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Stabilization of a fluid structure interaction with nonlinear damping*†

by

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Abstract: Asymptotic stability of *finite energy solutions* to a fluid-structure interaction with a static interface in a bounded domain $\Omega \in \mathbb{R}^n$, $n = 2$ is considered. It is shown that the undamped model subject to "partial flatness" geometric condition on the interface produces solutions whose *energy* converges *strongly* to zero; while with a stress type feedback control applied on the interface of the structure, the model produces solutions whose *energy* is exponentially stable. An addition of a static damping on the interface produces solutions whose *full* norm in the phase space is exponentially stable. Without a static damping an interesting phenomenon occurs that steady state solutions (equilibria) might generate genuinely growing in time solutions. This is purely nonlinear phenomenon captured by newly developed techniques amenable to handle instability of steady state solutions arising from nonlinearity.

Keywords: fluid structure interaction, interface, Navier Stokes equation, system of elasticity, feedback boundary control, strong stability, uniform stability, optimal control, passive damping, active damping, dynamic and static damping.

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1. Introduction

1.1. Description of the problem

We consider fluid-structure interaction described by a coupled system of partial differential equations (PDEs) comprising a nonlinear Navier-Stokes equation and a system of elasticity of wave equation. The coupling between two systems occurs on the boundary-interface between two environments: fluid and a solid. This model is well established in the literature and has numerous engineering applications that range from naval and aerospace engineering to cell biology and biomedical engineering see Moubachir & Zolesio (2006), Caputo & Hammer (2002), Khismatullin & Truskey (2005), Fernandez & Moubachir (2003), Du et al. (2003), and references therein.

However, due to mismatch of regularity between the particular hyperbolic component (dynamic system of elasticity) and parabolic component (fluid) the basic mathematical questions such as well-posedness of *finite energy* physical solutions had not been resolved until recently (Barbu et al., 2007, 2008; Kukavica et al., 2009, 2011; Coutand & Shokller, 2005). It is known by now that weak (finite energy) solutions corresponding to fluid structure interaction exist globally and they are *unique* when the dimension of the domain is equal to two. Thus, in the two dimensional case there exist a well defined semi-flow describing the associated dynamical system. In addition, it has been recently shown (Lasiecka & Lu, 2012) that the corresponding flow can be exponentially stabilized by a *linear* feedback placed at the interface between the solid and the fluid.

The main aim of this paper is to consider *nonlinear* feedback in a form of frictional damping affecting the solid-rather than the interface. We shall see that this configuration leads to new aspects of the theory exhibiting an interplay between the geometry of the domain (partial flatness of the contact area) and the effects of pressure in the N-S equation. The nonlinearity of the dissipation contributes to a more complex structure of decay rates for the energy which are still uniform but not necessarily exponential. They depend on the strength of the friction in the regime of small velocities.

1.2. The model

The model is defined on a bounded domain $\Omega \in \mathbb{R}^2$ that describes the interaction between an elastic body and a surrounding incompressible viscous fluid. Ω is a bounded simply connected domain, consisting of two open sub-domains Ω_s and Ω_f where Ω_f is the exterior domain, filled with fluid, and Ω_s is the interior doma, occupied by the elastic solid. The interaction between the fluid and the solid occurs at the interface $\Gamma_s \subset \partial\Omega_s$. The external boundary of Ω is denoted by Γ_f.

The dynamics of the fluid is described by the Navier-Stokes equation and the dynamics of the elastic body is described by an elasto-dynamic system of wave equations. $u(t,x) \in \mathbb{R}^2$ is a vector-valued function representing the ve-

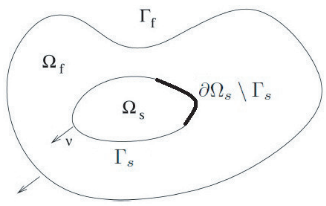

Figure 1. Geometry of Ω.

locity of the fluid and $p(t,x)$ is a scalar-valued function representing pressure. $w(t,x), w_t(t,x) \in \mathbb{R}^2$ denotes the displacement and the velocity functions of the elastic solid Ω_s. ν denotes, the unit outward normal vector on $\partial\Omega_s$ with respect to the region Ω_s. see Fig. 1.
This leads to the following interactive PDEs defined for the state variables $[u, w, w_t, p]$, Lions (1988):

$$
\begin{cases}
u_t - \triangle u + (u \cdot \nabla)u + \nabla p = 0 & \text{in } \ \Omega_f \times (0,\infty) \\
\text{div } u = 0 & \text{in } \ \Omega_f \times (0,\infty) \\
w_{tt} = \triangle w - g(w_t) & \text{in } \ \Omega_s \times (0,\infty) \\
\dfrac{\partial w}{\partial \nu} + \alpha w = \dfrac{\partial u}{\partial \nu} - p\nu + \dfrac{1}{2}(u \cdot \nu)u & \text{on } \ \Gamma_s \times (0,\infty) \\
\dfrac{\partial w}{\partial \nu} + \alpha w = \dfrac{\partial u}{\partial \nu} - p\nu + \dfrac{1}{2}(u \cdot \nu)u = 0 & \text{on } \ \partial\Omega_s \setminus \Gamma_s \times (0,\infty) \\
u = w_t & \text{on } \ \Gamma_s \times (0,\infty) \\
u = 0 & \text{on } \ \Gamma_f \times (0,\infty) \\
u(0,\cdot) = u_0 & \text{in } \ \Omega_f \\
w(0,\cdot) = w_0, \ w_t(0,\cdot) = w_1 & \text{in } \ \Omega_s
\end{cases}
\tag{1}
$$

where $g(w_t)$ represents an interior viscous-frictional nonlinear feedback - a typical damping mechanism present in oscillating structures. It will be assumed that $g(\cdot)$ is monotone, continuous and of polynomial growth, with $g(0) = 0$. In this model the fluid interacts with the solid on Γ_s, this fact being reflected by matching the velocities and matching the stresses. On $\partial\Omega_s \setminus \Gamma_s$ (possibly empty set) one assumes a hard contact.

The model considered accounts for small but rapid oscillations of the elastic displacements (Du et al., 2003). This allows one to assume that the interface is static.

It is convenient at this stage to introduce the energy of the system

$$E(t) \equiv \frac{1}{2}\int_{\Omega_f} |u|^2 dx + \frac{1}{2}\int_{\Omega_s} (|\nabla w|^2 + |w_t|^2) dx + \frac{1}{2}\alpha \int_{\partial\Omega_s} |w|^2 dx. \tag{2}$$

The main goal of this article is to establish asymptotic stability and energy decay rates for system (1). One should notice at the outset that without a static feedback the so called steady states corresponding to $w =$ *constant* and $u = 0$ always satisfy the equations (they are not "felt" by the dynamics). These data produce non-decaying in time solutions. Thus a challenge is to "filter out" these states by considering the complementary dynamics. In the linear case, it is a classical procedure to separate steady states from the state space (by orthogonal spectral decomposition) and then to consider the dynamics on the complementary-decaying phase space. The success of this procedure is due to local time invariance of the corresponding state spaces (due to linearity), Avalos & Triggiani (2007). However, in the nonlinear case, there is a possibility of "mixing" the states and such invariance no longer holds. As the result, the states that *in the linear dynamics* would lead to solutions constant in time, in the *nonlinear case* can be exploding in time. In fact, they can start growing polynomially or even exponentially. In order to illustrate this phenomenon it is constructive to look at the following benchmark problem involving wave equation only:

$$\begin{cases} w_{tt} - \triangle w + g(w_t) = 0, & \text{in } \Omega \\ \dfrac{\partial w}{\partial \nu} = 0, & \text{on } \partial\Omega \end{cases} \tag{3}$$

where Ω is a bounded domain in an Euclidean space and g is a continuous and monotone increasing function. The energy identity is given by $\widetilde{E}(t) + \int_0^t \int_\Omega g(w_t) w_t d\Omega ds = \widetilde{E}(0)$, with the energy functional $\widetilde{E}(t)$ defined as

$$\widetilde{E}(t) = \frac{1}{2}\left[\|\nabla w(t)\|^2_{L_2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)}\right].$$

Clearly, $w(t) \equiv C$, C: a constant, is a zero energy solution. However, it might generate solutions that will eventually blow up. To see this, let $g(s) = s^p$ ($p \neq 1$). Suppose $w(t,x) \equiv Cf(t)$ is a solution of (3). Then, $f(t)$ satisfies the equation

$$Cf_{tt} + g(Cf_t) = 0.$$

Solving the equation yields $f(t) = C^{-1}[C_p(t+c_1)^{\frac{p-2}{p-1}} + c_2]$, $p \neq 1$. Thus, where

$$w(t,x) = C_p(t+c_1)^{\frac{p-2}{p-1}} + c_2, \quad p \neq 1,$$

$w(t,x)$ is a finite energy solution if $p > 1$, since $w_t(t,x) \sim t^{-\frac{1}{p-1}} \to 0$ as $t \to \infty$, thus $\widetilde{E}(t) \to 0$ as $t \to \infty$. However, if $p > 2$, the solution will eventually blow up: $w(t,x) \sim t^{\frac{p-2}{p-1}} \to \infty$, as $t \to \infty$.

