + x^2)$ and $c(x) = 1 + 2.525x + x^2$, respectively, where $d(x) \le c(x)$. Since $\psi(x,u,u')$ is one sided Lipschitz in u with Lipschitz constant $L_1 = 0.47\,331$, which is obtained by using $[A_4]$. Also ψ is Lipschitz in u' therefore we derive from $[A_5]$, $L_2(x) = \frac{xe^P}{195}$, where $P > 0$ such that $\| u' \|_\infty \le P$ for all $x \in I$. We obtain $k \ge 0.4811$ and from $[A_6]$, $\phi(|s|) = \frac{e^{4.525} + e^{|s|}}{195}$, $P = 0.2154$. The range for k is computed by using these above results and Mathematica-11.3. Hence for every $k \in (\alpha, \beta) \subset (0, \frac{\pi^2}{4})$, all the inequalities are satisfied (from Figures 1–4) and the sequences are convergent (from Figures 5–8), where $\alpha = 0.4811$ and from Figures 1–4 we observe that $\beta < 0.9$.

Remark 2.18. In Figures 6 and 7 we take $k = 1$ for which inequality shown in Figure 3 is not valid but we are getting monotonic sequences.

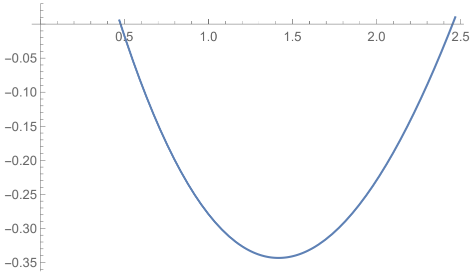

Fig. 1. $(L_1 - k)\cos\sqrt{k} + L_2(x)\sqrt{k}\sin\sqrt{k}$

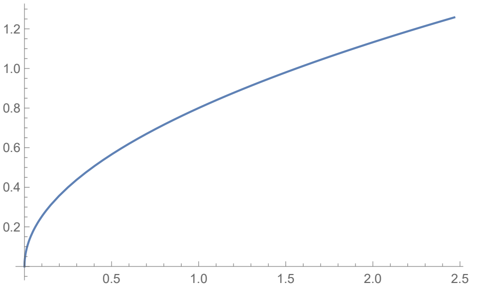

Fig. 2. $\sqrt{k} - \lambda_1 \sin\sqrt{k}\xi$

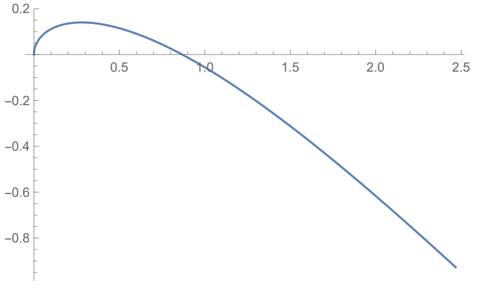

Fig. 3. $\sqrt{k}\cos\sqrt{k} - \lambda_2 \sin\sqrt{k}\eta$

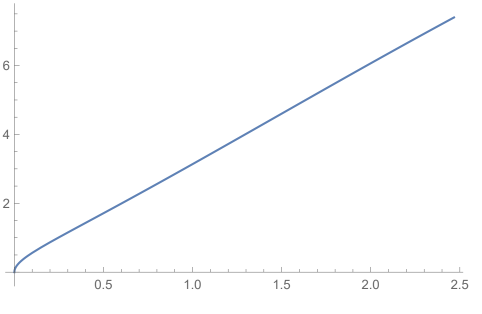

Fig. 4. D_k

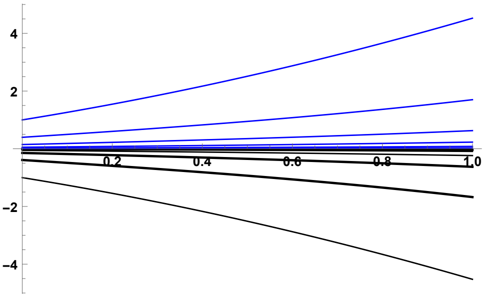

Fig. 5. $k = 0.49$

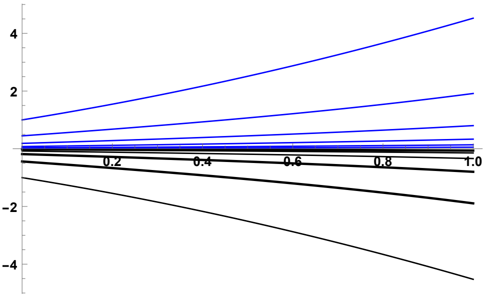

Fig. 6. $k = 0.6$

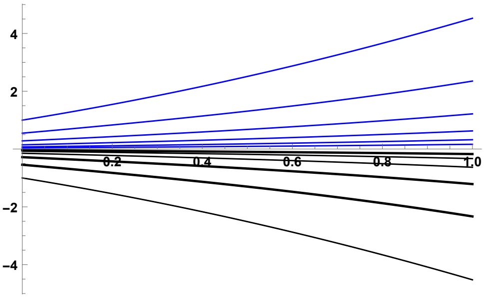

Fig. 7. $k = 0.89$

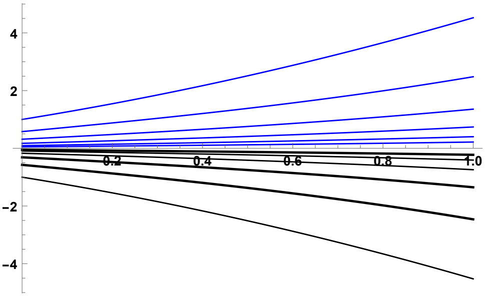

Fig. 8. $k = 1$

3. WHEN $k < 0$

In this section, for negative k, existence of NLBVPs (1.5)–(1.6) are studied. This section is similar to section two. It is also divided into five sub-sections. In first subsection we derive Green's function, sign of Green's function, solution of BVPs (1.7)–(1.8), and maximum principle. In the second subsection, we prove the existence of some differential inequalities