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THE FOURTH-ORDER ORDINARY DIFFERENTIAL EQUATION WITH THE FRACTIONAL INITIAL/BOUNDARY CONDITIONS

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Abstract. The initial/boundary value problem for the fourth-order homogeneous ordinary differential equation with constant coefficients is considered. In this paper, the particular solutions an ordinary differential equation with respect to the set of boundary conditions are studied. At least one of the boundary conditions is described by a fractional derivative. Finally, a few illustrative examples of particular solutions to the considered problem are shown.

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1. Introduction

The fourth-order ordinary differential equations are an important mathematical tool for modelling various phenomena occurring in dynamical systems. Examples of equations occur in almost all of the engineering sciences and appear in many fields of physics [1].

One of the types of these equations are the homogeneous linear equations that are usually used for modelling bending or deflection of elastic beams in, an equilibrium state, and two ends of a beam satisfy the specified boundary conditions [2, 3]. The set of boundary conditions allows for the establishment of the existence and uniqueness of solutions of a large class of boundary value problems. Usually, the initial/boundary conditions are described by functions and their derivatives of the integer order (the Dirichlet or natural boundary conditions). The problem of finding the exact analytical solution of these equations is studied in many works. The various approaches are used to find the particular solutions of differential equations, i.e. separation of variables, undetermined coefficients, variation of parameters or Green's function method [1-5].

In this work, the particular solutions to fourth-order homogeneous linear ordinary differential equation with linearly independent generalized boundary conditions are studied. Here, all or at least one of the boundary conditions are replaced by a fractional boundary condition, meaning that the function and/or integer order derivative in boundary conditions are replaced by fractional derivative [6-8]. In recent years, the various mathematical problems containing fractional derivatives have become an important topic. Such solutions of the considered generalized boundary value problems can be an important mathematical tool for modelling the phenomena occurring in dynamical systems or materials science. The above equation is an important mathematical tool for modelling the phenomena occurring in dynamical systems or materials science, i.e. for the deflection of a beam.

2. Governing equations

I analysed the initial/boundary value problem for the fourth-order homogeneous differential equation with a constant coefficient in the form

$$D^4 y(x) - k^4 y(x) = 0, \quad x \in [a,b], \quad k > 0 \tag{1}$$

The general solution of fourth-order Eq. (1) is in the form (i.e. [4])

$$y(x) = C_1 \cos(kx) + C_2 \sin(kx) + C_3 \cosh(kx) + C_4 \sinh(kx) \tag{2}$$

and containing four arbitrary independent constants of integration.

In this paper, Eq. (1) with respect to the following set of 4 linearly independent fractional boundary conditions is considered

$$\Phi_i \left(D^{\alpha}_{b^-} y(x)\Big|_{x=a}, D^{\beta}_{a^+} y(x)\Big|_{x=b} \right) = 0, \quad i = 1,\dots,4 \tag{3}$$

where $\alpha, \beta \in [0,4]$ and the operators $D^{\alpha}_{b^-}$, $D^{\beta}_{a^+}$ denote the left- and right-side Riemann-Liouville derivatives [6,7], defined by

$$D^{\alpha}_{b^-} y(x) := \begin{cases} \dfrac{(-1)^n}{\Gamma(n-\alpha)} \dfrac{d^n}{dx^n} \displaystyle\int_x^b \dfrac{y(\tau)}{(\tau - x)^{\alpha - n + 1}} d\tau, & \text{for } \alpha \notin \mathbb{N}_0, n = [\alpha] + 1 \\ (-1)^n D^n y(x), & \text{for } \alpha = n \in \mathbb{N}_0 \end{cases} \tag{4}$$

$$D^{\beta}_{a^+} y(x) := \begin{cases} \dfrac{1}{\Gamma(n-\beta)} \dfrac{d^n}{dx^n} \displaystyle\int_a^x \dfrac{y(\tau)}{(x - \tau)^{\beta - n + 1}} d\tau, & \text{for } \beta \notin \mathbb{N}_0, n = [\alpha] + 1 \\ D^n y(x), & \text{for } \beta = n \in \mathbb{N}_0 \end{cases} \tag{5}$$

and $D^n y(x) \equiv d^n y(x)/dx^n$.

