

Response of linear system to α -stable Levy input

Janusz Walczak, Seweryn Mazurkiewicz
Silesian University of Technology
44-100 Gliwice, ul. Akademicka 10, email: janusz.walczak@polsl.pl,
seweryn.mazurkiewicz@polsl.pl

This article suggests the method of analysis stochastic processes in deterministic linear SISO system of first order. Using the definition of α -stable random variable, the definition of α -stable Levy process has been introduced and presented their properties. Transformation of α -stable white noise process in the system has been investigated as well. Obtained results has been illustrated by an example.

1. Introduction

Stochastic process is a function which assigns to every moments of time, a random variable. One of the method to define stochastic process is present recursive formula uses summing increments [7].

If random variables $X_t^{\Delta t}$ has normal distribution of mean value equal to zero, variance equal to Δt ($X_t^{\Delta t} \sim N(0, \Delta t)$) and $Y(0) = 0$, then that process is called Wiener process. The example of Wiener process is Brownian motion. Brownian motion is a trajectory of particle motion in liquid.

On the Fig. 1 shown the sample trajectory of Wiener process.

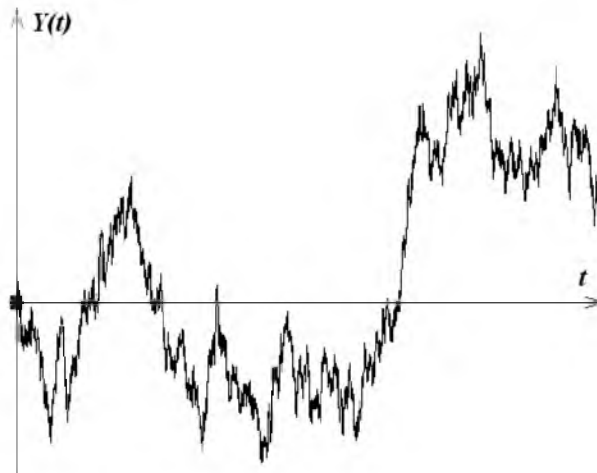


Fig. 1. Wiener process realization ($\Delta t = 0.0005$; $t_{max} = 2$)

Wiener process is continuous and not differentiable at every point [8]. Their realizations have a shape of “dense” sawtooth functions [7].

Monothematic conferences [1] as well as numerous articles in technical journals [2], [3], [4], [5], [7], [9], [10], [11], [12], [13], [14] are devoted to analysis of properties of Levy processes. This article is a continuation of the work [12] which investigates generation of α -stable Levy processes. Levy processes find applications in research of probabilistic models of physical systems [6] especially in information theory [5], reliable theory [14], signal processing [4], [9], [11], [12], automatic [13] and telecommunication [3], [10].

In further part of the article a definition of α -stable random variable has been introduced, needed to construct Levy processes, as well as the problem of transmission of these processes in deterministic SISO systems of first order has been presented. It has also shown the formulas which determine processes realizations considered in the system and determine quantile functions of these processes.

2. α -stable random variables

Random variable X has α -stable distribution, when for any constants a, b, c the relation is satisfied:

$$aY_1 + bY_2 + c =^D X, \quad (1)$$

where: Y_1, Y_2 – any independent random variables with equal distributions,

$=^D$ – means the equality of distributions.

α -stable random variable is described by four parameters [2] [3]:

- α – stability index,
- β – skewness parameter,
- σ – scale parameter,
- μ – placement parameter.

α -stable random variable distribution is signed as $S_\alpha(\sigma, \beta, \mu)$. Random variable X is a standard α -stable random variable when $X \sim S_\alpha(1, \beta, 0)$ and which is signed as $X \sim S_{\alpha, \beta}$ [3]. When $\beta = \pm 1$ and $\alpha \in (0, 1)$ then random variable X is called total skew [3]. When $\beta = 1$ and $\alpha \in (0, 1)$ then random variable X has only nonnegative realizations, whereas for $\beta = -1$ and $\alpha \in (0, 1)$ random variable X has only no positive realizations [3].

In generally, for α -stable random variable the probability density function in analytical form cannot be determined [4]. However it can be determined the characteristic function of α -stable random variable [6].