that are used to prove the monotonic behavior of sequences of well-ordered upper-lower solutions. In subsection three, MI-technique is developed. Some lemmas and propositions have also been given in this subsection that is used to prove the existence of solutions. In the fourth subsection, we obtain bounds for the derivative of the solution and we establish the existence theorem which is used to proves the existence of solutions between upper-lower solutions. In the last subsection, we give an example and compute the range of $k < 0$ for which all the sufficient conditions are true.

3.1. DEDUCTION OF GREEN'S FUNCTION

Define

$$G(x,s) = \frac{1}{\sqrt{|\,k\,|}D_{k'}}\cdot
\begin{cases}
\sqrt{|\,k\,|}\cosh\sqrt{|\,k\,|}x \\
\cdot(\lambda_2\sinh\sqrt{|\,k\,|}(\eta-s)-\sqrt{|\,k\,|}\cosh\sqrt{|\,k\,|}(s-1)) \\
+\lambda_1\sinh\sqrt{|\,k\,|}(s-x) \\
\cdot(\sqrt{|\,k\,|}\cosh\sqrt{|\,k\,|}(\xi-1)-\lambda_2\sinh\sqrt{|\,k\,|}(\eta-\xi)), & 0\le x\le s\le\xi, \\
-\sqrt{|\,k\,|}\cosh\sqrt{|\,k\,|}s \\
\cdot(\sqrt{|\,k\,|}\cosh\sqrt{|\,k\,|}(x-1)+\lambda_2\sinh\sqrt{|\,k\,|}(x-\eta)), & s\le x, s\le\xi, \\
-(\sqrt{|\,k\,|}\cosh\sqrt{|\,k\,|}x+\lambda_1\sinh\sqrt{|\,k\,|}(x-\xi)) \\
\cdot(\sqrt{|\,k\,|}\cosh\sqrt{|\,k\,|}(s-1) \\
+\lambda_2\sinh\sqrt{|\,k\,|}(s-\eta)), & \xi\le x\le s\le\eta, \\
-(\sqrt{|\,k\,|}\cosh\sqrt{|\,k\,|}s \\
+\lambda_1\sinh\sqrt{|\,k\,|}(s-\xi))(\sqrt{|\,k\,|}\cosh\sqrt{|\,k\,|}(x-1) \\
+\lambda_2\sinh\sqrt{|\,k\,|}(x-\eta)), & s\le x, s\le\eta, \\
-\sqrt{|\,k\,|}\cosh\sqrt{|\,k\,|}(s-1) \\
\cdot(\sqrt{|\,k\,|}\cosh\sqrt{|\,k\,|}x+\lambda_1\sinh\sqrt{|\,k\,|}(x-\xi)), & \eta\le x\le s\le 1, \\
\sqrt{|\,k\,|}\cosh\sqrt{|\,k\,|}(x-1) \\
\cdot(\lambda_1\sinh\sqrt{|\,k\,|}(\xi-s)-\sqrt{|\,k\,|}\cosh\sqrt{|\,k\,|}s) \\
+\lambda_2\sinh\sqrt{|\,k\,|}(s-x) \\
\cdot(\sqrt{|\,k\,|}\cosh\sqrt{|\,k\,|}\eta+\lambda_1\sinh\sqrt{|\,k\,|}(\eta-\xi)), & s\le x, s\le 1,
\end{cases} \tag{3.1}$$