Thus, there is no method available in the nonlinear case which separates the analysis into stable and unstable parts, according to the spectrum of the linear operator. However, it is conceivable that while some solutions may grow in time, the *energy* of the system should be conserved or decaying (as for the constant states). And it is the energy that is of main interest to applications. In view of the above, our goal is to develop a method, which allows for separating the *energy asymptotics* from the solution asymptotics. This being the case, our task becomes to show that the energy $E(t)$ given in (2) decays to zero (while some of the solutions may still blow up at infinity). Depending on the strength of the frictional damping $g(w_t)$ we shall consider both strong decay and uniform decay of the energy function given by (2) with $\alpha \geq 0$.

1.3. Challenges

The major mathematical difficulty stems from the mismatch between the boundary regularity of the hyperbolic wave equation and the parabolic Navier-Stokes equation, which does not provide sufficient regularity for the boundary traces. In dealing with this particular difficulty, several strategies have been developed in earlier mathematical literatures where either a structural damping is added to the wave equation or a very smooth local-in-time solution were considered. Only recently the existence, uniqueness (in two dimensions), of the solutions *in the natural energy level* were shown to hold, Barbu et al. (2007). This was accomplished by taking advantage of recently discovered hyperbolic trace theory (Lasiecka, Lions & Triggiani, 1986) applied on the interface of the structure. Regularity of weak solutions was subsequently developed in Barbu et al. (2008), and also in Kukavica, Tuffaha & Ziane (2009, 2011) for a slightly different topological setting. Smooth solutions with moving interface have been analyzed in Coutand & Shokller (2005).

In the context of stability, stability results are available for the *linearized* model with the presence of pressure: strong stability in Avalos & Triggiani (2007, 2008, 2009a). The main tool used to establish the strong stability results for linear models is spectral theory (Arendt & Batty, 1988), which has no extension to nonlinear models. Subsequently, the study of nonlinear effects has been undertaken. The *nonlinearity* of the fluid model and the presence of the *pressure* term in the fluid equation are two main new aspects and challenges of the analysis. First stability results for the nonlinear fluid-structure interaction (1) are obtained in Lasiecka & Lu (2011, 2012). In all these works only *linear damping* mechanisms were considered. As demonstrated by the example, nonlinear damping plays major role in building growing in time solutions. The analysis of this delicate situation is the main contribution of the present paper.

2. Preliminaries and main results

Before introducing the main results, we will review some preliminary definitions and results.

2.1. Phase space and energy functional

As in Barbu et al. (2007), we define the following key spaces:

$$H \equiv \{u \in [L_2(\Omega_f)]^2 : \text{div } u = 0\}, V \equiv H \cap [H^1_f(\Omega_f)]^2$$

and the finite energy space for state variables $[u, w, w_t]^T$:

$$\mathcal{H} \equiv H \times [H^1(\Omega_s)]^2 \times [L_2(\Omega_s)]^2$$

where $H^1_f(\Omega_f)$ denotes $H^1(\Omega_f)$, Sobolev space with zero boundary conditions imposed on the boundary Γ_f. Later on, we shall also use the space $H^1_d(\Omega_s)$ which denotes $H^1(\Omega_s)$ space with the compatibility condition $\int_{\Gamma_s} w|_{\Gamma_s} \cdot \nu ds = 0$

$$H^1_d(\Omega_s) \equiv \{w \in H^1(\Omega_s); \int_{\Gamma_s} w \cdot \nu d\Gamma_s = 0\}, \quad \mathcal{H}_d \equiv H \times [H^1_d(\Omega_s)]^2 \times [L_2(\Omega_s)]^2.$$

The following (standard) notations will be used:

$$(u, v)_f = \int_{\Omega_f} uv \, d\Omega_f, (u, v)_s = \int_{\Omega_s} uv \, d\Omega_s$$

$$\langle u, v\rangle = \int_{\Gamma_s} uv \, d\Gamma_s; \; \langle u, v\rangle_{\partial\Omega_s} = \int_{\partial\Omega_s} uv \, d\partial\Omega_s.$$

$$|u|_{\alpha,D} = |u|_{H^\alpha(D)}, |u|_f = |u|_{0,\Omega_f}, (u, v)_{1,f} \equiv \int_{\Omega_f} \nabla u \cdot \nabla v \, d\Omega_f$$

$$Q_s \equiv (0, T] \times \Omega_s; \; Q_f \equiv (0, T] \times \Omega_f; \; \Sigma_s \equiv (0, T] \times \Gamma_s; \; \Sigma_f \equiv (0, T] \times \Gamma_f.$$

2.2. Existence, uniqueness and regularity of finite energy solutions

Motivated by feedback stabilization results for the pure wave equation (Lagnese, 1983; Haraux, 2006; Lasiecka, 2002; Komornik, 1998), a natural feedback to consider is in the a form of a frictional damping subject to the following assumptions:

Assumption 1. *The function $g(s) = [g_i(s_i)]_{i=1,2}$, where $g_i(s_i), s_i \in R$ are monotone increasing, continuous functions, zero at the origin and subject to the following conditions for $|s| \geq 1$*

$$m|s_i|^2 \leq g_i(s_i)s_i, \quad |g(s)| \leq M|s|^p, p \geq 1$$

for some positive constants $m > 0, M < \infty$

REMARK 1. **1.** *Note that no growth conditions are imposed on the damping function at the origin. This is one of the issues when dealing with questions of stability and decay rates (Lasiecka & Tataru, 1993).*
2. *One could impose more general structure of monotone frictional damping allowing for mixing of the wave coordinates. However, the main challenges of the problem are present already in this special configuration. In order to focus reader's attention we shall consider this form only. For more general structures of frictional damping, acting on the wave vectors we refer the reader to Chueshov & Lasiecka (2010).*

Projecting the equations on H and utilizing the boundary conditions enables defining weak solutions of the fluid component to our PDE system:

$$\frac{d}{dt}(u,\phi)_f+\langle\frac{\partial w}{\partial\nu},\phi\rangle+\alpha\langle w,\phi\rangle+(\nabla u,\nabla\phi)_f+((u\cdot\nabla)u,\phi)_f-\langle\frac{1}{2}(u\cdot\nu)u,\phi\rangle_{\partial\Omega_s}=0,\forall\,\phi\in V \tag{4}$$

We recall some results describing well-posedness and regularity of finite energy solutions. Global-in-time existence of the weak solutions is obtained in Barbu et al. (2007).

Theorem 2.1. *(Existence and uniqueness of weak solutions Barbu et al., 2007) Given any initial condition $(u_0,w_0,w_1)\in\mathcal{H}$, and any $T>0$, there exists unique weak (finite energy) solution $(u,w,w_t)\in C_w([0,T],\mathcal{H})$ to the system (1) with the following additional properties:*

1. $u\in L_2(0,T;V), u_t\in L_2(0,T;V')$, $w_{tt}\in L_2(0,T;[[H^1(\Omega_s)]^2]')$, $u|_{\Gamma_s}=w_t|_{\Gamma_s}$
2. $g(w_t)w_t\in L_1(Q_s)$, $\qquad w_t|_{\Gamma_s}\in L_2((0,T);[H^{1/2}(\Gamma_s)]^2)$.

Moreover, the said solution depends continuously on the initial data (with respect to the topology induced by $\mathcal{H}$).

Finite energy solutions are constructed as limits of monotone approximations to Navier Stokes problem. More specifically, the nonlinear N-S term is truncated so that the resulting problem is maximally monotone. Weak solutions are shown to be strong limits of these approximations Barbu et al. (2007). In carrying out this argument for wave equation with nonlinear dissipation, critical role is played by monotonicity of function $g(s)$.