If random variable X has α -stable distribution [3]:

- $S_2(\sigma, 0, \mu)$ then random variable X has normal distribution $N(\mu, 2\sigma^2)$,

- $S_1(\sigma, 0, \mu)$ then random variable X has Cauchy distribution $Cauchy(\mu, \sigma)$,
- $S_{1/2}(\sigma, 1, \mu)$ then random variable X has Levy distribution $Levy(\mu, \sigma)$.

3. α -stable Levy motion process

Let $\{L_{\alpha,\beta}(t), t \geq 0\}$ be α -stable Levy motion process. The process starts at zero and has increments $L_{\alpha,\beta}(t + \Delta t) - L_{\alpha,\beta}(t)$, $\Delta t \rightarrow 0$ following the α -stable distribution $S_\alpha(\Delta t^{1/\alpha}, \beta, 0)$.

α -stable Levy process is defined:

$$L_{\alpha,\beta}(0) = 0, \tag{2}$$

$$L_{\alpha,\beta}(t + \Delta t) = L_{\alpha,\beta}(t) + L_t^{\Delta t}, \tag{3}$$

$$\Delta t \rightarrow 0, \tag{4}$$

where: $L_t^{\Delta t}$ - random variable with α -stable Levy distribution ($\mu = 0$, σ depend on Δt).

α -stable Levy process is continuous at every point $t \in \langle 0, \infty \rangle$ except countable number of discontinuous (jumps) of the first order. What is more, at every point this process is not differentiable.

Realizations of $\{L_{\alpha,\beta}(t), t \geq 0\}$ are shown in Fig. 2-3 for $\Delta t = 0.005$ and several values of α and β .