where

$$\begin{aligned} D_{k'} =&|\,k\,|\sinh\sqrt{|\,k\,|}-\lambda_2\sqrt{|\,k\,|}\cosh\sqrt{|\,k\,|}\eta-\lambda_1\lambda_2\sinh\sqrt{|\,k\,|}(\eta-\xi) \\ &+\lambda_1\sqrt{|\,k\,|}\cosh\sqrt{|\,k\,|}(\xi-1). \end{aligned}$$

Lemma 3.1. *$G(x,s)$ defined by (3.1) is the Green's function of the BVPs (2.1)–(2.2).*

The proof of Lemma 3.1 is similar to the proof described for $k<0$ in [45].

Let us assume that

$[A_1']$

$$\begin{aligned} &\sqrt{|\,k\,|}\sinh\sqrt{|\,k\,|}-\lambda_2\cosh\sqrt{|\,k\,|}\eta\ge 0, \\ &\sqrt{|\,k\,|}\sinh\sqrt{|\,k\,|}\xi+(\lambda_1-\sqrt{|\,k\,|})\cosh\sqrt{|\,k\,|}\xi\le 0, \\ &\sqrt{|\,k\,|}-\lambda_1\cosh\sqrt{|\,k\,|}\xi>0. \end{aligned}$$

In Section 3.5, we have shown graphically that above inequalities are satisfied when $k<0$.

Remark 3.2. $\sqrt{|\,k\,|}\sinh\sqrt{|\,k\,|}\xi+(\lambda_1-\sqrt{|\,k\,|})\cosh\sqrt{|\,k\,|}\xi\leq 0$ only if $\lambda_1-\sqrt{|k|}\leq 0$.

Lemma 3.3. *Suppose $[A_1']$ is true. Then $G(x,s)\leq 0$ for all $x\in I$.*

Proof. We first prove that $D_k'>0$. For this we have

$$\begin{aligned}D_{k'} =&|\,k\,|\sinh\sqrt{|\,k\,|}-\lambda_2\sqrt{|\,k\,|}\cosh\sqrt{|\,k\,|}\eta\\ &-\lambda_1\lambda_2\sinh\sqrt{|\,k\,|}(\eta-\xi)+\lambda_1\sqrt{|\,k\,|}\cosh\sqrt{|\,k\,|}(\xi-1)\\ =&(\sqrt{|\,k\,|}\sinh\sqrt{|\,k\,|}-\lambda_2\cosh\sqrt{|\,k\,|}\eta)(\sqrt{|\,k\,|}-\lambda_1\sinh\sqrt{|\,k\,|}\xi)\\ &+\lambda_1\cosh\sqrt{|\,k\,|}\xi(\sqrt{|\,k\,|}\cosh\sqrt{|\,k\,|}-\lambda_2\sinh\sqrt{|\,k\,|}\eta)>0.\end{aligned}$$

