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INTEGRALS OF THE ONE-DIMENSIONAL CONTINUITY EQUATION

Abstract

The authors analyze the method used by Cauchy and Lagrange to obtain the integral of continuity equation. The authors propose their own method of integration using Schwarz' theorem. As a result, the authors obtain a greater number of possible solutions with a higher level of generality while also being able to identify the basic disadvantages of the Cauchy-Lagrangian method. Further, the authors conducted a detailed interpretation of the results of their solution.

1. OBTAINING THE INTEGRAL OF THE ONE-DIMENSIONAL CONTINUITY EQUATION BY USING THE CAUCHY AND LAGRANGE METHOD

As we know, August Cauchy and Joseph Lagrange, when integrating the Euler equations of motion, used a particular method whose procedure is described in [1]. Let's try to apply this method to integrate a one-dimensional continuity equation.

Let us write this equation in the form:

$$\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_0}\right) = \frac{-\partial}{\partial x}\left(\frac{\rho}{\rho_0}\cdot\mathrm{u}\right) \tag{1.1}$$

The first step of the above-mentioned Cauchy-Lagrangian method will assume that in the space of one-dimensional transient flow there is, as yet to be defined, a variable *η* (Greek letter eta), as expressed through the function:

$$\eta=\eta(x,t) \tag{1.2}$$

which has its partial derivatives.

The next step of this method is an assumption that the following relationship occurs:

$$\frac{\rho}{\rho_0} = \frac{\partial\eta}{\partial x} \tag{1.3}$$

Next, for both sides of (1.3) we carry out a partial derivative operation with respect to time *t*. We get:

$$\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_0}\right) = \frac{\partial^2\eta}{\partial x\partial t} \tag{1.4}$$

And relationship (1.4) transforms into the form:

$$\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_0}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\eta}{\partial t}\right) \tag{1.5}$$

Now we substitute (1.5) into (1.1) and we get:

$$\frac{\partial}{\partial x}\left(\frac{\partial\eta}{\partial t}\right) = \frac{-\partial}{\partial x}\left(\frac{\rho}{\rho_0}\cdot\mathrm{u}\right) \tag{1.6}$$

Then, we group the terms of (1.6) on the left side of the equation:

$$\frac{\partial}{\partial x}\left(\frac{\partial\eta}{\partial t}\right) + \frac{\partial}{\partial x}\left(\frac{\rho}{\rho_0}\cdot\mathrm{u}\right) = 0 \tag{1.7}$$

where, after pulling the symbol of the operator before the brackets, we get the equation in the form that is prepared for integration:

$$\frac{\partial}{\partial x}\left(\frac{\partial\eta}{\partial t} + \frac{\rho}{\rho_0}\cdot\mathrm{u}\right) = 0 \tag{1.8}$$

The integral of the equation (1.8) is:

$$\frac{\partial\eta}{\partial t} + \frac{\rho}{\rho_0}\cdot\mathrm{u}=\mathrm{g}(t) \tag{1.9}$$

where *g(t)* is any function of time.

Therefore, similar to the result of the integration of Euler's equation of motion by the Cauchy-Lagrange method, we only obtain one integral of the continuity equation.

2. OBTAINING THE INTEGRALS OF THE ONE-DIMENSIONAL CONTINUITY EQUATION WITH SCHWARZ' THEOREM

The method of obtaining integrals with Schwarz' theorem was introduced and described in [1], based on the example of the Euler equation of motion. Let's now use it for the continuity equation.

Let the continuity equation have a form similar to (1.1) in this instance:

$$\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_0}\right) = \frac{-\partial}{\partial x}\left(\frac{\rho}{\rho_0}\cdot\mathrm{u}\right) \tag{2.1}$$

Assuming the existence of the flow space of the function:

$$\eta=\eta(x,t) \tag{2.2}$$

as in par. 1, having the partial derivatives of the corresponding order, let's now write Schwarz' theorem for this function:

$$\frac{\partial^2\eta}{\partial x\partial t} = \frac{\partial^2\eta}{\partial t\partial x} \tag{2.3}$$

which we transform to present it in a more spectacular form:

$$\frac{\partial}{\partial t}\left(\frac{\partial\eta}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\eta}{\partial t}\right) \tag{2.4}$$

Now by comparing the left and right sides of the relationships (2.1) and (2.4), respectively, we get two equations:

$$\frac{\partial}{\partial t}\left(\frac{\partial\eta}{\partial x}\right) = \frac{\partial}{\partial t}\left(\frac{\rho}{\rho_0}\right) \tag{2.5}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial\eta}{\partial t}\right) = \frac{-\partial}{\partial x}\left(\frac{\rho}{\rho_0}\cdot\mathrm{u}\right) \tag{2.6}$$