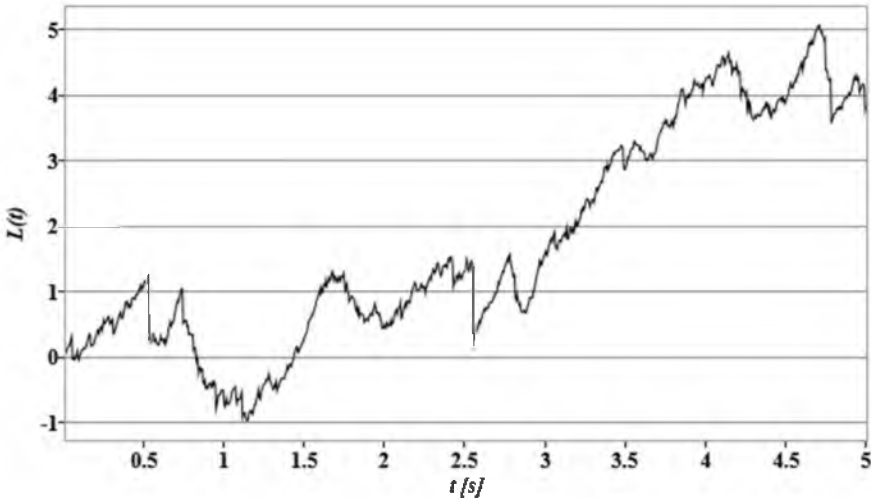


Fig. 2. Realization of Levy process for $\alpha = 1.5$ and $\beta = -1$

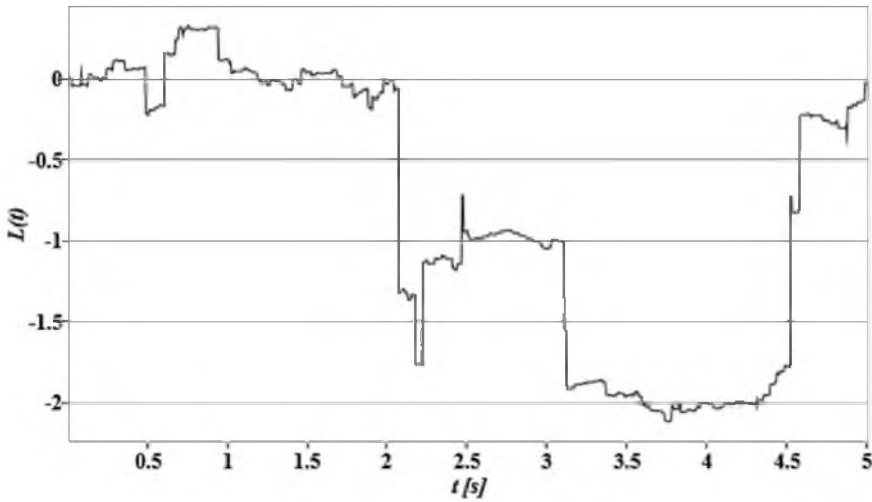


Fig. 3. Realization of Levy process for $\alpha = 0.8$ and $\beta = 0$

It is common in random vibration to interpret the Gaussian white noise as the formal derivative of Brownian motion process $\{B(t), t \geq 0\}$ [6]. Because $\{L_{\alpha,\beta}(t), t \geq 0\}$ has also independent increments, one can, by analogy, define a formal derivative:

$$W_{\alpha,\beta}(t) = \frac{dL_{\alpha,\beta}(t)}{dt} \quad (5)$$

of the α -stable Levy motion process. Process $W_{\alpha,\beta}(t)$ is called α -stable white noise [6].

Realizations of $\{W_{\alpha,\beta}(t), t \geq 0\}$ are shown in Fig. 6 and 8 for $\Delta t = 0.005$ and two values of α and β .

4. Formalization and solution of the problem

Considered system, that is shown on the Fig. 4, is described by stochastic differential equation, with deterministic initial condition. Let $\{Y(t), t \geq 0\}$ be the state of dynamic system and $X(t)$ be α -stable white noise.

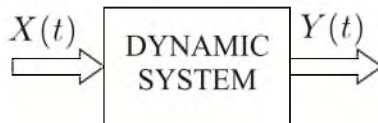


Fig. 4. SISO Random System

The state of the dynamic system satisfies the stochastic differential equation:

$$dY(t) = -\lambda Y(t)dt + \lambda dL_{\alpha,\beta}(t), \quad \lambda \in R \quad (6)$$

$$Y(0) = y_0 \quad (7)$$

It has the solution [6]:

$$Y(t) = y_0 e^{-\lambda t} + \lambda \int_0^t e^{-\lambda(t-s)} dL_{\alpha,\beta}(s) \quad (8)$$

Realizations of the $Y(t)$ can be generated from finite difference approximation:

$$Y(t + \Delta t) \cong (1 - \lambda H \Delta t) Y(t) + \Delta L_{\alpha,\beta}(t) \quad (9)$$

where:

$$\Delta L_{\alpha,\beta}(t) = L_{\alpha,\beta}(t + \Delta t) - L_{\alpha,\beta}(t) \quad (10)$$

It should be concluded, that recently the method of generation realizations (9) seems to be one right method applied to the computer simulations. The rate of convergence is not however precisely investigated [7].

5. Example

Considered system is shown on the Fig. 5.

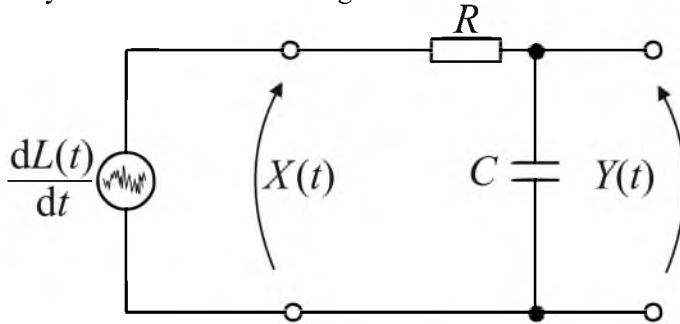


Fig. 5. RC Filter Circuit

This system is supplied by the voltage source that has been α -stable white noise process. The parameters $R = 1 \Omega$ and $C = 0.2 F$ are deterministic.

The state of the RC filter satisfies the stochastic differential equation:

$$dY(t) = -\frac{1}{RC} Y(t)dt + \frac{1}{RC} dL_{\alpha,\beta}(t) \quad (11)$$

$$Y(0) = y_0 \quad (12)$$

Realizations of $X(t)$ and $Y(t)$ are shown in Fig. 6-9 for $\Delta t = 0.005$ and several values of α and β .

