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RESPONSES OF STANDARD AND FRACTIONAL LINEAR SYSTEMS WITH DERIVATIVES OF THEIR INPUTS

Abstract

The responses of continuous-time and discrete-time linear systems with derivatives of their inputs are addressed. It is shown that the formulae for state vectors and outputs are also valid for their derivatives if the inputs and outputs and their derivatives of suitable order are zero for t = 0. Similar results are also shown for the discrete-time linear systems and for the fractional continuous-time and discrete-time linear systems.

INTRODUCTION

Derivation of the response formulae for linear systems is a classical problem of linear systems theory and it has been addressed in many books and papers [1-5, 10-12]. Mathematical fundamentals of fractional calculus and its some applications are given in the monographs [6-9]. Some problems of fractional systems theory and its applications have been considered in [3, 8].

In this paper the following problem is addressed. Under which conditions the well-known formulae for the solutions of the state equations and their outputs are also valid for derivatives of their inputs for standard and fractional continuous-time and discrete-time linear systems.

he paper is organized as follows. In section 2 the problem is analyzed for standard continuous-time linear systems and in section 3 for the standard discrete-time linear systems. An extension of these considerations to fractional continuous-time linear systems is given in section 4 and to the fractional discrete-time linear systems in section 5. Concluding remarks are presented in section 6.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices and $\Re^n = \Re^{n\times 1}$, Z_+ - the set of nonnegative integers, I_n - the $n\times n$ identity matrix.

1. CONTINUOUS-TIME LINEAR SYSTEMS

Consider the continuous-time linear system shown in Fig. 1 with the impulse response matrix $g(t) = \mathcal{L}^{-1}[G(s)]$,

$G(s) = \mathcal{L}[g(t)] = \int_0^\infty g(t)e^{-st}dt$, where $G(s) \in \Re^{p\times m}(s)$ is the transfer matrix, $\mathcal{L}^{-1}$ is the inverse Laplace transform and $\Re^{p\times m}(s)$ is the set of $p\times m$ rational matrices in s.

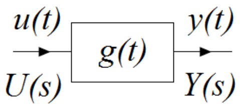

Fig. 1. *Continuous-time linear system.*

The output $y(t) \in \Re^p$ of the system for the input $u(t) \in \Re^m$ and zero initial conditions $x(0) = 0$ is given by

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau\ . \tag{1}$$

The following problem arises. Under which conditions the following equality also holds for the system

$$\dot{y}(t) = \int_0^t g(t-\tau)\dot{u}(\tau)d\tau\ , \tag{2}$$

where $\dot{y}(t) = \frac{dy(t)}{dt}$ and $\dot{u}(t) = \frac{du(t)}{dt}$.

We will prove that (1) implies (2) if and only if $u(0) = 0$ and $y(0) = 0$.

By assumption the initial conditions are zero and $u(0) = 0$ implies $y(0) = 0$. Multiplying the equality $Y(s) = G(s)U(s)$ by s and taking into account that $u(0) = 0$ and $y(0) = 0$ we have

$$sY(s) - y(0) = G(s)[sU(s) - u(0)]\ . \tag{3}$$

Applying the inverse Laplace transform to (3) we obtain (2) since

$$\mathcal{L}[\dot{y}(t)] = sY(s) - y(0) \text{ and } \mathcal{L}[\dot{u}(t)] = sU(s) - u(0)\ . \tag{4}$$

In general case we have the following theorem:

Theorem 1. The equality (1) implies

$$y^{(q)}(t) = \int_0^t g(t-\tau)u^{(q)}(\tau)d\tau\ ,\ q = 1,2,... \tag{5}$$

if and only if

$$u^{(k)}(0) = \left.\frac{d^k u(t)}{dt^k}\right|_{t=0} = 0\ ,\ y^{(k)}(0) = \left.\frac{d^k y(t)}{dt^k}\right|_{t=0} = 0\ ,\ k = 1,2,...\ . \tag{6}$$

Proof. Applying Laplace transform and the convolution theorem to (5) we obtain

$$\begin{aligned}\mathcal{L}[y^{(q)}(t)] &= s^q Y(s) - \sum_{j=1}^{q} s^{q-j}y^{(j-1)}(0) = G(s)\mathcal{L}[u^{(q)}(t)]\\ &= G(s)[s^q U(s) - \sum_{j=1}^{q} s^{q-j}u^{(j-1)}(0)].\end{aligned} \tag{7}$$

For zero initial conditions we have

$$s^q Y(s) = G(s)s^q U(s) = s^q G(s)U(s) \tag{8}$$

and (5) holds if and only if the conditions (6) are satisfied. □

Example 1. Consider the electrical circuit shown in Fig. 2 with given resistance *R*, capacitance *C*, and source voltage $u(t)$.