Which we transform into the form:

$$\frac{\partial}{\partial t}\left(\frac{\partial\eta}{\partial x}\right)-\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_0}\right)=0 \quad (2.7)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial\eta}{\partial t}\right)+\frac{\partial}{\partial x}\left(\frac{\rho}{\rho_0}\cdot\mathrm{u}\right)=0 \quad (2.8)$$

We then prepare them for integration:

$$\frac{\partial}{\partial t}\left(\frac{\partial\eta}{\partial x}-\frac{\rho}{\rho_0}\right)=0 \quad (2.9)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial\eta}{\partial t}+\frac{\rho}{\rho_0}\cdot\mathrm{u}\right)=0 \quad (2.10)$$

The integration gives us:

$$\frac{\partial\eta}{\partial x}-\frac{\rho}{\rho_0}=\mathrm{h}(x) \quad (2.11)$$

$$\frac{\partial\eta}{\partial t}+\frac{\rho}{\rho_0}\cdot\mathrm{u}=\mathrm{g}(t) \quad (2.12)$$

Thus, as a result of Schwarz' theorem, we get two integrals of the one-dimensional continuity equation, in which h(x) and g(t) are any functions of x and t.

The second of these integrals, i.e. (2.12), is of course identical to the integral of the equation obtained by the Cauchy-Lagrangian method.

3. OBTAINING A FURTHER INTEGRAL OF THE ONE-DIMENSIONAL CONTINUITY EQUATION WITH THE CAUCHY-LAGRANGIAN METHOD

For this purpose, let's use the continuity equation in the form presented in (1.1), but with its parties multiplied by (-1):

$$\frac{\partial}{\partial t}\left(\frac{-\rho}{\rho_0}\right)=\frac{\partial}{\partial x}\left(\frac{\rho}{\rho_0}\cdot\mathrm{u}\right) \quad (3.1)$$

The difference between equations (1.1) and (3.1) is very important, because the integration of the equation (3.1) with the Cauchy-Lagrangian method gives an integral different than that of (1.9).
For this reason we assume that in a space of one-dimensional transient flow there is, as yet to be defined, a variable ξ (Greek letter ksi) expressed with the function:

$$\xi=\xi(\mathrm{x,t}) \quad (3.2)$$

which has its partial derivatives.
The next step of this method is an assumption that the following relationship occurs:

$$\frac{-\rho}{\rho_0}=\frac{\partial\xi}{\partial x} \quad (3.3)$$

Next, for both sides of (3.3) we carry out a partial derivative operation with respect to time *t*. We get:

$$\frac{\partial}{\partial t}\left(\frac{-\rho}{\rho_0}\right)=\frac{\partial^2\xi}{\partial x\partial t} \quad (3.4)$$

Relationship (3.4) is transformed into the form:

$$\frac{\partial}{\partial t}\left(\frac{-\rho}{\rho_0}\right)=\frac{\partial}{\partial x}\left(\frac{\partial\xi}{\partial t}\right) \quad (3.5)$$

Now we substitute (3.5) to (3.1) and we get:

$$\frac{\partial}{\partial x}\left(\frac{\partial\xi}{\partial t}\right)=\frac{\partial}{\partial x}\left(\frac{\rho}{\rho_0}\cdot\mathrm{u}\right) \quad (3.6)$$

Then, we group the terms of (3.6) on the left side of the equation:

$$\frac{\partial}{\partial x}\left(\frac{\partial\xi}{\partial t}\right)-\frac{\partial}{\partial x}\left(\frac{\rho}{\rho_0}\cdot\mathrm{u}\right)=0 \quad (3.7)$$

where, after pulling the symbol of the operator before the brackets, we get the equation in the form that is prepared for integration:

$$\frac{\partial}{\partial x}\left(\frac{\partial\xi}{\partial t}-\frac{\rho}{\rho_0}\cdot\mathrm{u}\right)=0 \quad (3.8)$$

The integral of the equation (3.8) is:

$$\frac{\partial\xi}{\partial t}-\frac{\rho}{\rho_0}\cdot\mathrm{u}=\mathrm{i}(t) \quad (3.9)$$

where *i(t)* is any function of time.

Therefore, similar to the result of the integration of Euler's equation of motion by the Cauchy-Lagrange method, we only obtain one integral of the continuity equation. However, when comparing the integrals (1.9) and (3.9), we find that they are quite different.