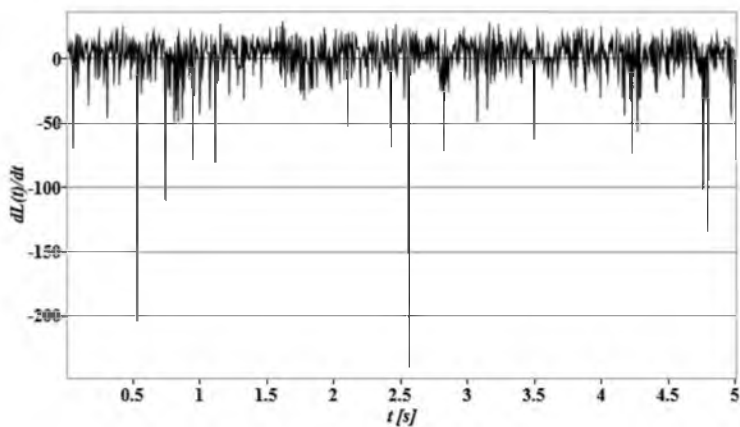


Fig. 6. Realization of $X(t)$ α -stable white noise process for $\alpha = 1.5$ and $\beta = -1$

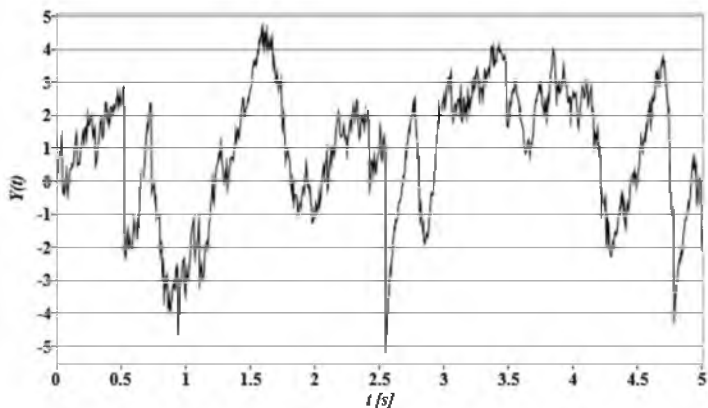


Fig. 7. Realization of $Y(t)$ process for $\alpha = 1.5$ and $\beta = -1$

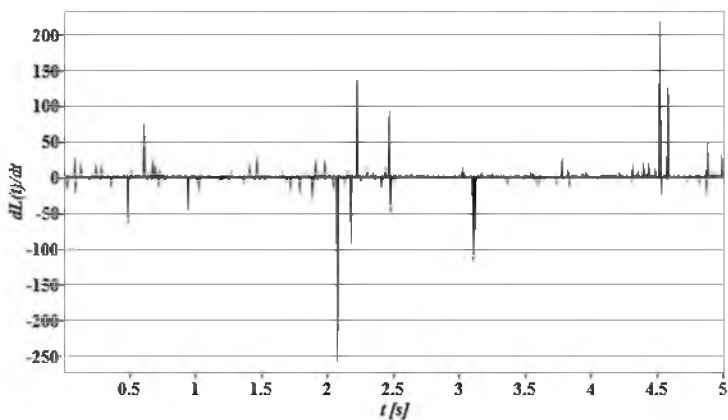


Fig. 8. Realization of $X(t)$ α -stable white noise process for $\alpha = 0.8$ and $\beta = 0$