To prove that $G(x,s)\leq 0$ for all $x\in I$ we simplify $G(x,s)$ given in Lemma 3.1 for each subintervals of I individually and using $[A_1']$. □

Lemma 3.4. *If $g(x)\in C(I)$ and c is any constant, then the solution $u(x)\in C^2(I)$ of BVPs* (1.7)–(1.8) *is given by*

$$u(x)=\frac{c}{D_k'}\big(\sqrt{|k|}\cosh\sqrt{|k|}x+\lambda_1\sinh\sqrt{|k|}(x-\xi)\big)-\int_0^1 G(x,s)g(s)ds. \tag{3.2}$$

The proof of Lemma 3.4 is similar to the proof described in Lemma 3.4 of [45].

Lemma 3.5. *If $G(x,s)$ is Green's function of BVPs* (1.7)–(1.8)*, then $\frac{\partial G}{\partial x}\leq 0$, $x\neq s$.*

Proof. Since $G(x,s)$ satisfies the equation

$$u''(x)+ku(x)=0,\quad x\in(0,1), \tag{3.3}$$

$$u'(0)=\lambda_1 u(\xi),\quad u'(1)=\lambda_2 u(\eta), \tag{3.4}$$

integrating equation (3.3) from 0 to x we have

$$\int_0^x G''(x,s)dx=\int_0^x -k\;G(x,s)dx$$

which gives

$$\frac{\partial G(x,s)}{\partial x}=\lambda_1 G(\xi,s)-k\int_0^x G(x,s)dx\leq 0,\quad x\neq s,$$

as $k<0$ and $G(x,s)\leq 0$ for all $x\in I$. □

Proposition 3.6 (Maximum principle)**.** *Let $[A_1']$ be satisfied, $c\geq 0$, $g(x)\geq 0$ and $g(x)\in C(I)$, then the solution $u(x)$ of BVPs* (1.7)–(1.8) *is non-negative.*

Proof. Given that $g(x)\geq 0$, $c\geq 0$ and $[A_1']$ is satisfied. Now (3.2) can be written as

$$u(x)=\cosh\sqrt{|k|}\xi\big(|k|-\lambda_1\sinh\sqrt{|k|}\xi\big)+\lambda_1\sinh\sqrt{|k|}x\cosh\sqrt{|k|}x-\int_0^1 G(x,s)g(s)ds.$$

Applying $[A_1']$ and Lemma 3.3 to the above equation we can easily obtain the required result. □

3.2. EXISTENCE OF SOME DIFFERENTIAL INEQUALITIES

Lemma 3.7. *Suppose* $L_1 \in \mathbb{R}^+$, $k < 0$ *is such that* $(L_1 + k) \leq 0$, *and* $L_2 : I \to \mathbb{R}^+$ *is such that* $L_2(0) = 0$. *Then the following assertions hold:*

(a) *if* $(L_1 + k) + \sup(L_2'(x) + L_2(x)\sqrt{| k |}) \leq 0$, *then*

$$F(x) = (L_1 + k) \sinh \sqrt{| k |}x + L_2(x)\sqrt{| k |} \cosh \sqrt{| k |}x \leq 0, \quad \forall x \in I.$$

(b) $(L_1 + k) \cosh \sqrt{| k |}x + L_2(x)\sqrt{| k |} \sinh \sqrt{| k |}x \leq 0, \quad \forall x \in I.$

Proof. (a) Since $F(0) = 0$, and $F'(x) \leq 0$ whenever

$$(L_1 + k) + \sup(L_2'(x) + L_2(x)\sqrt{| k |}) \leq 0, \quad \forall x \in I$$

and therefore $F(x) \leq 0$ for all $x \in I$. This completes the proof.