REMARK 2. *The boundary condition representing matching of velocities on Γ_s: $w_t=u$, on Γ_s implies that the subspace $\mathcal{H}_d\subset\mathcal{H}$ is invariant under the flow. Thus, the statement of Theorem 2.1 holds true with $\mathcal{H}$ replaced by $\mathcal{H}_d$.*

REMARK 3. *When $g(w_t)\in L_2(\Omega)$, one can also show, Barbu et al. (2007) that weak solutions satisfy $\dfrac{\partial w}{\partial\nu}\in L_2((0,T);[H^{-1/2}(\Gamma_s)]^2)$.*

Additional regularity including differentiability of weak solutions is asserted in Barbu et al. (2008) (see also Kukavica, Tuffaha & Ziane, 2009, for different topological configuration) for solutions with more regular initial data. In fact, in that case (u, w, w_t) satisfies the variational form of the original PDE equation a.e. in $t \in (0, T)$

$$(u_t, \phi)_f + \langle \frac{\partial w}{\partial \nu} + \alpha w, \phi \rangle + (\nabla u, \nabla \phi)_f + ((u \cdot \nabla) u, \phi)_f - \langle \frac{1}{2}(u \cdot \nu) u, \phi \rangle_{\partial \Omega_s} = 0, \ \forall \phi \in V \quad (5)$$

$$(w_{tt}, \psi)_s - \langle \frac{\partial w}{\partial \nu}, \psi \rangle + \alpha \langle w, \psi \rangle_{\partial \Omega_s \setminus \Gamma_s} + (\nabla w, \nabla \psi)_s + (g(w_t), \psi)_s = 0, \ \forall \psi \in [H^1(\Omega_s)]^2. \quad (6)$$

Theorem 2.2. *(Regularity Barbu et al., 2008) Let* $(u_0, w_0, w_1) \in \mathcal{H} \cap \{([H^2(\Omega_f)]^2 \cap V \times [H^2(\Omega_s)]^2 \times [H^1(\Omega_s)]^2)\}$ *satisfy the usual boundary compatibility conditions imposed on the boundary. Then, for any* $T > 0$, (u, w, w_t) *satisfies the variational form (5), (6), and we have :*

1. $(u, p) \in L_2((0, T); [H^2(\Omega_f)]^2 \times H^1(\Omega_f))$
2. $(u_t, w_t, w_{tt}) \in L_\infty((0, T); \mathcal{H})$, $w \in L_\infty((0, T); [H^2(\Omega_s)]^2)$.

Theorems 2.1 and 2.2 were proved in Barbu et al. (2007, 2008) without the damping $g(w_t)$. However, the same proof can be carried out in the presence of frictional damping that is assumed *monotone* and subject to polynomial growth condition when dimension of Ω is equal to two.

2.3. Energy functional and energy identity

Let u, w be regular solutions obtained in Theorem 2.2. Choose as the test functions $\phi = u$ and $\psi = w_t$ in the formulation (5)-(6). Noticing cancellation occurring in the nonlinear term

$$((u \cdot \nabla) u, u)_f - \langle \frac{1}{2}(u \cdot \nu) u, u \rangle_{\partial \Omega_s} = 0$$

and utilizing the transmission condition $u = w_t$ on Γ_s, one obtains the following energy identity for $0 \leq s \leq t$

$$E(t) + \int_s^t (|\nabla u|_f^2 + (g(w_t), w_t)_s) d\tau = E(s), \ 0 \leq s < t \quad (7)$$

where $E(t)$ is the energy functional defined in (2). Denote the dissipation terms in (7) as

$$D(t) = |\nabla u(t)|_f^2 + (g(w_t), w_t(t))_s.$$

The energy identity can be rewritten as

$$E(t) + \int_s^t D(\tau)d\tau = E(s), \; 0 \leq s < t \tag{8}$$

this relation revealing the dissipative character of the evolution. Few observations are in order.

1). The energy identity (7) reveals that there are two potential sources of dissipation: one propagated from the Navier-Stokes equation and the other from the dynamic damping $g(w_t)$. Without the dynamic damping, the energy does not decay. This is the case, even in the linear case when one can show that there are infinitely many eigenvalues with real parts on the imaginary axis (Avalos & Triggiani, 2007, 2009b). Strong stability has been shown in Avalos & Triggiani (2007) under a suitable geometric condition and for initial data restricted to a closed subspace of $\mathcal{H}$, which eliminates a subspace corresponding to zero eigenvalue of the linear generator. The corresponding condition is the following: Γ_s contains a flat portion Γ_0 with positive measure. We note that the above condition fails when Ω is a ball. Thus, the aforementioned condition is not compatible with a perfect symmetry of the domain.

2). The energy functional $E(t)$ with $\alpha = 0$ is only a *semi-norm* on the phase space $\mathcal{H}$. Thus, there could be zero energy solutions which might have nonzero displacement of the solid. One could eliminate these by adding some static damping (Lasiecka & Lu, 2012) or by assuming that $\alpha > 0$. In this latter case the energy functional determines a full norm on the phase space $\mathcal{H}$, a result of Poincaré's inequality and trace theory. How to eliminate nonzero steady states, *without* adding static damping will be one of the major issues we will contend with in this paper.

2.4. Main results

Detailed statements of our main results are given in the following theorems. We begin by formulating the following Geometric Assumption.

Assumption 2. *(a)* Γ_s *contains a flat portion* Γ_0 *with positive measure;*
(b) EITHER Γ_s *is a strict subset of* $\partial\Omega_s$ *such that* $\displaystyle\int_{\Gamma_s} \nu d\Gamma_s \neq 0$, *OR* $\Gamma_s = \partial\Omega_s$.

Theorem 2.3. *(***Strong Stability of Energy***) Let* $\alpha \geq 0$ *and let impose geometric Assumption 2. Then, for any initial data* $(u_0, w_0, w_1) \in \mathcal{H}$ *when* $\Gamma_s \subset \partial\Omega_s$*, (respectively* $\mathcal{H}_d$ *when* $\Gamma_s = \partial\Omega_s$*), one obtains that the energy functional for the system (1) tends to 0 as* $t \to \infty$*. This is to say:*

$$E(t) \to 0 \;\; \textit{as} \;\; t \to \infty. \tag{9}$$

Theorem presented above is an extension of the corresponding result obtained in Lasiecka & Lu (2011), where the structure equation was assumed linear.

Knowing that the energy converges strongly to zero, one would like to know how fast this convergence occurs. This corresponds to the question of decay rates. Under additional assumption, imposed on the damping, one obtains uniform decay of energy with the rates determined by a solution of nonlinear ODE.

Theorem 2.4. *(***Uniform Decay Rates for the Energy***). Let $\alpha \geq 0$. We assume Assumption 1 and part (b) of geometric Assumption 2.*
Then, there exists constant $T_0 > 0$, such that the energy satisfies

$$E(t) \leq S(t), \ \ for \ t > T_0$$

where $S(t)$ satisfies the following ODE:

$$\frac{d}{dt}S(t) + q(S(t)) = 0, S(0) = E(0) \tag{10}$$

with $q(s) \sim \hat{h}^{-1}(s)$ where $\hat{h}(s) = (meas \ Q_s) \ h(\frac{s}{meas \ Q_s})$ with h monotone increasing, continuous, $h(0) = 0$, concave and determined from the inequality $s^2 \leq h(sg(s)), |s| \leq 1$.
When $\Gamma_s = \partial\Omega_s$, we take the initial data in $\mathcal{H}_d$.

REMARK 4. *Note that owing to monotonicity of $g(s)$ function $h(s)$, can be always constructed as a concave envelope (Lasiecka & Tataru, 1993). Function $h(s)$ captures the behavior of the dissipation $g(s)$ at the origin. This is the most sensitive region with respect to the decay rates. Thus, the task of finding decay rates is reduced to solving an ODE (10) with a given function $h(s)$ (hence $q(s)$) .*

In fact, when $g(s) = as$, then $h(s) = a^{-1}s$ and the decay rates are exponential of the type e^{-at}. For polynomial $g(s)$ at the origin, the decay rates are polynomial as well (algebraic) $t^{-\frac{2}{p-1}}$. These are optimal algebraic decay rates. See Lasiecka & Tataru (1993) for many examples.