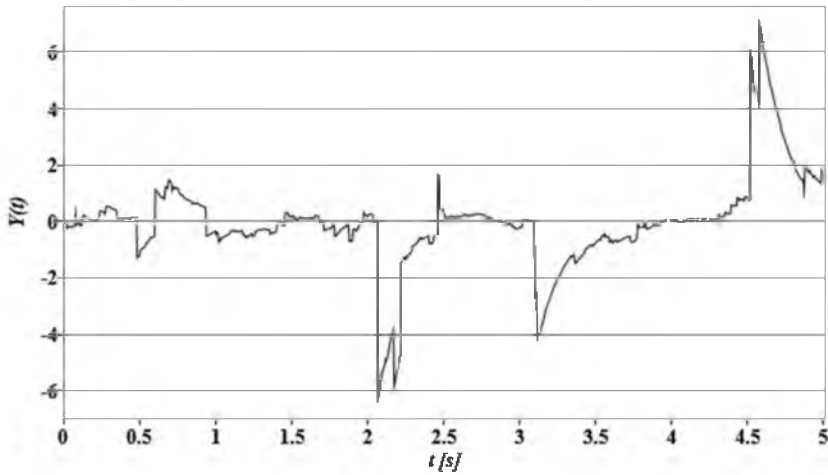


Fig. 9. Realization of $Y(t)$ process for $\alpha = 0.8$ and $\beta = 0$

6. Quantile function

The moment of order l of the α -stable Levy process is bounded for $l < \alpha$. For $l \geq \alpha$ the moment of order l tends to infinity. Hence the moment functions are not appropriate tools to describes the α -stable Levy processes. The convenient tool to describes the α -stable Levy processes is a quantile function [7]. The quantile function is a kind of the distribution.

For constant $p \in (0, 1)$ and for the stochastic process $X(t) = \{X(t) : t \in [0, T]\}$, p -quantile function is a deterministic function $q_p(t)$ defined on the interval $t \in [0, T]$ following relation [7]:

$$P\{X(t) \leq q_p(t)\} = p \quad (13)$$

The analytical determination of the quantile function for α -stable Levy process is probably impossible because the probability density function is not given in the analytical form.

In this article, an estimation of the quantile function has been used by applying the Monte Carlo method.

On the Fig. 10 - 15 are shown the estimations of quantile functions of α -stable Levy process, α -stable white noise process and output of RC filter process (see an example). Assuming that the parameter p is equal to 0.1, 0.2... 0.9. To estimate quantile functions 20000 realizations of processes have been used.

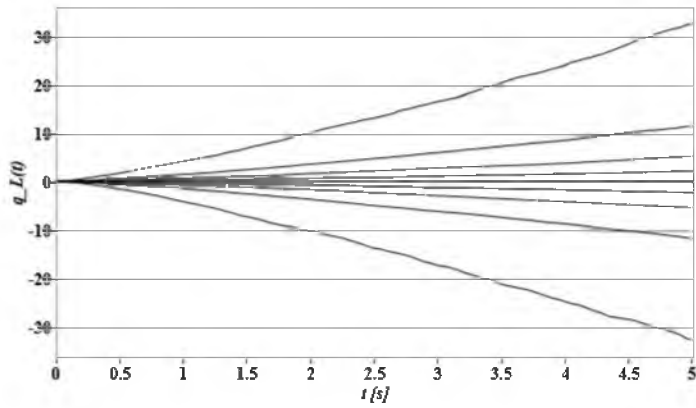


Fig. 10. Estimation of $L_{\alpha,\beta}(t)$ quantile functions for $\alpha = 0.8$ and $\beta = 0$

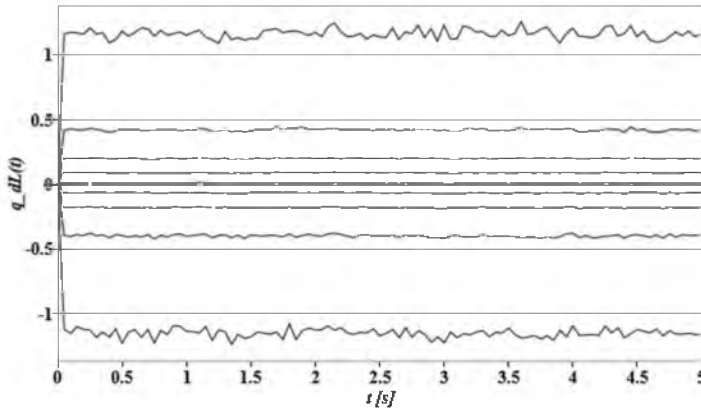


Fig. 11. Estimation of $dL_{\alpha,\beta}(t)/dt$ quantile functions for $\alpha = 0.8$ and $\beta = 0$