4. OBTAINING A FURTHER INTEGRAL OF THE ONE-DIMENSIONAL CONTINUITY EQUATION WITH SCHWARZ' THEOREM

Like in par. 3 of this study, let's integrate the continuity equation (3.1) once more, but this time with Schwarz' theorem [1]. For methodological reasons, let's rewrite the equation here again, assigning it an appropriate and current paragraph number:

$$\frac{\partial}{\partial t}\left(\frac{-\rho}{\rho_0}\right)=\frac{\partial}{\partial x}\left(\frac{\rho}{\rho_0}\cdot\mathrm{u}\right) \quad (4.1)$$

Assuming the existence of the flow space of the function:

$$\xi=\xi(\mathrm{x,t}) \quad (4.2)$$

which has its partial derivatives.
Let us write Schwarz' theorem for this function:

$$\frac{\partial^2\xi}{\partial x\partial t}=\frac{\partial^2\xi}{\partial t\partial x} \quad (4.3)$$

which we transform to present it in a more spectacular form:

$$\frac{\partial}{\partial t}\left(\frac{\partial\xi}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial\xi}{\partial t}\right) \quad (4.4)$$

Now by comparing the left and right sides of the relationships (4.1) and (4.4), respectively, we get two equations:

$$\frac{\partial}{\partial t}\left(\frac{\partial\xi}{\partial x}\right)=\frac{\partial}{\partial t}\left(\frac{-\rho}{\rho_0}\right) \quad (4.5)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial\xi}{\partial t}\right)=\frac{\partial}{\partial x}\left(\frac{\rho}{\rho_0}\cdot\mathrm{u}\right) \quad (4.6)$$

Which we transform into the form:

$$\frac{\partial}{\partial t}\left(\frac{\partial \xi}{\partial x}\right)+\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_0}\right)=0 \qquad (4.7)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial t}\right)-\frac{\partial}{\partial x}\left(\frac{\rho}{\rho_0}\cdot \mathrm{u}\right)=0 \qquad (4.8)$$

To further transform them into the form prepared for integration:

$$\frac{\partial}{\partial t}\left(\frac{\partial \xi}{\partial x}+\frac{\rho}{\rho_0}\right)=0 \qquad (4.9)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial t}-\frac{\rho}{\rho_0}\cdot \mathrm{u}\right)=0 \qquad (4.10)$$

The integration gives us:

$$\frac{\partial \xi}{\partial x}+\frac{\rho}{\rho_0}=\mathrm{j}(x) \qquad (4.11)$$

$$\frac{\partial \xi}{\partial t}-\frac{\rho}{\rho_0}\cdot \mathrm{u}=\mathrm{i}(t) \qquad (4.12)$$

Thus, as a result of Schwarz' theorem, we get two integrals of the one-dimensional continuity equation, in which *j(x)* and *i(t)* are any functions of *x* and *t*.

The second of these integrals, i.e. (4.12), is of course identical to the integral (3.9) of this equation obtained by the Cauchy-Lagrangian method.

It should be noted that the two integrals of the one-dimensional continuity equation obtained in the current section are actually two further integrals of the continuity equation. This is due to the explicit comparison of the complete results of the integration in (2.11), (2.12) and in (4.11), (4.12), respectively. The presence of these two sets of results from the integration of the one-dimensional continuity equation shows that the equation is satisfied by two different functions: ξ and η.

5. IDENTIFICATION OF THE FUNCTIONS KSI AND ETA. VERIFICATION OF THE RESULTS OF THE INTEGRATION OF THE ONE-DIMENSIONAL CONTINUITY EQUATION

Unlike the historical Cauchy-Lagrangian integral of the Euler equations of motion, in which the indistinct physical function φ was interpreted[1], we are currently in a quite comfortable situation, because in par. 3 of this study we have obtained a set of two one-dimensional integrals of the continuity equation that were previously known [2].

In ref [2], on the basis of independent reasoning that led to the articulation of some of the properties of the one-dimensional continuity equation, formulas were derived for the partial derivatives of the function ξ; see (3.11) and (3.13) in [2].

In addition, here it turns out that the function ξ is a identifiable physical quantity. It is the function that expresses the position of liquid planes in a flow with respect to the position occupied by those planes while the fluid is at rest.

Also, the function η can be identified as expressing the initial position of the liquid planes – counted starting from 0 on the *x* axis – for the system of coordinates *x, t*. The following Figure 1 contains an explanation of the situation described herein.