The values of left- and right-side fractional derivatives of solution $y(x)$ (see Eq. (2)) determined on the boundaries of interval $[a, b]$ are equal to

$$\begin{aligned} D^{\beta}_{a^+} y(x)\Big|_{x=b} &= D^{\beta}_{a^+} \left\{C_1 \cos(kx) + C_2 \sin(kx) + C_3 \cosh(kx) + C_4 \sinh(kx)\right\}\Big|_{x=b} \\ &= C_1 D^{\beta}_{a^+} \cos(kx)\Big|_{x=b} + C_2 D^{\beta}_{a^+} \sin(kx)\Big|_{x=b} \\ &\quad + C_3 D^{\beta}_{a^+} \cosh(kx)\Big|_{x=b} + C_4 D^{\beta}_{a^+} \sinh(kx)\Big|_{x=b} \end{aligned} \tag{6}$$

$$\begin{aligned} D^{\alpha}_{b^-} y(x)\Big|_{x=a} &= D^{\alpha}_{b^-} \left\{C_1 \cos(kx) + C_2 \sin(kx) + C_3 \cosh(kx) + C_4 \sinh(kx)\right\}\Big|_{x=a} \\ &= C_1 D^{\alpha}_{b^-} \cos(kx)\Big|_{x=a} + C_2 D^{\alpha}_{b^-} \sin(kx)\Big|_{x=a} \\ &\quad + C_3 D^{\alpha}_{b^-} \cosh(kx)\Big|_{x=a} + C_4 D^{\alpha}_{b^-} \sinh(kx)\Big|_{x=a} \end{aligned} \tag{7}$$

Details on the numerical approximation of the left- and right-sided Riemann--Liouville fractional derivatives of the sine, cosine, hyperbolic sine and hyperbolic cosine functions can be found in [9]. Now, I present the final form of the fractional derivatives of these functions. The left-sided Riemann-Liouville derivatives for $\alpha \geq 0$, have the following forms

$$D^{\alpha}_{a^+} \sin(kx) = \begin{cases} (x-a)^{-\alpha} \left(\cos(ka) \sum_{i=0}^{\infty} \dfrac{(-1)^i \left(k(x-a)\right)^{2i+1}}{\Gamma(2i+2-\alpha)} + \sin(ka) \sum_{i=0}^{\infty} \dfrac{(-1)^i \left(k(x-a)\right)^{2i}}{\Gamma(2i+1-\alpha)} \right), & \text{for } \alpha \notin \mathbb{N}_0 \\ k^n \sin\left(kx + \dfrac{n\pi}{2}\right) & \text{for } \alpha = n \in \mathbb{N}_0 \end{cases} \tag{8}$$

$$D^{\alpha}_{a^+} \cos(kx) = \begin{cases} (x-a)^{-\alpha} \left(\cos(ka) \sum_{i=0}^{\infty} \dfrac{(-1)^i \left(k(x-a)\right)^{2i}}{\Gamma(2i+1-\alpha)} - \sin(ka) \sum_{i=0}^{\infty} \dfrac{(-1)^i \left(k(x-a)\right)^{2i+1}}{\Gamma(2i+2-\alpha)} \right), & \text{for } \alpha \notin \mathbb{N}_0 \\ k^n \cos\left(kx + \dfrac{n\pi}{2}\right) & \text{for } \alpha = n \in \mathbb{N}_0 \end{cases} \tag{9}$$

$$D_{a^+}^{\alpha}\sinh(kx)=\begin{cases}(x-a)^{-\alpha}\left(\cosh(ka)\sum_{i=0}^{\infty}\frac{(k(x-a))^{2i+1}}{\Gamma(2i+2-\alpha)}+\sinh(ka)\sum_{i=0}^{\infty}\frac{(k(x-a))^{2i}}{\Gamma(2i+1-\alpha)}\right), & \text{for } \alpha\notin\mathbb{N}_0\\ k^n\begin{cases}\cosh(kx) & \text{for} \quad n=1,3,5...\\ \sinh(kx) & \text{for} \quad n=0,2,4...\end{cases} & \text{for } \alpha=n\in\mathbb{N}_0\end{cases}\tag{10}$$