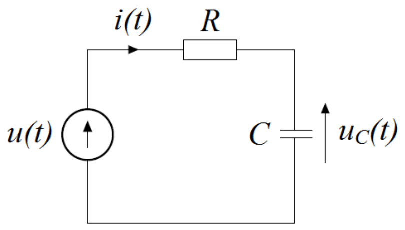

Fig. 2. Electrical circuit

Using Kirchhoff's law and Laplace transform to the electrical circuit we obtain

$$U(s) = sRCU_C(s) + U_C(s) \text{ for } u_C(0) = 0, \tag{9}$$

where $U(s) = \mathcal{L}[u(t)]$, $U_C(s) = \mathcal{L}[u_C(t)]$.

From (9) we have

$$U_C(s) = \frac{U(s)}{RC}\frac{1}{s + \frac{1}{RC}} = \frac{1}{RC}U(s)\mathcal{L}\left[e^{\frac{-t}{RC}}\right]. \tag{10}$$

Using the convolution theorem and inverse Laplace transform to (10) we obtain

$$u_C(t) = \frac{1}{RC}\int_0^t e^{-\frac{(t-\tau)}{RC}} u(\tau)d\tau \tag{11}$$

and

$$\dot{u}_C(t) = \frac{1}{RC}\int_0^t e^{-\frac{(t-\tau)}{RC}} \dot{u}(\tau)d\tau \text{ for } u(0) = 0. \tag{12}$$

Note that for

$$u(t) = U\sin t \tag{13}$$

$u(0) = 0$, but for

$$u(t) = U\cos t \tag{14}$$

$u(0) = U \neq 0$.

Using (12) for (13) we obtain

$$i(t) = C\frac{du_C(t)}{dt} = \frac{U}{R}\int_0^t e^{-\frac{(t-\tau)}{RC}} \cos\tau d\tau. \tag{15}$$

Consider the linear continuous-time system described by the state equations

$$\dot{x} = Ax + Bu, \tag{16}$$

$$y = Cx + Du, \tag{17}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors, respectively and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

The solution to the equation (16) for zero initial conditions $x(0) = x_0 = 0$ has the form

$$x(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau. \tag{18}$$

Substitution of (18) into (17) yields

$$y(t) = C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t). \tag{19}$$

Theorem 2. The equalities (18) and (19) imply, respectively

$$x^{(q)}(t) = \int_0^t e^{A(t-\tau)}Bu^{(q)}(\tau)d\tau,\ q = 1,2,... \tag{20}$$

and

$$y^{(q)}(t) = C\int_0^t e^{A(t-\tau)}Bu^{(q)}(\tau)d\tau + Du^{(q)}(t),\ q = 1,2,... \tag{21}$$

if and only if the condition (6) is satisfied.

Proof. Proof is similar to the proof of Theorem 1.

2. DISCRETE-TIME LINEAR SYSTEMS

Consider the discrete-time linear system shown in Fig. 3 with given the impulse response matrix $g(i) = \mathcal{Z}^{-1}[G(z)]$, $G(z) = \mathcal{Z}[g(i)] = \sum_{i=0}^{\infty} g(i)z^{-i}$, where $G(z) \in \Re^{p\times m}(z)$ is the transfer matrix of the discrete-time system and $\Re^{p\times m}(z)$ is the set of $p\times m$ rational matrices in *z*.

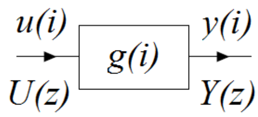

Fig. 3. Discrete-time linear system

The output $y(i) \in \Re^p$ of the system for the input $u(i) \in \Re^m$ and zero initial conditions $x(0) = 0$ is given by

$$y(i) = \sum_{j=0}^{i} g(i-j)u(j). \tag{22}$$

The following problem arises. Under which conditions the following equality holds

$$\Delta y(i) = \sum_{j=0}^{i} g(i-j)\Delta u(j), \tag{23}$$

where $\Delta y(i) = y(i+1) - y(i)$ and $\Delta u(j) = u(j+1) - u(j)$.

We will prove that (22) implies (23) if and only if $u(0) = 0$ and $y(0) = 0$.