REMARK 5. *Note that Theorem 2.4 does not require partial flatness assumption (part (a) of Assumption 2). This is due to the fact that the presence of damping $g(s)$ under the Assumption 1 eliminates the dynamics from the ω limit set.*

REMARK 6. *In the case when $\Gamma_s = \partial\Omega_s$ and the initial data are not restricted to $\mathcal{H}_d$ (so that condition (b) of Assumption 2 is violated), the proof of Theorem 2.3 shows convergence of each energy solution to a one dimensional manifold spanned by a vector $(0, w^*, 0) \in \mathcal{H}$, where w^* is a solution to the following elliptic* *problem:*

$$\begin{aligned}
&\Delta w^* = 0, \ \textit{in } \Omega_s \\
&\frac{\partial w^*}{\partial \nu} = \nu, \ \textit{on } \Gamma_s.
\end{aligned} \tag{11}$$

Remark 7. *The dimensionality of the domain is restricted to two for different reasons. For strong stability, the argument depends on the locally Lipschitz property of the flow on the phase space $\mathcal{H}$, which does not hold in three dimensional space. For uniform stability, the method used does not depend on the dimensionality of the domain. However, when $n = 3$, weak solutions are not known to be unique, thus the decay rates obtained for strong solutions only can not be extended to all weak solutions. In that case the result remains valid for smooth solutions which are global (e.g. corresponding to small initial data-as shown in Barbu et al., 2008).*

3. Strong stability with weak or without frictional damping - Proof of Theorem 2.3.

In this section, we will establish strong stability for the model with weak frictional damping. We will show that under the geometric condition, as specified by Theorem 2.3, the kinetic and potential energy $E(t)$ decays to zero when time goes to infinity.

3.1. General comments

The model without the frictional damping possesses a few distinct features in the context of strong stability: (a) the dissipation is *weak*; (b) the resolvent operator is *not* compact; (c) the dynamics is partially hyperbolic. These features render the standard tools used for the study of strong stability of nonlinear systems not applicable in the present situation. Indeed, a classical tool is LaSalle's Invariance Principle (La Salle, 1976). A key hypothesis assumed by this principle (and its variants) is the compactness of the orbits, often secured by the compactness of the resolvent of the semigroup generated by the flow. However, this latter property, while typical in parabolic flows, does not hold in hyperbolic-like dynamics, e.g. the wave equation component in the system we consider. Some known nonlinear methods (Ball, 1977, 1978; Brezis, 1978) for studying asymptotic stability require one of the following conditions to be satisfied: (i) semigroup associated with the linearization be "smoothing" (parabolic like situation), or (ii) the nonlinear generator be m-monotone, or (iii) linearization be exponentially stable, or (iv) linear generator be monotone and nonlinear perturbation *weakly sequentially compact*. In the case of the model under consideration neither of these options is available.

The approach we develop is motivated by a *relaxed* version of LaSalle's Invariance Principle (Slemrod, 1989), based on the concept of 'relaxed' ω-limit set, which yields strong stability in a suitable weakened topology. In order to follow this route, as mentioned earlier, we will first transform the system, following the method introduced in Lasiecka & Seidman (2003). Once the correct dynamical system is identified, we shall show that this system admits a "relaxed" ω-limit set containing only the trivial solution. The main technical difficulties that need to be addressed are: (1) to improve weak into strong convergence - a challenging endeavor in the absence of compactness, and (2) to identify ω-limit sets with suitable equilibria of coupled dynamics. The first task will be handled by exploiting suitable multipliers that are harmonic extensions of the Stokes operator. The second task relies on the geometric conditions ensuring an appropriate version of unique continuation property for the overdetermined on the boundary system.

3.2. Change of variables

We focus on a more challenging case when $\alpha = 0$. We shall mainly treat the case, when $\Gamma_s \subset \partial\Omega_s$ and the initial data are taken from $\mathcal{H}$. The (minor) differences in the arguments for the case when the initial data are in $\mathcal{H}_d$ and $\Gamma_s = \partial\Omega_s$ will be commented in the corresponding Remarks.

Since the energy relation provides information only on the gradient of the displacement (without controlling the entire L_2 norm, where the latter may increase in time), we will construct a new dynamical system which accounts for the "degeneracy" of the energy. To achieve this we shall use the approach introduced in Lasiecka & Seidman (2003). We consider the space defined as

$$\mathcal{H}_0 \equiv H \times U \times [L_2(\Omega_s)]^2$$

where

$$U \equiv L^2_\nabla(\Omega_s) \equiv \{\nabla h, h \in [H^1(\Omega_s)]^2\}.$$

Note, that $L^2_\nabla(\Omega_s)$ is the space of vector tensors of order four, i.e. $\nabla h = \begin{pmatrix} \nabla h_1 \\ \nabla h_2 \end{pmatrix}$. As shown in Lasiecka & Seidman (2003), $L^2_\nabla(\Omega_s)$ is a closed subspace of $[L_2(\Omega_s)]^2 \times [L_2(\Omega_s)]^2$ and so is a Hilbert space. With the above notation (in the sequel we shall omit explicit writing of multiple copies of the vector spaces), we shall rewrite the original system as a dynamical system governed by the variables $(u(t), \Xi(t), v(t)) \in \mathcal{H}_0$ which satisfy: *fluid equation in the variable* $u \in H$:

$$(u_t, \phi)_f + \langle \Xi \cdot \nu, \phi \rangle + (\nabla u, \nabla \phi)_f + ((u \cdot \nabla)u, \phi)_f - \langle \frac{1}{2}(u \cdot \nu)u, \phi \rangle_{\partial\Omega_s} = 0, \quad \forall\ \phi \in V \quad (12)$$

and solid equation in the variable $(\Xi, v) \in L^2_\nabla(\Omega_s) \times L_2(\Omega_s)$:

$$\begin{cases} \Xi_t = \nabla v, & \text{in } \Omega_s \times (0,\infty) \\ v_t = \nabla \cdot \Xi + g(v), & \text{in } \Omega_s \times (0,\infty) \\ v|_{\Gamma_s} = u|_{\Gamma_s}, & \text{on } \Gamma_s \times (0,\infty) \\ \Xi \cdot \nu = 0, & \text{on} \partial\Omega_s \setminus \Gamma_s \end{cases}. \tag{13}$$

The equivalent variational form is the following:

$$(u_t, \phi)_f - \langle \Xi \cdot \nu, \phi \rangle + (\nabla u, \nabla \phi)_f + ((u \cdot \nabla)u, \phi)_f - \langle \frac{1}{2}(u \cdot \nu)u, \phi \rangle_{\partial\Omega_s} = 0, \quad \forall\ \phi \in V$$
$$(\Xi_t, \Psi)_s = (\nabla v, \Psi)_s, \quad \forall \Psi \in L^2_\nabla(\Omega_s)$$
$$(v_t, \psi)_s = \langle \Xi \cdot \nu, \psi \rangle - (\Xi, \nabla \psi)_s + (g(v), \psi)_s, \quad \forall \psi \in H^1(\Omega_s)$$
$$\text{with the transmission condition } v|_{\Gamma_s} = u|_{\Gamma_s}. \tag{14}$$

(14) is supplied with the initial conditions:

$$u(0) = u_0 \in H, \Xi(0) = \Xi_0 \in L^2_\nabla(\Omega_s), v(0) = v_0 \in L_2(\Omega_s)$$

Time derivatives are defined distributionally. It is clear that every solution (u, w, w_t) of the original problem corresponds to (u, Ξ, v) with the identification: $\Xi = \nabla w$, $v = w_t$. Also, having given variable (u, Ξ, v), we can easily reconstruct w from $w(t) = w(0) + \int_0^t v(s)ds$. Of course, the latter quantity may not be bounded in time when $t \to \infty$.
In what follows we consider the system defined by (14).

• **Energy identity and energy functional for the transformed dynamics.** Energy method applied to strong solutions of (14), i.e. taking $\phi = u, \Psi = \Xi, \psi = v$ gives

$$E_0(t) + \int_0^t [|\nabla u|_f^2 + (g(v), v)_s] ds = E_0(0)$$

where

$$E_0(t) \equiv \frac{1}{2}\left[|u|_f^2 + |\Xi|_s^2 + |v|_s^2\right].$$

Thus, the energy function defines a norm on $\mathcal{H}_0$ defined above. In fact, "both" energies for the original system and the transformed one are the same. The original system defined in the variables (u, w, w_t) corresponds in a one to one manner to a "new" system (u, Ξ, v) where $u = u, \Xi = \nabla w, v = w_t$.