$$D_{a^+}^{\alpha}\cosh(kx)=\begin{cases}(x-a)^{-\alpha}\left(\cosh(ka)\sum_{i=0}^{\infty}\frac{(k(x-a))^{2i}}{\Gamma(2i+1-\alpha)}+\sinh(ka)\sum_{i=0}^{\infty}\frac{(k(x-a))^{2i+1}}{\Gamma(2i+2-\alpha)}\right), & \text{for } \alpha\notin\mathbb{N}_0\\ k^n\begin{cases}\sinh(kx) & \text{for} \quad n=1,3,5...\\ \cosh(kx) & \text{for} \quad n=0,2,4...\end{cases} & \text{for } \alpha=n\in\mathbb{N}_0\end{cases}\tag{11}$$

while, the left-sided Riemann-Liouville derivatives for $\alpha\geq 0$ are as follows:

$$D_{b^-}^{\alpha}\sin(kx)=\begin{cases}(b-x)^{-\alpha}\left(\cos(kb)\sum_{i=0}^{\infty}\frac{(-1)^i(k(b-x))^{2i}}{\Gamma(2i+1-\alpha)}+\sin(kb)\sum_{i=0}^{\infty}\frac{(-1)^i(k(b-x))^{2i+1}}{\Gamma(2i+2-\alpha)}\right), & \text{for } \alpha\notin\mathbb{N}_0\\ (-1)^n k^n\sin\left(kx+\frac{n\pi}{2}\right) & \text{for } \alpha=n\in\mathbb{N}_0\end{cases}\tag{12}$$

$$D_{b^-}^{\alpha}\cos(kx)=\begin{cases}(b-x)^{-\alpha}\left(\cos(kb)\sum_{i=0}^{\infty}\frac{(-1)^i(k(b-x))^{2i}}{\Gamma(2i+1-\alpha)}+\sin(kb)\sum_{i=0}^{\infty}\frac{(-1)^i(k(b-x))^{2i+1}}{\Gamma(2i+2-\alpha)}\right), & \text{for } \alpha\notin\mathbb{N}_0\\ (-1)^n k^n\cos\left(kx+\frac{n\pi}{2}\right) & \text{for } \alpha=n\in\mathbb{N}_0\end{cases}\tag{13}$$

$$D_{b^-}^{\alpha}\sinh(kx)=\begin{cases}(b-x)^{-\alpha}\left(-\cosh(kb)\sum_{i=0}^{\infty}\frac{(k(b-x))^{2i+1}}{\Gamma(2i+2-\alpha)}\right.\\ \qquad\left.+\sinh(kb)\sum_{i=0}^{\infty}\frac{(k(b-x))^{2i}}{\Gamma(2i+1-\alpha)}\right), & \text{for } \alpha\notin\mathbb{N}_0\\ (-1)^n k^n\begin{cases}\cosh(kx) & \text{for} \quad n=1,3,5...\\ \sinh(kx) & \text{for} \quad n=0,2,4...\end{cases} & \text{for } \alpha=n\in\mathbb{N}_0\end{cases} \tag{14}$$

$$D_{b^-}^{\alpha}\cosh(kx)=\begin{cases}(b-x)^{-\alpha}\left(\cosh(kb)\sum_{i=0}^{\infty}\frac{(k(b-x))^{2i}}{\Gamma(2i+1-\alpha)}\right.\\ \qquad\left.-\sinh(kb)\sum_{i=0}^{\infty}\frac{(k(b-x))^{2i+1}}{\Gamma(2i+2-\alpha)}\right), & \text{for } \alpha\notin\mathbb{N}_0\\ (-1)^n k^n\begin{cases}\sinh(kx) & \text{for} \quad n=1,3,5...\\ \cosh(kx) & \text{for} \quad n=0,2,4...\end{cases} & \text{for } \alpha=n\in\mathbb{N}_0\end{cases} \tag{15}$$

3. Examples of calculations

In this section, I display a method of construction of particular solutions to the considered fourth-order linear differential equation. The initial/boundary conditions, written in the general form in Eq. (2), can be used on both sides of the domain $[a, b]$ in many combinations. Now, I show two selected examples of their use.