(b) Clearly,

$$(L_1 + k) \cosh \sqrt{| k |}x + L_2(x)\sqrt{| k |} \sinh \sqrt{| k |}x \leq F(x).$$

The result is obvious. □

Remark 3.8. From Lemma 3.7 (a) the inequality

$$(L_1 + k) + \sup(L_2'(x) + L_2(x)\sqrt{| k |}) \leq 0,$$

gives an upper bound for k with

$$k \leq -\sup\left(L_1 + L_2'(x) + \frac{L_2(x)^2}{2} + \frac{L_2(x)}{2}\sqrt{L_2^2(x) + 4(L_1 + L_2'(x))}\right), \quad \forall x \in I.$$

Lemma 3.9. *Suppose* $[A_1']$ *and conditions of Lemma 3.7 are satisfied. Then for all* $x \in I$ *the following inequalities hold:*

(a)

$$\begin{aligned}&(L_1 + k)(\sqrt{| k |} \cosh \sqrt{| k |}x + \lambda_1 \sinh \sqrt{| k |}(x - \xi))\\ &\pm L_2(x)\sqrt{| k |} \cdot (\sqrt{k} \sinh \sqrt{| k |}x - \lambda_1 \cosh \sqrt{| k |}(x - \xi)) \leq 0,\end{aligned}$$

(b)

$$(L_1 + k)G(x, s) \pm L_2(x)\frac{\partial G(x, s)}{\partial x} \geq 0, \quad x \neq s,$$

suppose to be the case for $(L_1 + k) + \sup L_2(x)(\lambda_1 - k) \leq 0$.

Proof. (a) Using Lemma 3.7 and $[A_1']$, it is easy to prove the result.

(b) To prove $(L_1 + k)G(x, s) + L_2(x)\frac{\partial G(x,s)}{\partial x} \geq 0$, we proceed similar to Lemma 2.6 given in Section 2.2. □

Remark 3.10. From Lemma 3.9 (b) it can be observed that if $(1 - \sup\ L_2(x)) > 0$, then $k \leq \frac{L_1 + \lambda_1 \sup\ L_2(x)}{1 - \sup L_2(x)}$.

From Remarks 2.18, 3.2, and 3.10 we conclude that

$$[A_2'] \ \ k \leq \min \bigg\{ -L_1, -\lambda_1^2, \frac{L_1 + \lambda_1 \sup \ L_2(x)}{1 - \sup \ L_2(x)}, \\ -\sup \bigg(L_1 + L_2'(x) + \frac{L_2(x)^2}{2} + \frac{L_2(x)}{2}\sqrt{L_2^2(x) + 4(L_1 + L_2'(x))} \bigg) \bigg\}.$$

3.3. WELL ORDERED CASE: CONSTRUCTION OF UPPER-LOWER SOLUTIONS

In this section, we provide some conditions based on upper-lower solutions and nonlinear term $\psi(x, u, u')$. We develop MI-technique based on functions $\{c_n(x)\}_n$ and $\{d_n(x)\}_n$. We discuss some lemmas and propositions to shows that upper solutions are monotonically decreasing and lower solutions are monotonically increasing. We develop a theorem which shows that these sequences uniformly converge to the solution of NLBVPs (1.5)–(1.6) under some sufficient conditions.

Assume the following properties:

- $[A_3']$ there exist upper-lower solutions $c(x), d(x) \in C^2(I)$ of NLBVP (1.5)–(1.6) such that $c(x) \leq d(x)$ for all $x \in I$,
- $[A_4']$ $\psi(x, v, w) : E \to \mathbb{R}$ is a continuous function on

$$E := \{(x, v, w) \in I \times \mathbb{R}^2 : c(x) \leq v \leq d(x)\},$$

- $[A_5']$ for all $(x, v_1, w), (x, v_2, w) \in E$ there exists a constant $L_1 \geq 0$ such that

$$v_1 \leq v_2 \Rightarrow \psi(x, v_2, w) - \psi(x, v_1, w) \geq -L_1(v_2 - v_1),$$

- $[A_6']$ for all $(x, v, w_1), (x, v, w_2) \in E$ there exists a function $L_2(x) \geq 0$ such that

$$| \ \psi(x, v, w_1) - \psi(x, v, w_2) \ | \leq L_2(x) \ | \ (w_1 - w_2) \ | \ .$$

Lemma 3.11. *If $c_n(x)$ is a lower solution of* (1.5)–(1.6)*, then $c_n(x) \leq c_{n+1}(x)$ for all $x \in I$, where $c_{n+1}(x)$ is given by equation* (2.6)–(2.7)*.*

The proof of Lemma 3.11 is similar to Lemma 2.9.