Due to the invariance of the flow, defined by (14) on $\mathcal{H}_0$, by Theorem 2.1 we can construct a nonlinear semigroup $S_0(t) : \mathcal{H}_0 \to \mathcal{H}_0$ such that

$$S_0(t)(u_0, \Xi_0, v_0) = (u(t), \Xi(t), v(t)), \forall (u_0, \Xi_0, v_0) \in \mathcal{H}_0.$$

Then, $(S_0(t), \mathcal{H}_0)$ defines a dynamical system, Chueshov & Lasiecka (2008). Similarly, we define nonlinear semigroup on $\mathcal{H}_{0d}$ where the U component is replaced by $U_d = \{\nabla h, h \in [H^1_d(\Omega_s)]^2\}$.

We define the following weak ω-limit set and the set $\mathcal{D}$-invariant under the flow and representing smooth data.

Definition 1. *(weak ω-limit set) Let $(u(t), w(t), w_t(t))$ be weak solution of (1), specified in Theorem 2.1, corresponding to the initial data $(u, \Xi, v) \in \mathcal{H}_0$. We say that a point $(\overline{u_0}, \overline{\Xi_0}, \overline{v_0}) \in \mathcal{H}_0$ is in the weak ω-limit set $\omega(u_0, \Xi_0, v_0)$ if there exists a sequence $t_n \to \infty$ such that $(u(t_n), v(t_n)) \to (\overline{u}_0, \overline{v}_0)$ strongly in $L_2(\Omega_f) \times L_2(\Omega_s)$ and $\Xi(t_n) \rightharpoonup \overline{\Xi}_0$ in $L^2_\nabla(\Omega_s)$.*

Definition 2. *(Smooth data) We say the data $(u_0, \Xi_0, v_0) \in \mathcal{H}$ is* smooth*, if it is contained in the following set $\mathcal{D}$:*

$$\mathcal{D} = \{(u_0, \Xi_0, v_0) \in V \times L^2_\nabla(\Omega_s) \times H^1(\Omega_s) \text{ such that}$$

$$P_H \Delta u_0 \in L_2(\Omega_f), \quad div\ \Xi_0 \in L_2(\Omega_s), \quad \Xi_0 \cdot \nu \in H^{-1/2}(\Gamma_s), \Xi_0 \cdot \nu = 0, \text{ on } \partial\Omega_s \backslash \Gamma_s$$

$$u_0|_{\Gamma_s} = v_0|_{\Gamma_s}, \langle \Xi_0 \cdot \nu - \frac{\partial u_0}{\partial \nu} + \frac{1}{2}(u_0 \cdot \nu)u_0, \phi\rangle = 0, \phi \in V\}$$

where P_H denotes a projection operator on H. Exploiting monotonicity of the damping $g(v)$, polynomial growth condition which gives $g(v) \in L_2(\Omega_s)$ for $v \in V \subset L_r(\Omega_s)$ with any $r < \infty$ and the energy method applied to time derivatives of solutions gives

Lemma 3.1. *Let $(u_0, \Xi_0, v_0) \in \mathcal{D}$. Then $(u_t, \Xi_t, v_t) \in L_\infty((0, \infty), \mathcal{H}_0)$.*

Set $\mathcal{D}$ is invariant under the dynamics. This claim can be easily seen from Lemma 3.1 after performing energy calculations for time derivatives of solutions and appealing to monotonicity of $g(s)$ and Sobolev's embeddings controlling invariance of nonlinear terms.

3.3. Weak ω-limit is $\{0\}$ for smooth initial data in $\mathcal{D}$

In this section, we will show that the dynamical system (14) admits a weak ω-limit set which is zero in the topology of $\mathcal{H}_0$. We should first point out that the weak ω-limit set is not empty, since $V \times H^1(\Omega_s)$ is strongly compact in $H \times L_2(\Omega_s)$ and boundedness of Ξ_n in $L^2_\nabla(\Omega_s)$ implies existence of weakly convergent subsequence $\Xi_{n,k}$ in $L^2_\nabla(\Omega_s)$.

Let $(\overline{u_0}, \overline{\Xi_0}, \overline{v_0})$ be an element in $\omega(u_0, \Xi_0, v_0)$ for $(u_0, \Xi_0, v_0) \in D$. By definition, there exists a sequence $t_n \to \infty$ such that $(u(t_n), v(t_n)) \to (\overline{u_0}, \overline{v_0})$ strongly in $L_2(\Omega_f) \times L_2(\Omega_s)$ and $\Xi(t_n) \rightharpoonup \overline{\Xi_0}$ weakly in $L^2_\nabla(\Omega_s)$, where $X(t; X_0) := (u(t), \Xi(t), v(t))$ is a solution with the initial data $X_0 = (u_0, \Xi_0, v_0)$. For this sequence t_n, consider the translate $X_n(t) := X(t+t_n; X_0)$. From energy identity (7),

$$\|X_n(t)\|_{\mathcal{H}_0} \leq \|X_0\|_{\mathcal{H}_0}, \ \textit{for all } t \in \mathbb{R}^+$$

X_n is a bounded sequence in $L_\infty((0,\infty); \mathcal{H}_0)$. Thus, X_n has a subsequence, which we still denote by $X_n := (u_n, \Xi_n, v_n)$, such that X_n converges to $\overline{X} := (\overline{u}, \overline{\Xi}, \overline{v})$ weakly in $L_2((0,T); \mathcal{H}_0)$ and weak* in $L_\infty((0,\infty); \mathcal{H}_0)$. We will first show that $\overline{u} = 0$. Energy inequality (7) and invariance of the dynamics on $\mathcal{D}$ imply the following:

Lemma 3.2. *$u_n \to 0$ in $C((0,T], V)$ and $g(v_n)v_n \to 0$ in $L_1(0,T,L_1(\Omega))$ for each $T \geq 0$.*

The convergence of u_n obtained in Lemma 3.2 allows us to pass to the limit in the weak formulation (14) and it turns out that $[\overline{\Xi}, \overline{v}]$ satisfies a special Dirichlet -Stokes problem stated in the following lemma whose proof is technical (see Lasiecka & Lu, 2011).

Lemma 3.3. *$[\overline{\Xi}, \overline{v}]$ is a weak solution of the following problem:*

$$\begin{cases} \overline{\Xi}_t = \nabla \overline{v} & in \ \ \Omega_s \times (0,T_1) \\ \overline{v}_t = \nabla \cdot \overline{\Xi} & in \ \ \Omega_s \times (0,T_1) \\ \overline{v} = 0, \ \overline{\Xi} \cdot \nu = p(t)\nu & on \ \ \Gamma_s \times (0,T_1) \\ \overline{\Xi} \cdot \nu = 0 & on \ \ \partial\Omega_s \setminus \Gamma_s \times (0,T_1) \end{cases} \tag{15}$$

with initial condition $[\overline{u}(0), \overline{\Xi}(0), \overline{v}(0)] = [\overline{u}_0, \overline{\Xi}_0, \overline{v}_0] \in \mathcal{D}$ and $p \in L_\infty(0,T_1)$, where T_1 is arbitrary.

Our next step is to analyze the overdetermined problem (15) and show that the solution to (15) is stationary. We have the following Lemma:

Lemma 3.4. *With reference to system (15) the overdetermined on the boundary the following hold:*

- *Under part (a) of the Assumption 2 the energy $E_0(t)$ is a strict Lyapunov function on $\mathcal{H}_0$. Solutions to (15) are stationary.*
- *Under the full strength of Assumption 2 the only solution of (15) is the trivial one.*

Proof. Let (Ξ, v) be a solution to the overdetermined problem specified in (15). Let D_τ denote the tangential derivative applied to the flat portion of the boundary $\Gamma_0 \subset \Gamma_s$. D_τ is orthogonal to ν and commutes with ν on Γ_0 (flatness assumption). D_τ can be naturally extended into a small collar near Γ_s, denoted by $\Omega_0 \subset \Omega_s$. We denote

$$\Xi_\tau \equiv D_\tau \Xi, \ \ v_\tau \equiv D_\tau v, \ \ in \ \ \Omega_0.$$

Exploiting the flatness of the boundary Γ_0 we obtain the following system satisfied for the new variables (Ξ_τ, v_τ) in Ω_0,

$$\begin{aligned} \Xi_{\tau,t} &= \nabla v_\tau,\ \ in\ \Omega_0 \\ v_{\tau,t} &= \nabla \cdot \Xi_\tau,\ \ in\ \Omega_0 \\ \Xi_\tau \cdot \nu = 0,\quad v_\tau &= 0,\quad on\ \ \Gamma_0. \end{aligned} \tag{16}$$

The above system can be reduced to the wave equation with the overdetermined boundary data on Γ_0:

$$\begin{aligned} v_{\tau,tt} &= \Delta v_\tau,\ \ in\ \Omega_0 \\ v_\tau = 0, \frac{\partial v_\tau}{\partial \nu} &= 0,\ \ on\ \Gamma_0. \end{aligned} \tag{17}$$

By the unique continuation property (Littman, 2000; Lions, 1988), we conclude that $v_\tau = 0,\ in\ \Omega_0$. Applying now the classical Holmgren's Uniqueness Theorem we extend local uniqueness to the global, claiming

$$v_\tau \equiv 0,\ \ in\ \Omega_s.$$

The above condition implies that v is constant in y. Therefore, for any fixed $x \in \Omega_s$, $v(x,y,t) = v(x,y^*,t)$ for any $y^* \in \Gamma_s$, $y \in \Omega_s$ and $t \in \mathbb{R}^+$. But on the boundary Γ_s, v is identically zero for all t. Thus,

$$v \equiv 0,\ \text{ in } \Omega_s \times \mathbb{R}^+.$$

Going back to the original system we obtain that $\Xi_t = \nabla \cdot v \equiv 0$, which then implies that $E_0(t)$ is a strict Lyapunov's function on $\mathcal{H}_0$. This proves the first part of the Lemma.