Example 1: Eq. (1) with two boundary conditions given at boundary $x = a$ and with two boundary conditions at boundary $x = b$ are presented. Here, I assume the following set of boundary conditions in the form

$$\begin{aligned} D_{b^-}^{\alpha_1}y(x)\Big|_{x=a}&=L_1\\ D_{b^-}^{\alpha_2}y(x)\Big|_{x=a}&=L_2\\ D_{a^+}^{\beta_3}y(x)\Big|_{x=b}&=L_3\\ D_{a^+}^{\beta_4}y(x)\Big|_{x=b}&=L_4\end{aligned} \tag{16}$$

I substitute the general solution (2) into (16) and I have

$$
\begin{aligned}
& C_1 D_{b^-}^{\alpha_1} \cos(kx)\Big|_{x=a} + C_2 D_{b^-}^{\alpha_1} \sin(kx)\Big|_{x=a} + C_3 D_{b^-}^{\alpha_1} \cosh(kx)\Big|_{x=a} \\
& \qquad\qquad + C_4 D_{b^-}^{\alpha_1} \sinh(kx)\Big|_{x=a} = L_1 \\
& C_1 D_{b^-}^{\alpha_2} \cos(kx)\Big|_{x=a} + C_2 D_{b^-}^{\alpha_2} \sin(kx)\Big|_{x=a} + C_3 D_{b^-}^{\alpha_2} \cosh(kx)\Big|_{x=a} \\
& \qquad\qquad + C_4 D_{b^-}^{\alpha_2} \sinh(kx)\Big|_{x=a} = L_2 \\
& C_1 D_{a^+}^{\beta_3} \cos(kx)\Big|_{x=b} + C_2 D_{a^+}^{\beta_3} \sin(kx)\Big|_{x=b} + C_3 D_{a^+}^{\beta_3} \cosh(kx)\Big|_{x=b} \\
& \qquad\qquad + C_4 D_{a^+}^{\beta_3} \sinh(kx)\Big|_{x=b} = L_3 \\
& C_1 D_{a^+}^{\beta_4} \cos(kx)\Big|_{x=b} + C_2 D_{a^+}^{\beta_4} \sin(kx)\Big|_{x=b} + C_3 D_{a^+}^{\beta_4} \cosh(kx)\Big|_{x=b} \\
& \qquad\qquad + C_4 D_{a^+}^{\beta_4} \sinh(kx)\Big|_{x=b} = L_4
\end{aligned}
\tag{17}
$$

The constants C_i, $i = 1,\ldots 4$, occurring in the general solution (2), can be determined from the solution of the following system of equations

$$
\mathbf{A} \cdot \mathbf{C} = \mathbf{L} \tag{18}
$$

where

$$
\mathbf{C} = \left[C_1, C_2, C_3, C_4\right]^{\mathrm{T}}, \quad \mathbf{L} = \left[L_1, L_2, L_3, L_4\right]^{\mathrm{T}} \tag{19}
$$

$$
\mathbf{A} = \begin{bmatrix}
D_{b^-}^{\alpha_1} \cos(kx)\Big|_{x=a} & D_{b^-}^{\alpha_1} \sin(kx)\Big|_{x=a} & D_{b^-}^{\alpha_1} \cosh(kx)\Big|_{x=a} & D_{b^-}^{\alpha_1} \sinh(kx)\Big|_{x=a} \\
D_{b^-}^{\alpha_2} \cos(kx)\Big|_{x=a} & D_{b^-}^{\alpha_2} \sin(kx)\Big|_{x=a} & D_{b^-}^{\alpha_2} \cosh(kx)\Big|_{x=a} & D_{b^-}^{\alpha_2} \sinh(kx)\Big|_{x=a} \\
D_{a^+}^{\beta_3} \cos(kx)\Big|_{x=b} & D_{a^+}^{\beta_3} \sin(kx)\Big|_{x=b} & D_{a^+}^{\beta_3} \cosh(kx)\Big|_{x=b} & D_{a^+}^{\beta_3} \sinh(kx)\Big|_{x=b} \\
D_{a^+}^{\beta_4} \cos(kx)\Big|_{x=b} & D_{a^+}^{\beta_4} \sin(kx)\Big|_{x=b} & D_{a^+}^{\beta_4} \cosh(kx)\Big|_{x=b} & D_{a^+}^{\beta_4} \sinh(kx)\Big|_{x=b}
\end{bmatrix}
\tag{20}
$$

In Figures 1 and 2, the plots of example particular solutions of Eq. (1) with above boundary conditions have been presented.