By assumption the initial conditions are zero and $u(0) = 0$. Multiplying the equality $Y(z) = G(z)U(z)$ by $(z-1)$ and taking into account that $y(0) = G(z)u(0) = 0$ for $u(0) = 0$ we obtain

$$(z-1)Y(z) - zy(0) = G(z)[(z-1)U(z) - zu(0)]. \tag{24}$$

Applying the inverse Z-transform to (24) we obtain (23) since

$\mathcal{Z}[\Delta y(i)] = (z-1)Y(z) - zy(0)$ and

$$\mathcal{Z}[\Delta u(i)] = (z-1)U(z) - zu(0). \quad (25)$$

In general case we have the following theorem.

Theorem 3. The equality (23) implies

$$\Delta^{(q)} y(i) = \sum_{j=0}^{i} q(i-j)\Delta^{(q)} u(j),\ q = 1,2,... \quad (26)$$

if and only if

$$\Delta^{(k)} u(0) = \sum_{j=0}^{k} (-1)^j \frac{k!}{(k-j)!\,j!} u(k-j) = 0,$$
$$\Delta^{(k)} y(0) = \sum_{j=0}^{k} (-1)^j \frac{k!}{(k-j)!\,j!} y(k-j) = 0 \ \text{for}\ k = 1,2,..., q-1 \quad (27a)$$

or

$$u(i) = 0,\ y(i) = 0 \ \text{for}\ i = 0,1,..., q-1. \quad (27b)$$

Proof. Proof is similar to the proof of Theorem 1.

Consider the linear discrete-time system described by the state equations

$$x(i+1) = Ax(i) + Bu(i),\ i \in Z_+ = \{0,1,...\} \quad (28a)$$

$$y(i) = Cx(i) + Du(i), \quad (28b)$$

where $x(i) \in \Re^n$, $u(i) \in \Re^m$, $y(i) \in \Re^p$, $i \in Z_+$ are the state, input and output vectors, respectively and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

The solution to the equation (28a) for zero initial conditions $x(0) = 0$ has the form

$$x(i) = \sum_{j=0}^{i-1} A^{i-j-1} Bu(j),\ i \in Z_+. \quad (29)$$

Substitution of (29) into (28b) yields

$$y(i) = C\sum_{j=0}^{i-1} A^{i-j-1} Bu(j) + Du(i),\ i \in Z_+. \quad (30)$$

Theorem 4. The equalities (29) and (30) imply, respectively

$$\Delta^{(q)} x(i) = \sum_{j=0}^{i-1} A^{i-j-1} B\Delta^{(q)} u(j),\ i \in Z_+,\ q = 1,2,... \quad (31)$$

and

$$\Delta^{(q)} y(i) = C\sum_{j=0}^{i-1} A^{i-j-1} B\Delta^{(q)} u(j) + D\Delta^{(q)} u(i),\ q = 1,2,... \quad (32)$$

if and only if the condition (27) is satisfied.

Proof. Proof is similar to the proof of Theorem 3.

Example 2. Given the discrete-time linear system (28) with the matrices

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C = [0 \quad 1],\ D = 0 \quad (33)$$

input

$$u(i) = 2(1 - e^{-i}) \quad (34)$$

and zero initial conditions.

The transfer function of the system is equal to

$$G(z) = C[Iz - A]^{-1} B + D$$
$$= [0 \quad 1]\begin{bmatrix} z & -1 \\ 6 & z+5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{6}{z^2 + 5z + 6} \quad (35)$$

and

$$g(i) = \mathcal{Z}^{-1}[G(z)] = \mathcal{Z}^{-1}\left[\frac{6}{z^2 + 5z + 6}\right] = -3(-2)^i + 2(-3)^i. \quad (36)$$

Using (22) and (36) we obtain

$$y(i) = \sum_{j=0}^{i} g(i-j)u(j)$$
$$= \sum_{j=0}^{i} [-3(-2)^{i-j} + 2(-3)^{i-j}] \cdot 2(1 - e^{-j}). \quad (37)$$

Note that (34) satisfies the condition $u(0) = 0$, but $u(1) = 2(1 - e^{-1}) \neq 0$. Therefore, the equalities (31) and (32) are satisfied only for $q = 1$ but are not satisfied for $q = 2,3,...$. From (32) and (37) for $q = 1$ we have

$$\Delta y(i) = \sum_{j=0}^{i} g(i-j)\Delta u(j)$$
$$= \sum_{j=0}^{i} [-3(-2)^{i-j} + 2(-3)^{i-j}] \cdot 2e^{-j}(1 - e^{-1}). \quad (38)$$

3. FRACTIONAL CONTINUOUS-TIME LINEAR SYSTEMS

In this section the following Caputo definition of the fractional derivative will be used [3, 6-9]

$$ {}_0D_t^{\alpha} f(t) = \frac{d^{\alpha} f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau,$$
$$n - 1 < \alpha < n \in N = \{1,2,...\}, \quad (39a)$$

where $\alpha \in \Re$ is the order of the derivative,

$$f^{(n)}(\tau) = \frac{d^n f(\tau)}{d\tau^n} \quad (39b)$$

and

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{s-1} dt \quad (39c)$$

is the Euler gamma function.