For the second part of the Lemma, we are led to consider the stationary problem:

$$\text{div }\Xi = 0, in\ \Omega_s,\ \ \Xi \cdot \nu = p\nu, on\ \Gamma_s,\ \ \Xi \cdot \nu = 0\ on\ \partial\Omega_s \setminus \Gamma_s \tag{18}$$

with p being now just a constant. We shall show that p must be zero. This can be seen as follows. The divergence theorem implies

$$\int_{\partial\Omega_s} \Xi \cdot \nu dx = 0.$$

Zero boundary conditions on the complement of $\partial\Omega_s \setminus \Gamma_s$ imply

$$\int_{\Gamma_s} \Xi \cdot \nu d\Gamma_s = 0.$$

Hence, $p \int_{\Gamma_s} \nu d\Gamma_s = 0$, which is impossible (due to the geometric condition $\int_{\Gamma_s} \nu d\Gamma_s \neq 0$) unless $p = 0$. So we have div $\Xi = 0$, and $\Xi \cdot \nu = 0$ on the boundary Γ_s . Since $\Xi \in L^2_\nabla(\Omega_s)$ we have that $\Xi = \nabla h$ for some $h \in H^1(\Omega)$. This, along with (15), imply

$$\Delta h = 0, \quad in \quad \Omega_s, \ \ \frac{\partial h}{\partial \nu} = 0, \ on\ \partial\Omega_s.$$

The above can happen only if $h = constant$. But then $\Xi \equiv 0$, proving that both $v \equiv 0$, $\Xi \equiv 0$. This completes the proof of the second part of the Lemma. ∎

Remark 8. *When $\Gamma_s = \partial\Omega_s$ and $w \in H_d(\Omega_s)$, the divergence theorem applied to (18) implies*

$$0 = (div\ \Xi, w)_s = -|\Xi|_s^2 + \langle \Xi \cdot \nu, w\rangle = -|\Xi|_s^2 + p \int_{\Gamma_s} w \cdot \nu d\Gamma_s = -|\Xi|_s^2$$

hence $\Xi = constant$ and the remaining part of the argument is the same as before .

Remark 9. *In the case when the frictional damping is strong enough, so that*

$$g(s)s = 0 \Rightarrow s = 0$$

one obtains the first part of Lemma 3.4 without any flattness geometric condition. However, we do not assume this property for the problem under consideration. In fact, $g(s)$ can be equal 0 .

Lemmata 3.4 and 3.3 imply the following important Corollary:

Corollary 1. *Under the geometric Assumption 2 we have that weak ω limit set for the dynamical system $(S_0(t), \mathcal{H}_0)$ consists of zero element only. This is to say*

$$(\overline{u}, \overline{\Xi}, \overline{v}) \equiv 0, \ in\ \mathcal{H}_0, \quad (respectively \mathcal{H}_{0,d}).$$

Remark 10. *In the case when $\partial\Omega_s = \Gamma_s$, so that part (b) of the Assumption 2 fails, solutions to the overdetermined system (18) are characterized by:*

$$div\ \Xi = 0, \ in\ \Omega_s, \Xi \cdot \nu = k\nu, on\ \partial\Omega_s \tag{19}$$

where $k \in R$. These solutions are eigenvectors corresponding to the zero eigenvalue of the original linear generator (Avalos & Triggiani, 2007, 2009b), Thus, in that case the weak ω limit set consists of a one dimensional manifold spanned by solution to $div\ \Xi = 0, \ in\ \Omega_s, \Xi \cdot \nu = \nu, on\ \partial\Omega_s$

3.4. Strong ω-limit set is $\{0\}$ for smooth initial data in $\mathcal{D}$

Our goal in this section is to improve weak convergence of $\Xi_n(s)$ in $L^2_\nabla(\Omega_s)$ to the strong convergence.

Lemma 3.5. *Assume the geometric conditions imposed by Assumption 2. Then for all initial data in D, we have that $(u_n(t), \Xi_n(t), v_n(t)) \to 0$, strongly in $\mathcal{H}_0$, for all $t \in [0, T]$.*

Proof. It suffices to show that the convergence of Ξ_n to zero is *strong* in $L^2_\nabla(\Omega)$. Here the idea is to utilize certain harmonic extensions associated with the Stokes operator. To this aim we define the following Stokes extension of the Dirichlet map D

$$z = Dg \;\Leftrightarrow\; \begin{cases} \triangle z = \nabla q, \quad \operatorname{div} z = 0 & \text{in } \Omega_f \\ z = g & \text{on } \partial\Omega_s \\ z|_{\Gamma_f} = 0 & \text{on } \Gamma_f \end{cases} \tag{20}$$

where we assume the compatibility $\int_{\partial\Omega_s} g \cdot \nu ds = 0$. Stokes theory (Temam, 1977) gives that $D : H^\alpha(\partial\Omega_s) \to H^{\alpha+\frac{1}{2}}(\Omega_f)$ is well defined and continuous. In particular, D is continuous from $H^{\frac{1}{2}}(\partial\Omega_s)$ to V.

By the definition of $L^2_\nabla(\Omega_f)$, we have that $\Xi_n = \nabla h_n$ for some $h_n \in H^1(\Omega_f)$. Let Pw denote the average operator given by

$$Pw = \frac{1}{|\Omega_s|} \int_{\Omega_s} w(x)dx, \forall w \in L_1(\Omega_s).$$

Define

$$w_n \equiv h_n - Ph_n \to \nabla w_n = \Xi_n.$$

With this selection we have that

$$\nabla w_n(t) = \Xi_n(t) \rightharpoonup 0 \;\; in \;\; L^2_\nabla(\Omega_s), \forall t \in [0, T]. \tag{21}$$

By Poincaré-Wirtenberg type inequality we infer that for all $t \in [0, T]$

$$|w_n(t)|_s \le C|\nabla w_n(t)|_s = C|\Xi_n(t)|_s. \tag{22}$$

Indeed, the latter follows from the usual compactness-uniqueness argument, where uniqueness depends on the property that $\Xi_n = 0 \to w_n = 0$. Thus, by Lemma 3.4, (22) and Rellich-Kondratiev Theorem, we infer that for all $t \in [0, T]$

$$w_n(t) \to 0 \; strongly \; in \; H^{1-\epsilon}(\Omega_f), \forall \epsilon > 0. \tag{23}$$

In (14) choose for test functions $\psi = w_n$ and $\phi = Dg_n$, where $g_n \equiv w_n|_{\Gamma_s}$ *on* Γ_s with smooth extension to $\partial\Omega_s$, so that the compatibility condition $\int_{\partial\Omega_s} g_n \cdot \nu ds = 0$ holds and $H^{1/2}(\partial\Omega_s)$ norms of the extension are controlled by the same norms $w_n|_{\Gamma_s}$.

We thus have: $Dg_n \in C(0, T; H^1(\Omega_f))$ and since $w_{nt}|_{\Gamma_s} = v_n|_{\Gamma_s} - Pv_n = u_n|_{\Gamma_s} - Pv_n$, we also have $Dg_{nt} \in C(0, T; H^1(\Omega_f))$.