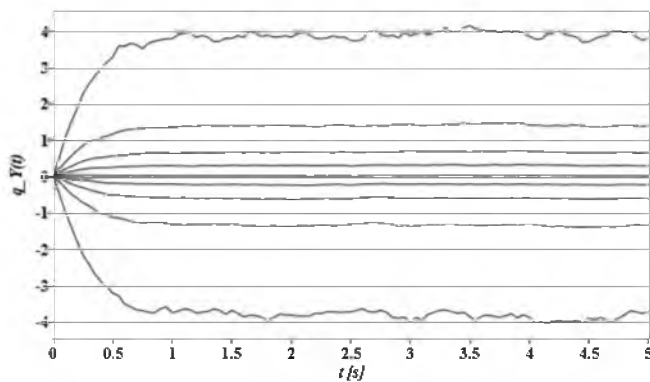


Fig. 12. Estimation of $YY(t)$ quantile functions for $\alpha = 0.8$ and $\beta = 0$

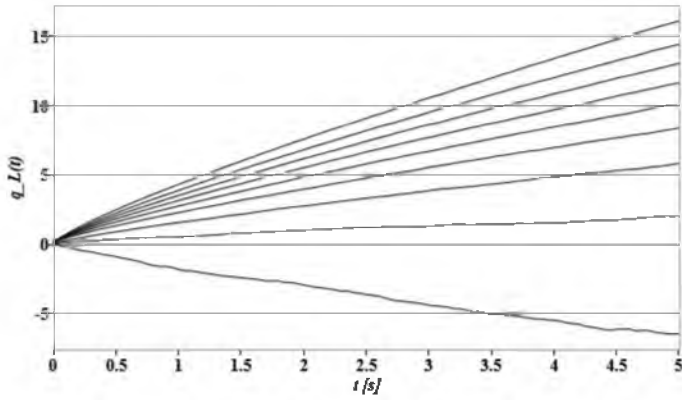


Fig. 13. Estimation of $L_{\alpha, \beta}(t)$ quantile functions for $\alpha = 1.2$ and $\beta = -1$

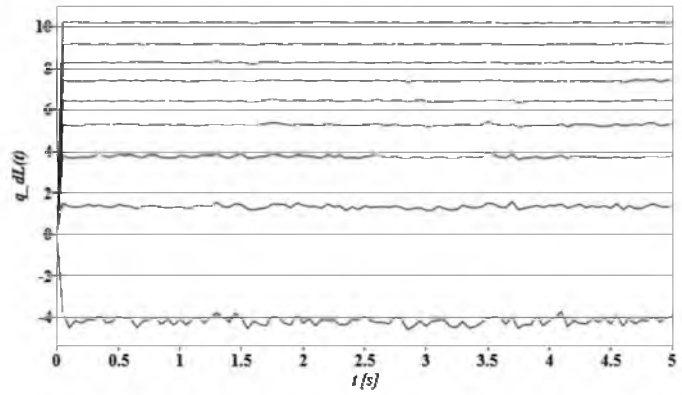


Fig. 14. Estimation of $L_{\alpha, \beta}(t)$ quantile functions for $\alpha = 1.2$ and $\beta = -1$

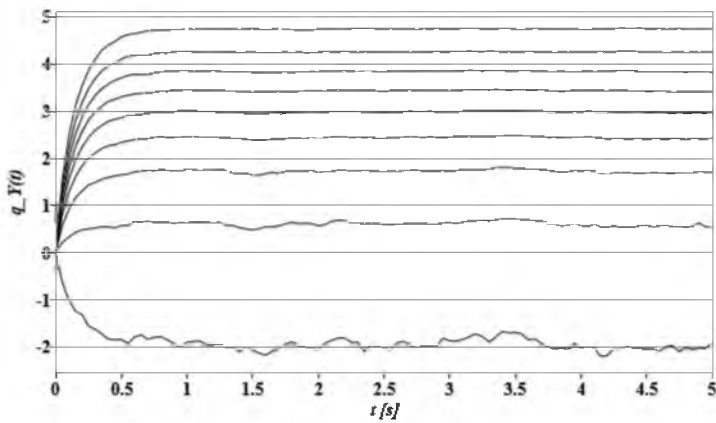


Fig. 15. Estimation of $Y(t)$ quantile functions for $\alpha = 1.2$ and $\beta = -1$

The probability that realization of any process will be between two quantile functions is equal to 0.1 because the parameter p (see eq. (13)) changes with steep 0.1. From the simulation results that quantile functions undergo the same transformation by the system, as the signal transformation (see Fig. 4 and 16).