Proposition 3.12. *Assume $[A_1']$–$[A_6']$ are true. Let $L_1 \in \mathbb{R}^+$, $L_2 : I \to \mathbb{R}^+$ be such that $(L_1 + k) \leq 0$, $L_2(0) = 0$, and conditions of Lemma 3.7 hold. Then for all $x \in I$, if $c_n(x)$ is a lower solution of* (1.5)–(1.6)*, then*

$$(L_1 + k)u(x) + L_2(x)(\operatorname{sign} u'(x))u'(x) \leq 0,$$

where u is any solution of NLBVPs (1.7)–(1.8)*.*

The proof of Proposition 3.12 is similar to Proposition 2.10 in Section 3.

Lemma 3.13. *Assume* $[A_1']$–$[A_6']$ *are true. Let* $L_1 \in \mathbb{R}^+$, $L_2 : I \to \mathbb{R}^+$ *be such that* $(L_1 + k) \leq 0$, $L_2(0) = 0$, *and conditions of Lemma 3.7 hold. Then the functions* $c_n(x)$ *given by equation* (2.6)–(2.7) *satisfy:*

(a) $c_n(x) \leq c_{n+1}(x)$,
(b) $c_n(x)$ *is a lower solution of* (1.5)–(1.6).

The proof of Lemma 3.13 is similar to Lemma 2.11 of Section 2.

Lemma 3.14. *If* $d_n(x)$ *is an upper solution of* (1.5)–(1.6), *then* $d_n(x) \geq d_{n+1}(x)$ *for all* $x \in I$, *where* $d_{n+1}(x)$ *is given by equation* (2.8)–(2.9).

Lemma 3.15. *Assume* $[A_1']$–$[A_6']$ *are true. Let* $L_1 \in \mathbb{R}^+$, $L_2 : I \to \mathbb{R}^+$ *be such that* $(L_1 + k) \leq 0$, $L_2(0) = 0$, *and conditions of Lemma 3.7 hold. Then the functions* $d_n(x)$ *given by equation* (2.8)–(2.9) *satisfy:*

(a) $d_n(x) \geq d_{n+1}(x)$,
(b) $d_n(x)$ *is an upper solution of* (1.5)–(1.6).

The proof of 3.15 is similar to Lemma 2.12 of Section 2.

Proposition 3.16. *Assume* $[A_1']$–$[A_6']$ *are true. Let* $L_1 \in \mathbb{R}^+$, $L_2 : I \to \mathbb{R}^+$ *be such that* $(L_1 + k) \leq 0$, $L_2(0) = 0$, *and conditions of Lemma 3.7 hold, and*

$$\psi(x, d(x), d'(x)) - \psi(x, c(x), c'(x)) - k(d(x) - c(x)) \geq 0, \quad \forall x \in I.$$

Then $c_n(x) \leq d_n(x)$, *where* c_n *and* d_n *are given by equation* (2.6)–(2.7) *and* (2.8)–(2.9), *respectively.*

The proof is the same as Proposition 2.14 of Section 2.