As a consequence

$$Dg_n \rightharpoonup 0 \;\; in \;\; H^1(\Omega_s), \forall t \in [0, T]. \tag{24}$$

REMARK 11. *Note that when $\partial\Omega_s = \Gamma_s$, then for $h_n \in H^1_d(\Omega_s)$ one automatically obtains compatibility condition $\int_{\Gamma_s} w_n \cdot \nu = \int_{\Gamma_s}(h_n - Ph_n)\cdot \nu = 0$.*

Applying variational inequalities with test functions $\phi = Dg_n$ and $\psi = w_n$, adding the resulting equalities and accounting for a cancellation of the boundary terms gives for $t \in \mathbb{R}^+$:

$$(u_{t,n}(t), Dg_n(t))_f + (\nabla u_n(t), \nabla Dg_n(t))_f + ((u_n(t)\cdot\nabla)u_n(t), Dg_n(t))_f$$
$$-\frac{1}{2}\langle u_n \cdot \nu u_n, g_n\rangle_{\partial\Omega_s} + (g(v_n(t)), w_n(t))_s + (v_{t,n}(t), w_n(t))_s + |\Xi_n(t)|^2_s = 0. \quad (25)$$

The task left is to prove that the first six terms go to zero as $n \to \infty$. This follows from Lemma 3.2, (24), (23) and Lemma 3.1, which implies uniform in time convergence

$$(u_{nt}(t), Dg_n(t))_f \to 0, (v_{tn}(t), w_n(t))_s \to 0.$$

Growth condition imposed on $g(s)$ along with Sobolev's embedings imply $|g(v_n(t))|_s \leq M$, hence on the strength of (23) $(g(v_n(t)), w_n(t))_s \to 0$. Feeding this information into (25) yields

$$|\Xi_n(t)|^2_s \to 0$$

as desired to complete the proof of Lemma 3.5.

■

3.5. Strong ω-limit set is $\{0\}$ for any initial data in $\mathcal{H}_0$

The final step in the proof is to show strong stability for arbitrary initial data in $\mathcal{H}_0$. Theorem 2.1 defines semigroup $S_0(t) : \mathcal{H}_0 \to \mathcal{H}_0$, so that for any data $x \in \mathcal{H}_0$, $S_0(t)x$ is weak solution of (14). Lemma 3.5 asserts that this semigroup, when restricted to a dense set $\mathcal{D} \subset \mathcal{H}_0$, is strongly stable. Thus, the proof of strong stability on $\mathcal{H}_0$ entails proving that the nonlinear semigroup $S_0(t)$ describing the flow is locally Lipschitz on $\mathcal{H}_0$. This last property depends critically on two-dimensionality of the domain. The required argument calls for appropriate estimates applied to the difference of two solutions. In the case of two dimensional domains Sobolev's embeddings provide the following needed estimate valid for any elements $X_n, X_0 \in \mathcal{D}$

$$|S(t)X_n - S(t)X_0|_{\mathcal{H}_0} \leq C(|X_0|_{\mathcal{H}_0})|X_n - X_0|_{\mathcal{H}_0}.$$

The uniform boundedness of the semigroup and above local Lipschitz estimate allows to extend by density strong stability from $\mathcal{D}$ onto $\mathcal{H}_0$.

4. Uniform stability under the frictional damping - Proof of Theorem 2.4

The proof of Theorem 2.4 is based on the multiplier's method with projecting operators. As usual, the critical step in proving Theorem 2.4 is the following estimate:

Theorem 4.1. *Under the conditions of Theorem 2.4, there exist a time $T > 0$ and a constant $C_T > 0$, such that the energy at $t = T$ is dominated by the dissipation for all initial condition $(u_0, w_0, w_1) \in \mathcal{H}$:*

$$E(T) \leq H_T \left(\int_0^T D(t)dt \right) \tag{26}$$

where $H_T(s) : R^+ \to R^+$ is a concave, monotone increasing function and zero at the origin. Once Theorem 4.1 is established, using the energy identity (7) and following the nonlinear version of an inductive argument in Lasiecka & Tataru (1993), one is able to demonstrate Theorem 2.4. Indeed, using the fact that the system is autonomous we reiterate the same estimate on the multiple T, which gives

$$E((m+1)T) \leq H_T \left(\int_{mT}^{(m+1)T} D(t)dt \right), m = 0, 1 \dots$$

By the energy identity (8)

$$E((m+1)T) \leq H_T(E(m(T)) - E((m+1)T)$$

$$H_T^{-1}(E((m+1)T)) \leq E(mT) - E(m+1)T$$
$$H_T^{-1}(E((m+1)T)) + E((m+1)T) \leq E(mT).$$

The rest of the argument rests on ODE comparison theorem Lasiecka & Tataru (1993). Thus, the main task is to establish the validity of Theorem 4.1.

5. Proof of Theorem 4.1.

In this step, we will show the *uniform stability* result for the model without static damping. While in the case of strong stability the main mechanism for dissipation of the energy is viscosity of the fluid, for the uniform stability we also exploit frictional damping imposed on the solid. Our goal is to establish the inequality

$$E(T) \leq H_T \left(\int_0^T D(t)dt \right) \tag{27}$$

where $H(s)$ is a concave, continuous function, monotone and zero at the origin. The existence of such function will allow to calculate the decay rates as in Lasiecka & Tataru (1993).

We note that the energy identity gives

$$\int_0^T D(t)dt = \int_0^T |\nabla u|_f^2 dt + \int_0^T (g(v), v)_s dt = E(0) - E(T). \tag{28}$$

This form controls kinetic energy of the wave. Indeed

$$\int_0^T |v|_s^2 dt = \int_{Q_s \cap |v| \geq 1} |v|_s^2 dt + \int_{Q_s \cap |v| \leq 1} |v|_s^2 dt \leq \frac{1}{m} \int_{Q_s} (g(v), v)_s dt + \int_{Q_s} h(g(v)v)dt.$$

Hence, by Jensen's inequality and the monotonicity of h

$$\int_0^T |\nabla u|_f^2 dt + \int_0^T |v|_s^2 dt \leq [\hat{h} + m^{-1} I] \left(\int_0^T (g(v), v)_s dt \right) + E(0) - E(T) \tag{29}$$

where $\hat{h} = \frac{1}{measQ_s} h(measQ_s \cdot)$ on the strength of Jensen's inequality.

The next step is the control of potential energy. The key point of the proof is that rather than using w as the multiplier in equipartition of energy, inspired by Haraux (2006) we will use a different multiplier that is linked to projection on unstable manifold.

In order to achieve equipartition of the energy, we shall use projector operator which allows to separate steady states from the solution, the idea employed for the damped wave equation in Haraux (2006). Let $\{\phi_i\}$ be the orthonormal basis of $L_2(\Omega_s)$, formed by the eigenfunctions of the eigenvalue problem $-\triangle \phi = \lambda \phi$ with Neumann boundary condition $\dfrac{\partial \phi}{\partial \nu} = 0$ satisfying the condition that $0 \leq \lambda_1 \leq ... \lambda_i \leq ...$, where λ_i is the corresponding eigenvalue of ϕ_i. Recall that ϕ_1 is constant, thus let P be the projection from $L_2(\Omega_s)$ to the subspace expanded by ϕ_1, then,

$$Pw = \frac{1}{|\Omega_s|} \int_{\Omega_s} w(x)dx, \ \forall \ w \in L_2(\Omega_s). \tag{30}$$

And a classical version of the Poincaré's inequality states that there exists a constant $C > 0$ depending only on Ω such that

$$|w - Pw|_{L_2} + |w - Pw|_{H^1} \leq C |\nabla w|_{L_2}, \ \forall \ w \in H^1(\Omega_s).$$

Applying the multiplier $w - Pw$ to the equation $v_t = \mathrm{div}\ \Xi + g(v)$ along with integration by parts and $\dfrac{\partial w}{\partial \nu} = 0,\ on\ \partial\Omega_s - \Gamma_s$, yields:

$$(v, w - Pw)_s|_0^T - \int_0^T (v, w_t - Pw_t)_s dt + \int_0^T (g(v), w - Pw)_s dt$$

$$= \int_0^T \langle \Xi \cdot n, w - Pw \rangle dt - \int_0^T (\Xi, \nabla(w - Pw))_s dt.$$

Noting that since Pw is constant in space, $\nabla(w - Pw) = \nabla w = \Xi$, and recalling that $w_t = v$ gives:

$$\int_0^T [|\Xi|^2_{L_2(\Omega_s)} - |v|^2_{L_2(\Omega_s)}] dt = -(v, w - Pw)_s|_0^T - \int_0^T (v, Pv)_s dt + \int_0^T \langle \Xi \cdot \nu, w - Pw \rangle dt + \int_0^T (g(v), w - Pw)_s dt. \tag{31}$$