Consider the fractional continuous-time linear system

$$\frac{d^{\alpha} x}{dt^{\alpha}} = Ax + Bu,\ 0 < \alpha < 1 \quad (40a)$$

$$y = Cx + Du, \quad (40b)$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors, respectively and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

Applying the Laplace transform to (40) and taking into account that

$$\mathcal{L}\left[\frac{d^\alpha x}{dt^\alpha}\right] = s^\alpha X(s) - s^{\alpha-1}x(0),$$

$$X(s) = \mathcal{L}[x(t)] = \int_0^\infty x(t)e^{-st}dt,\ 0 < \alpha < 1 \tag{41}$$

for zero initial conditions $x(0) = 0$, we obtain

$$X(s) = [I_n s^\alpha - A]^{-1} BU(s),\ U(s) = \mathcal{L}[u(t)]. \tag{42}$$

Taking into account that [3]

$$[I_n s^\alpha - A]^{-1} = \sum_{k=0}^\infty A^k s^{-(k+1)\alpha} \tag{43}$$

we obtain

$$X(s) = \sum_{k=0}^\infty A^k s^{-(k+1)\alpha} BU(s). \tag{44}$$

Using the inverse Laplace transform and the convolution theorem to (44) we obtain [3]

$$x(t) = \int_0^t \Phi(t-\tau)Bu(\tau)d\tau, \tag{45}$$

where

$$\Phi(t) = \sum_{k=0}^\infty \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}. \tag{46}$$

Substitution of (45) into (40b)

$$y(t) = C\int_0^t \Phi(t-\tau)Bu(\tau)d\tau + Du(t). \tag{47}$$

Theorem 5. The equalities (4.7) and (4.9) imply, respectively

$$\frac{d^\beta x(t)}{dt^\beta} = \int_0^t \Phi(t-\tau)Bu^\beta(\tau)d\tau \tag{48}$$

and

$$\frac{d^\beta y(t)}{dt^\beta} = C\int_0^t \Phi(t-\tau)Bu^\beta(\tau)d\tau + Du^\beta(t) \tag{49}$$

if and only if $u(0) = 0$, $y(0) = 0$.

Proof. Multiplying (44) by s^β we obtain

$$s^\beta X(s) - s^{\beta-1}x(0) = \sum_{k=0}^\infty A^k s^{-(k+1)\alpha} B[s^\beta U(s) - s^{\beta-1}u(0)] \tag{50}$$

since by assumption $x(0) = 0$ and $u(0) = 0$.

Applying the inverse Laplace transform to (50) we obtain (48) if and only if $u(0) = 0$. Proof of (49) is similar. □

4. FRACTIONAL DISCRETE-TIME LINEAR SYSTEMS

Consider the fractional discrete-time linear system

$$\Delta^\alpha x(i+1) = Ax(i) + Bu(i),\ i \in Z_+ = \{0,1,...\} \tag{51a}$$

$$y(i) = Cx(i) + Du(i), \tag{51b}$$

where $x(i) \in \Re^n$, $u(i) \in \Re^m$, $y(i) \in \Re^p$, $i \in Z_+$ are the state, input and output vectors, respectively, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$ and the fractional difference of the order α is defined by

$$\Delta^\alpha x(i) = \sum_{j=0}^i (-1)^j \binom{\alpha}{j} x(i-j), \tag{51c}$$

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for} \quad j = 0 \\ \dfrac{\alpha(\alpha-1)...(\alpha-j+1)}{j!} & \text{for} \quad j = 1,2,.... \end{cases} \tag{51d}$$

Substituting (51c) into (51a) we obtain

$$x(i+1) = A_\alpha x(i) + \sum_{j=2}^{i+1} (-1)^{j+1} \binom{\alpha}{j} x(i-j+1) + Bu(i), \tag{52a}$$

where

$$A_\alpha = A + I_n\alpha. \tag{52b}$$

The solution of the equation (52a) has the form [3]

$$x(i) = \Phi(i)x(0) + \sum_{j=0}^{i-1} \Phi(i-j+1)Bu(j), \tag{53}$$

where

$$\Phi(j+1) = A_\alpha\Phi(j) + \sum_{k=2}^{j+1} (-1)^{k+1}\binom{\alpha}{k}\Phi(j-k+1),\ \Phi(0) = I_n. \tag{54}$$