3.4. BOUND ON DERIVATIVE OF SOLUTION

$[A_7']$ Let $|\psi(x, v, w)| \leq \phi(|w|)$ for all $(x, v, w) \in E$, where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and satisfies

$$\int_\gamma^\infty \frac{s ds}{\phi(s)} \geq \max_{x\in I} d(x) - \min_{x\in I} c(x),$$

such that

$$\gamma = 2 \max \Big\{ \sup_{x\in[0,1]} |c(x)|, \sup_{x\in[0,1]} |d(x)| \Big\}.$$

Lemma 3.17. *Let* $[A_7']$ *be true. Then there exists* $P > 0$ *such that* $\| u' \|_\infty \leq P$ *for all* $x \in I$, *where* u *is any solution of the inequality*

$$-u''(x) \geq \psi(x, u, u'), \quad x \in (0, 1),$$
$$u'(0) = \lambda_1 u(\xi), \quad u'(1) \geq \lambda_2 u(\eta),$$

such that $c(x) \leq u(x) \leq d(x)$ *for all* $x \in I$.

Lemma 3.18. *Let $[A_7']$ be true. Then there exists $P > 0$ such that $\| u' \|_\infty \leq P$ for all $x \in I$, where u is any solution of the inequality*

$$-u''(x) \leq \psi(x, u, u'), \quad x \in (0,1),$$
$$u'(0) = \lambda_1 u(\xi), \quad u'(1) \leq \lambda_2 u(\eta),$$

such that $c(x) \leq u(x) \leq d(x)$ for all $x \in I$.

The proof of above two lemmas are similar to the proof described in Lemma 2.15 of Section 2.4.

Theorem 3.19. *Assume $[A_1']$–$[A_6']$ are true. Let $L_1 \in \mathbb{R}^+$, $L_2 : I \to \mathbb{R}^+$ be such that $(L_1 + k) \leq 0$, $L_2(0) = 0$, and conditions of Lemma 3.7 hold, and*

$$\psi(x, d(x), d'(x) - \psi(x, c(x), c'(x)) - k(d - c) \geq 0, \quad \forall x \in I.$$

Then $(c_n)_n \to y$ and $(d_n)_n \to z$ uniformly in $C^2(I)$, and $c \leq z \leq y \leq d$, where y and z are solutions of (1.7)–(1.8).

3.5. NUMERICAL ILLUSTRATION

In this section, we show numerically and graphically that for well-ordered case sequences of upper-lower solutions are uniformly convergent and converge to the solution. We also give some range for $k < 0$ which will validate our results.

3.5.1. Example

Consider four-point BVPs

$$-u''(x) = \frac{e^x - 1}{40}\left(u'^2 - u - \frac{\cos x}{4}\right),$$
$$u'(0) = \frac{1}{4}u(0.2), \quad u'(1) = \frac{1}{9}u(0.3),$$

where $\psi(x, u, u') = \frac{e^x-1}{40}\left(u'^2 - u - \frac{\cos x}{4}\right)$, $\xi = 0.2$, $\eta = 0.3$, $\lambda_1 = \frac{1}{4}$ and $\lambda_2 = \frac{1}{9}$. We consider initial upper-lower solutions $c(x) = -1.905 - \frac{x}{2} + \frac{x^2}{8}$, $d(x) = 1.9 + \frac{x}{2}$, respectively, where $d(x) \geq c(x)$. Since $\psi(x, u, u')$ is one-sided Lipschitz in u with a Lipschitz constant $L_1 = 0.042\,957$. Also f is Lipschitz in u', therefore we derive from $[A_5']$, $L_2(x) = \frac{2P(e^x-1)}{40}$, where $P > 0$ is such that $\| u' \|_\infty \leq P$ for all $x \in I$. Here we obtained $\phi(|s|) = 0.042\,957(|s^2| + 2.65)$. By using $[A_6']$ we obtain $P = 5.868\,826$. From $[A_2']$ we have $k \leq -1.447\,171$. The range of k is $(-\alpha_0, -\beta_0)$, where $\beta_0 = 1.447\,171$, and $\alpha_0 = -125$ which is computed by using $[A_1']$ and Figure 10. We observe that α_0 is not sharp. The graph of convergent sequences and all the conditions are given in Figures 9–14.

Remark 3.20. We observe that for $k = -1.0698$ the sequences of upper-lower solutions are monotone but we have obtained that the range for k is $(-\alpha_0, -\beta_0)$, where $\beta_0 = 1.447\,171$.