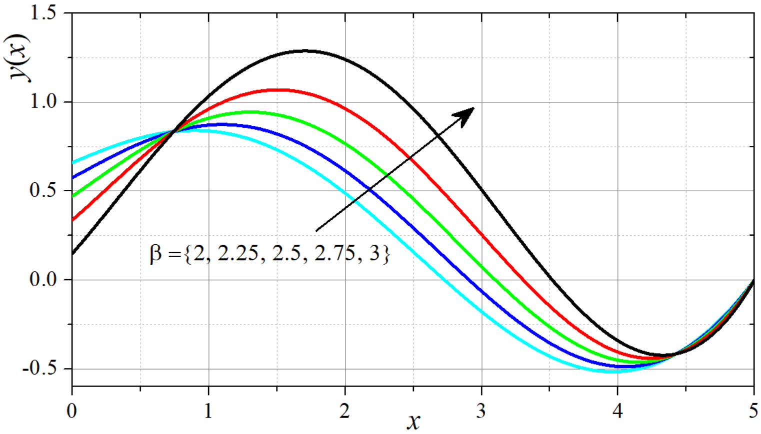

Fig. 1. The particular solutions of Eq. (1) for $a = 0$, $b = 5$, $k = 1$ and selected set of boundary conditions $D^{0.5}_{b^-}y(x)\big|_{x=a}=1,\ D^{1.2}_{b^-}y(x)\big|_{x=a}=0,\ D^{\beta}_{a^+}y(x)\big|_{x=b}=1,\ D^{0}_{a^+}y(x)\big|_{x=b}=0$

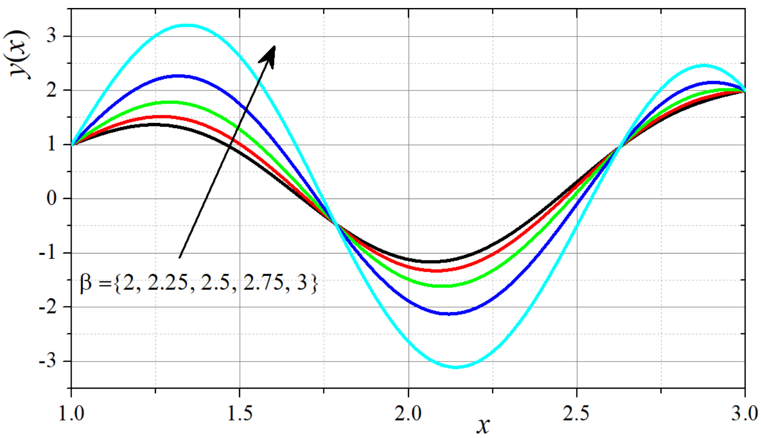

Fig. 2. The particular solutions of Eq. (1) for $a = 1$, $b = 3$, $k = 4$ and selected set of boundary conditions $D^{0}_{b^-}y(x)\big|_{x=a}=1,\ D^{2}_{b^-}y(x)\big|_{x=a}=0,\ D^{0}_{a^+}y(x)\big|_{x=b}=2,\ D^{\beta}_{a^+}y(x)\big|_{x=b}=0$

Example 2: In this example, the following set of four initial conditions given at boundary $x = a$ are considered

$$\begin{aligned} D^{\alpha_1}_{b^-}y(x)\big|_{x=a} &= L_1 \\ D^{\alpha_2}_{b^-}y(x)\big|_{x=a} &= L_2 \\ D^{\alpha_3}_{b^-}y(x)\big|_{x=a} &= L_3 \\ D^{\alpha_4}_{b^-}y(x)\big|_{x=a} &= L_4 \end{aligned} \tag{21}$$