Step 1: interior damping term:

$$\Big|(g(v), w - Pw)_s\Big| \leq I + II \tag{32}$$

where

$$I = \Big|\int_{\Omega_s, |v| \leq 1} g(v)(w - P(w)) d\Omega_s\Big| \leq C \int_{\Omega_s} |v||w - Pw| d\Omega_s \leq C_\epsilon |v|_s^2 + \epsilon |\Xi|_s^2$$

$$II = \Big|\int_{\Omega_s, |v| \geq 1} g(v)(w - P(w)) d\Omega_s\Big| \leq |g(v)|_{L_r} |w - Pw|_{L_{\overline{r}}} \leq |g(v)|_{L_r} |\Xi|_s$$

$$\leq (E(0))^{\frac{1}{2}} \Big|\int_{\Omega_s} g(v)(g(v))^{r-1} d\Omega_s\Big|^{r^{-1}} \leq (E(0))^{\frac{1}{2}} \Big|\int_{\Omega_s} g(v)(v)^{p(r-1)} d\Omega_s\Big|^{r^{-1}},$$

taking $p(r-1) = 1$

$$\leq (E(0))^{\frac{1}{2}} \Big|\int_{\Omega_s} (g(v))(v) d\Omega_s\Big|^{r^{-1}} \leq C_{E(0)} \Big|\int_{\Omega_s} g(v)(v) d\Omega_s\Big|,$$

In the last step, we used $r = 1 + \dfrac{1}{p} \geq 1$, which is true for any $p > 0$.

Step 2: the boundary term $\int_0^T \langle \Xi \cdot \nu, w - Pw \rangle dt$:

we construct Stoke's solver

$$\begin{aligned} \Delta D g^* &= \nabla q, \ in \ \Omega_f \\ div \ D g^* &= 0, \ in \ \Omega_f \\ D g^*|_{\Gamma_f} &= 0 \\ D g^* &= g^*, on \ \partial\Omega_s \end{aligned}$$

for any g^* such that $\int_{\Gamma_s} g^* \cdot \nu d\Gamma_s = 0$. We define the boundary term g^* as before. $g^* = w|_{\Gamma_s} - Pw$ on Γ_s and then extended smoothly (keeping $H^{1/2}(\Gamma_s)$ norms) and onto $\partial\Omega_s$ while retaining compatibility condition $\int_{\partial\Omega_s} g^* \cdot \nu d\Gamma_s = 0$.

REMARK 12. *When $\partial\Omega_s = \Gamma_s$ and the initial data are in $\mathcal{H}_d$, then $g^* = w|_{\Gamma_s} - Pw$ automatically satisfies the compatibility condition $\int_{\Gamma_s} g^* \cdot \nu = 0$. Thus, there is no need for extension.*

We apply flow inequality with the test function $\phi = Dg^*$. The compatibility condition is satisfied due to compatibility of the initial data and the space $H^1(\Omega_s)$.

$$(u_t, Dg^*)_f - \langle \Xi \cdot \nu, w|_{\Gamma_s} - Pw\rangle + (\nabla u, \nabla Dg^*)_f + \\ ((u \cdot \nabla)u, Dg^*)_f - \langle \frac{1}{2}(u \cdot \nu)u, Dg^*\rangle_{\partial\Omega_s} = 0. \tag{33}$$

Integration by parts in time and taking advantage of the matching on the interface yields:

$$- \int_0^T [(u, D(g_t^*))_f - \langle \Xi \cdot \nu, w - Pw\rangle + (\nabla u, \nabla Dg^*)_f]dt + \\ \int_0^T [((u \cdot \nabla)u, Dg^*)_f - \langle \frac{1}{2}(u \cdot \nu)u, g^*\rangle_{\partial\Omega_s}]dt = (u, Dg^*)_f|_0^T. \tag{34}$$

From here the estimate for the boundary term becomes:

$$\int_0^T \langle \Xi \cdot \nu, w - Pw\rangle dt \leq \int_0^T (u, Dg_t^*)_f + (\nabla u, \nabla Dg^*)_f]dt + \\ \int_0^T [((u \cdot \nabla)u, Dg^*)_f - \langle \frac{1}{2}(u \cdot \nu)u, w - Pw\rangle]dt + (u, Dg^*)_f|_0^T. \tag{35}$$

Step 3: nonlinear terms.

Introducing the bilinear form

$$b(u, v, w) \equiv ((u \cdot \nabla)v, w)_f - \frac{1}{2}\langle (u \cdot \nu)v, w\rangle$$

we obtain by virtue of Sobolev's embedding

$$|b(u, v, w)| \leq C[|u|_{1/2,f}|v|_{1,f}|w|_{1/2,\Omega_f} + |u|_{1/2,f}|v|_{3/4,f}|w|_{3/4,f}]. \tag{36}$$

This gives

$$|b(u, u, Dg^*)| \leq C|u|_{1/2,f}|u|_{1,f}|Dg^*|_{1/2,f} + C|u|_{1/2,f}|u|_{3/4,f}|Dg^*|_{3/4,f}$$

where we recall $g^* = w - Pw$, on Γ_s. Elliptic theory, Temam (1977), and interpolation thus imply

$$|Dg^*|_{1/2,f} \leq |g^*|_{0,\Gamma_s} \leq C|w - Pw|_s^{1/2}|w - Pw|_{1,s}^{1/2}.$$

Similarly

$$|Dg^*|_{3/4,f} \leq |g^*|_{1/4,\Gamma_s} \leq C|w - Pw|_s^{1/4}|w - Pw|_{1,s}^{3/4}.$$

Combining the above

$$\int_0^T |b(u,u,Dg^*)|dt \leq \epsilon \int_0^T |\Xi|_s^2 dt + C_{\epsilon,E(0)} \int_0^T |\nabla u|_f^2 dt. \tag{37}$$

Step 4: time derivatives.

We have

$$g_t^* = v|_{\Gamma_s} - Pv = u|_{\Gamma_s} - Pv, \ \text{on } \Gamma_s.$$

Hence, $|g_t^*|_{1/2,\Gamma_s} \leq C[|\nabla u|_f + |v|_s]$ and

$$|(u, Dg_t^*)_f| \leq C[|\nabla u|_f^2 + |v|_s^2]. \tag{38}$$

Substituting inequalities (37), (38) into (35) gives the final bound for the boundary interface coupling term:

$$\begin{aligned} &\int_0^T \langle \Xi \cdot \nu, w - Pw\rangle dt \\ \leq \ & C\int_0^T |v|_s^2 dt + C\int_0^T [|u|_f(|v|_s + |u|_\Gamma) + |u|_{1,f}|\Xi| + |u|_{1,f}^2|\Xi|_s]dt + E(0) + E(T) \\ \leq \ & C\int_0^T |v|_s^2 dt + C(E(0))[C_\epsilon \int_0^T |u|_{1,f}^2 + \epsilon|\Xi|_s^2]dt + CE(0) + CE(T). \end{aligned} \tag{39}$$

Combining with (31) and taking suitably small ϵ yields

$$\int_0^T |\Xi|_{L_2(\Omega_s)}^2 dt \leq C\int_0^T |v|_{L_2(\Omega_s)}^2 dt + C(E(0))\int_0^T |u|_{1,f}^2 dt + CE(0) + CE(T). \tag{40}$$

(29) and (31) and energy identity and Jensen's inequality imply

$$\int_0^T [|\Xi|_{L_2(\Omega_s)}^2 + |v|_s^2 + |u|_{1,f}^2]dt \leq C(E(0))\int_0^T D(t)dt + h\left(\int_0^T D(t)dt\right) + CE(T). \tag{41}$$

Hence,

$$\int_0^T E(t)dt \leq C_{E(0)}[1 + \hat{h}]\left(\int_0^T D(t)dt\right) + CE(T)$$

$$\frac{1}{2}E(T)T + \frac{1}{2}\int_0^T E(t)dt \le C_{E(0)}[1 + m^{-1} + \hat{h}]\left(\int_0^T D(t)dt\right) + CE(T)$$

and taking $T > 2C$ yields

$$E(T) \le C(E(0))H\left(\int_0^T D(t)dt\right) \tag{41}$$

where $H(s) = s + m^{-1}s + \hat{h}(s) \sim \hat{h}(s)$ for small s. This determines the asymptotic behavior of the ODE.

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