Substitution of (53) into (51b) yields

$$y(i) = C\Phi(i)x(0) + \sum_{j=0}^{i-1} C\Phi(i-j+1)Bu(j) + Du(i). \tag{55}$$

Theorem 6. The equalities (53) and (55) for zero initial condition $x(0) = 0$ imply, respectively

$$\Delta x(i) = \sum_{j=0}^{i-1} \Phi(i-j+1)B\Delta u(j),\ i \in Z_+ \tag{56}$$

and

$$\Delta y(i) = \sum_{j=0}^{i-1} C\Phi(i-j+1)B\Delta u(j) + D\Delta u(i),\ i \in Z_+ \tag{57}$$

if and only if $u(0) = 0$, $y(0) = 0$.

Proof. Using (53) for $x(0) = 0$ we obtain

$$\Delta x(i) = x(i+1) - x(i) = \sum_{j=0}^{i} \Phi(i-j)Bu(j) - \sum_{j=0}^{i-1} \Phi(i-j+1)Bu(j) = \sum_{j=0}^{i-1} \Phi(i-j+1)B\Delta u(j) \tag{58}$$

if and only if $u(0) = 0$. The proof of (57) is similar. □

The considerations can be easily extended to higher order difference.

Theorem 7. The equalities (53) and (55) for zero initial conditions $x(0)=0$ imply

$$\Delta^\alpha x(i)=\sum_{j=0}^{i-1}\Phi(i-j+1)B\Delta^\alpha u(j)\,,\ i\in Z_+\,,\ 0<\alpha<1 \quad (59)$$

and

$$\Delta^\alpha y(i)=\sum_{j=0}^{i-1}C\Phi(i-j+1)B\Delta^\alpha u(j)+D\Delta^\alpha u(i)\,,\ i\in Z_+\,,\ 0<\alpha<1 \quad (60)$$

if and only if $u(0)=0$, $y(0)=0$.

Proof. Using the z-transform to (51c) for zero initial conditions and the convolution theorem we obtain

$$\mathcal{Z}[\Delta^\alpha x(i)]=\sum_{i=0}^{\infty}\Delta^\alpha x(i)z^{-i}=\sum_{i=0}^{\infty}\left[\sum_{j=0}^{i}(-1)^j\binom{\alpha}{j}x(i-j)\right]z^{-i} \quad (61)$$
$$=(1-z^{-1})^\alpha X(z)$$

where $X(z)=\mathcal{Z}[x(i)]$.

The z- transform to (51a) and (51b) for zero initial conditions yields

$$X(z)=[I_n(1-z^{-1})^\alpha-A]^{-1}BU(z) \quad (62)$$

and

$$Y(z)=\{C[I_n(1-z^{-1})^\alpha-A]^{-1}B+D\}U(z)\,, \quad (63)$$

where $U(z)=\mathcal{Z}[u(i)]$.

Multiplying (62) and (63) by $(1-z^{-1})^\alpha$ and using the inverse zet transform and the convolution theorem we obtain (59) and (60), respectively. □

CONCLUDING REMARKS

The responses of continuous-time and discrete-time linear systems with derivatives of their inputs have been addressed. It has been shown that the formulae for state vectors and outputs are also valid for their derivatives if the inputs and outputs and their derivatives of suitable order are zero for t = 0 (Theorem 2). Similar results are also valid for discrete-time linear systems (Theorem 3) and fractional linear systems (Theorem 5 and Theorem 7). The considerations have been illustrated by examples of continuous-time and discrete-time linear systems. The considerations can be extended to fractional positive linear systems.

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ODPOWIEDZI UKŁADÓW LINIOWYCH RZĘDÓW CAŁKOWITYCH I NIE CAŁKOWITYCH Z POCHODNYMI WYMUSZEŃ

Streszczenie

W artykule rozpatrywane są ciągłe układy liniowe oraz dyskretne układy liniowe z pochodnymi (i odpowiednio różnicami) wymuszeń. Pokazano, że wzory określające pochodne wyjścia układów i wektorów stanu są również prawdziwe dla ich pochodnych jeżeli odpowiednie warunki początkowe i ich pochodnych są zerowe. Analogiczne wyniki zostały również wyprowadzone dla układów dyskretnych rzędów całkowitych i niecałkowitych.

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