The approach of determining of constants C_i, $i = 1,\dots4$, in Eq. (2) is very similar as in the preview example, and leads to solution of the system of linear equations (18), where matrix **A** is of form

$$\mathbf{A} = \begin{bmatrix} D_{b^-}^{\alpha_1}\cos(kx)\big|_{x=a} & D_{b^-}^{\alpha_1}\sin(kx)\big|_{x=a} & D_{b^-}^{\alpha_1}\cosh(kx)\big|_{x=a} & D_{b^-}^{\alpha_1}\sinh(kx)\big|_{x=a} \\ D_{b^-}^{\alpha_2}\cos(kx)\big|_{x=a} & D_{b^-}^{\alpha_2}\sin(kx)\big|_{x=a} & D_{b^-}^{\alpha_2}\cosh(kx)\big|_{x=a} & D_{b^-}^{\alpha_2}\sinh(kx)\big|_{x=a} \\ D_{b^-}^{\alpha_3}\cos(kx)\big|_{x=a} & D_{b^-}^{\alpha_3}\sin(kx)\big|_{x=a} & D_{b^-}^{\alpha_3}\cosh(kx)\big|_{x=a} & D_{b^-}^{\alpha_3}\sinh(kx)\big|_{x=a} \\ D_{b^-}^{\alpha_4}\cos(kx)\big|_{x=a} & D_{b^-}^{\alpha_4}\sin(kx)\big|_{x=a} & D_{b^-}^{\alpha_4}\cosh(kx)\big|_{x=a} & D_{b^-}^{\alpha_4}\sinh(kx)\big|_{x=a} \end{bmatrix} \tag{22}$$

In Figure 3, the plots of particular solutions of Eq. (1) with four initial conditions given on the boundary $x = a$ are shown.

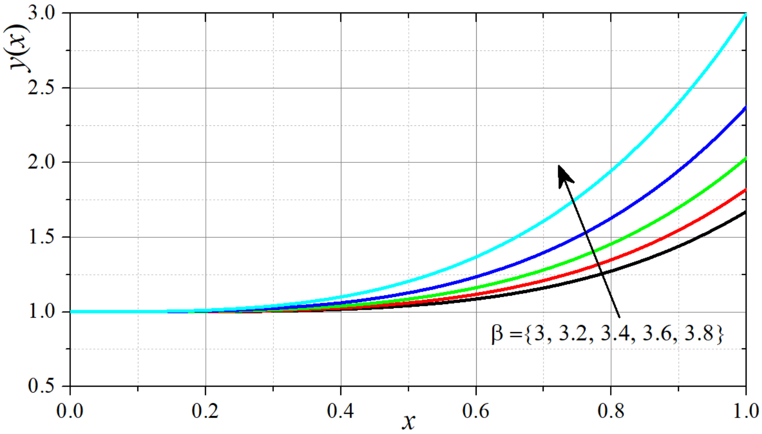

Fig. 3. The particular solutions of Eq. (1) for $a = 0$, $b = 1$, $k = 2$ and selected set of boundary conditions $D_{b^-}^{0}y(x)\big|_{x=a} = 1,\ D_{b^-}^{1}y(x)\big|_{x=a} = 0,\ D_{b^-}^{2}y(x)\big|_{x=a} = 0,\ D_{b^-}^{\beta}y(x)\big|_{x=a} = 0$

Other combinations of the particular boundary/initial conditions can be easily adopted by the Reader (in a similar way).

4. Conclusions

In this paper, the initial or/and boundary value problem for the fourth-order homogeneous differential equations with constant coefficients has been studied. The general solutions of such equations are widely known in the literature. The main aim of considerations was to find the particular solutions to this problem, meaning to determine the values of arbitrary constants in the general solution which satisfy the generalised boundary conditions given both by the set of fractional derivatives and integer ones (including the Dirichlet, Neumann and Robin types).

It should be noted that the application of the fractional boundary conditions in the considered initial or/and boundary value problem required the fractional differentiation of the general solution. In this way, analytical formulas for the left and right fractional Riemann-Liouville derivatives of the sine, cosine, hyperbolic sine and hyperbolic cosine functions were obtained, which were used to determine the values of integration constants.

Introducing fractional boundary conditions to the classical fourth-order boundary value problem gives new possibilities in physical phenomena modelling, among others, the modeling deflections in beams or the oscillator modelling.

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