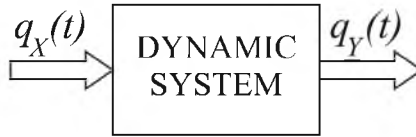


Fig. 16. Transformation of quantile function

Quantile function of the α -stable white noise asymptotically tends to Heaviside function (see Fig. 11 and Fig. 14). Whereas the response of RC system tends to functions shown on Fig. 12 and 15. This is due to the fact, that distribution also undergo the same transformation (and the quantile function is a kind of the distribution).

7. Summary

The analysis of dynamical system, by excitation being α -stable white noise, is impossible by applying methods of moments (for $l \geq \alpha$) because moments of this process in generally not exists. It is possible to analyze the system by applying realizations method (see chapter 5). However single realizations do not give precise information about the character of the system. The analysis by using a new tool as it is the quantile function [7], let us obtain characteristic of the system, having well physic interpretation. Depend on choosing the parameter p , one can obtain intervals (p -tube) in which probability, that the realization of process is contained in them, is advance assumed.

Obtaining effective results can be achieved by estimating quantile functions using relatively simple algorithms (see chapter 6). Furthermore, quantile function is undergoes the same transformation as the realizations of the system input.

References

- [1] 5-th International Conference on Levy Processes. Theory and Applications. August 13 – 17, Copenhagen, 2007.
- [2] Chambers J. M., Mallows C. L., Stuck B.W.: A Method for Simulating Stable Random Variables, Journal of American Statistical Association, June 1976, Vol. 71, No. 354, pp. 340-344.
- [3] Corvaja R., Pupolin S. :Phase Noise Effects in QAM Systems, 8-th Int. Symp. an Personal, Indoor and Mobile Radio Communication, Sept. 1 – 4, Helsinki, Finland.

- [4] Dimakis A. G., Maragos P.: Modelling Resonances with Phase-Modulated Self-Similar Processes, IEEE Int. Conference on Acoustics, Speech and Signal Processing, 2004, pp. 877 – 880.
- [5] Duncan T. E.: Mutual Information for Stochastic Signals and Levy Processes. IEEE Trans. on Information Theory, Vol. 56, No 1, 2010, pp. 18 – 24.
- [6] Grigoriu M.: Applied Non-Gaussian Processes, Prentice Hall, Inc., New Jersey, 1995.
- [7] Janicki A., Izodarczyk A.: Komputerowe metody w modelowaniu stochastycznym, WNT 2001.
- [8] Janicki A., Weron A.: Simulation and Chaotic Behavior of α -Stable Stochastic Processes, MARCEL DEKKER 1994.
- [9] Kogon S. M., Manolakis D. G.: Signal Modelling with Self-Similar α -stable processes. IEEE Trans. on Signal Processing Vol. 44, No 4, 1996, pp. 1006-1010.
- [10] Pacheco R. A., Hatzinakos D.: BER analysis of self-heterodyne OFDM transmission scheme. Canadian Conference on Electrical and Computer Engineering, May 2 – 5, Ontario, Canada.
- [11] Patel A., Kosko B.: Levy Noise Benefits in Neural Signal Detection, IEEE Int. Conference on Acoustics, Speech and Signal Processing ICASSP'07, April 15 – 20, Honolulu, Hawai'i, USA, 2007, pp. III 1413 – III 1416.
- [12] Walczak J., Mazurkiewicz S.: Programowy generator procesów stochastycznych α -stabilnych Levy'ego. Journal of Electrical Engineering 69, Poznań 2012, pp. 169-174.
- [13] Yin X. Lin Z.: Controlability Driven by Levy Processes In Hilbert Space. Int. Conference on Intelligent Computation Technology and Automation ICICTA'2010, May 11 – 15, Changsha, China.
- [14] Yoonjung Y., Klutke G.: Lifetime-characteristics and inspection-schemes for Levy degradation processes. IEEE Trans. on Reliability, Vol. 49, No. 4, 2000, pp. 377 – 382.