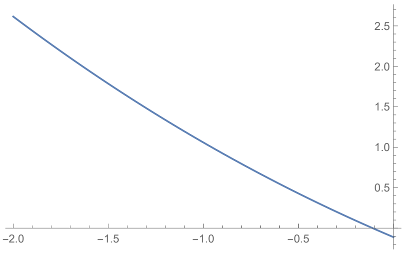

Fig. 9. $\sqrt{|\,k\,|}\sinh\sqrt{|\,k\,|} - \lambda_2\cosh\sqrt{|\,k\,|}\eta$

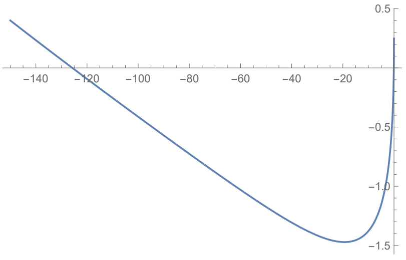

Fig. 10. $\sqrt{|\,k\,|}\sinh\sqrt{|\,k\,|}\xi$
$+(\lambda_1 - \sqrt{|\,k\,|})\cosh\sqrt{|\,k\,|}\xi$

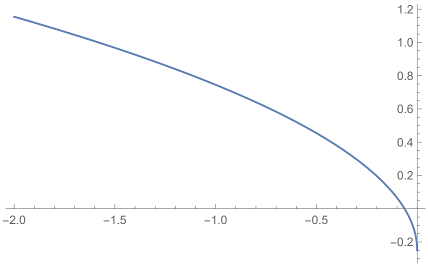

Fig. 11. $\sqrt{|\,k\,|} - \lambda_1\cosh\sqrt{|\,k\,|}\xi$

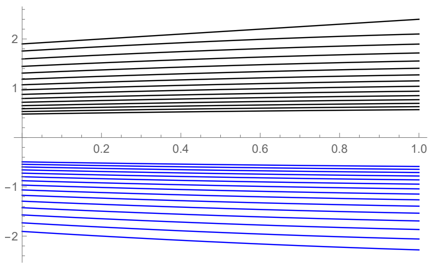

Fig. 12. $k = -1.4472$

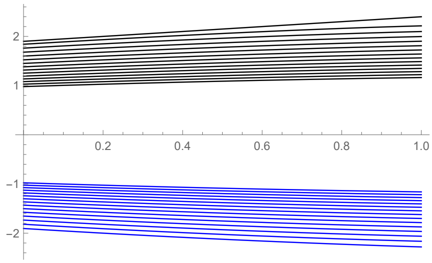

Fig. 13. $k = -3$

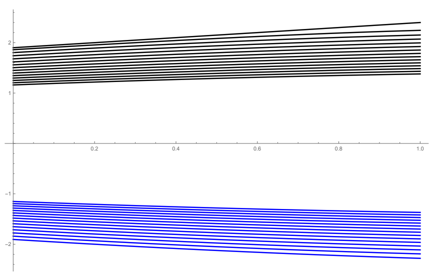

Fig. 14. $k = -4$

4. CONCLUSION

In this paper, we have considered four-point NLBVPs and developed a technique which is called MI-technique with upper-lower solutions. We have dealt with both cases reverse and well-ordered cases. We have defined upper-lower solutions, the maximum, anti-maximum principle, and obtained the sign of $G(x, s)$. We observed that in order to prove propositions 2.14 and 3.16, $\psi(x, u, u')$ should be onesided Lipschitz in u and Lipschitz in u'. We have shown that if we are able to construct monotone sequences of upper-lower solutions with the help of initial guesses, then these sequences of upper-lower solutions will give a guarantee to the existence of the solution. From the numerical point of view this technique is easy to handle. We have illustrated one example each for $k > 0$ and $k < 0$ and have shown that the sequences converge uniformly to the solution. For this, Mathematica-11.3 is used. We have considered k as a constant and not equal to zero. We have also obtained some range of k for both the cases in which MI-technique satisfies all the conditions of our problem which we have considered.

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