In the figure, the following are represented on plane *x,t*:

- The line t_0E'A' is the trajectory of the piston, which begins its movement at time t_0, when time $t = \theta$ (Greek letter theta) is at the point E' with coordinates $\xi_1(\theta)$, and at time *t* reaches point A', with coordinates $\xi_1(t)$;
- The line t_0CD is the trajectory of the flow front, which at time $t = \theta$ is at point C with coordinates $x_2(\theta)$, and at time *t* reaches point D, with coordinates $x_2(t)$;

The area between lines t_0E'A' i t_0CD is the area of flow. The area above line t_0CD is occupied by the fluid at rest.

At any point of B'(x,t) – in the area of flow on the plane *x,t* – a liquid plane is located, in which the fluid at rest has coordinates $x_2(\theta)\equiv\eta$, at point B' has coordinates with respect to $x_2(\theta)$ equal to $\xi(x,t)$, while its coordinate – calculated starting from 0 on the *x* axis – is *x*. It is obvious that the following relationship exists:

$$x_2(\vartheta)=x-\xi(x,t) \qquad (5.1)$$

which – because of the arbitrary choice of point B' – will be preserved throughout the flow area and therefore can be written as a relation of general relevance:

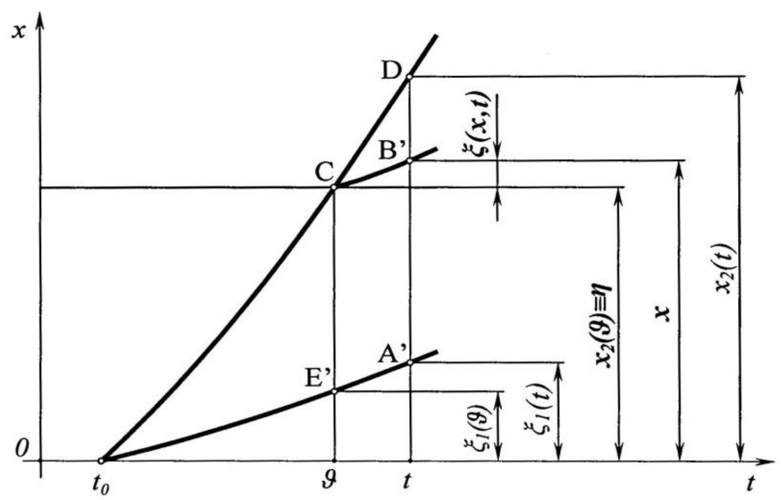

Fig. 1. *The system of coordinates x, t with an interpretation of functions ξ and η.*

[1] Function φ is well-known as a potential of velocity.

$$\eta = x - \xi \tag{5.2}$$

Let us rewrite the following formulas (3.11) and (3.13) from [2], giving them the current numbering:

$$\frac{\partial \xi}{\partial x} = 1 - \frac{\rho}{\rho_0} \tag{5.3}$$

$$\frac{\partial \xi}{\partial t} = \frac{\rho}{\rho_0} \cdot \mathrm{u} \tag{5.4}$$

In this study, these formulas - with simple transformations - can be presented in the following forms:

$$\frac{\partial}{\partial x}(x - \xi) = \frac{\rho}{\rho_0} \tag{5.5}$$

$$\frac{\partial}{\partial t}(x - \xi) = \frac{-\rho}{\rho_0} \cdot \mathrm{u} \tag{5.6}$$

For the full effect, using the relationship (5.2) let's transform formulas (5.5) and (5.6) to take the form of:

$$\frac{\partial \eta}{\partial x} = \frac{\rho}{\rho_0} \tag{5.7}$$

$$\frac{\partial \eta}{\partial t} = \frac{-\rho}{\rho_0} \cdot \mathrm{u} \tag{5.8}$$

In this way, by transforming the integrals of the one-dimensional continuity equation (5.3) and (5.4) by using (5.2), we get a second (further) set of two integrals of the equation.
It appears that the result of integrating the one-dimensional continuity equation consists of two sets of two integrals expressing the partial derivatives of the two functions ξ and η.
As you can see, these functions meet the one-dimensional continuity equation in an elementary way.
Let's write down here the basic properties of function η:

- this function, like function ξ, is expressed in units of length,
- along the trajectories of liquid planes has constant values, i.e. $\eta = const.$,
- along the trajectory of piston [2] resets, i.e. $\eta = 0$, because on this line $x = \xi$,
- along the trajectory of flow front [2] $\eta = x$, because on this line $\xi = 0$.

The results of integrating the one-dimensional continuity equation are obtained in the form of general integrals, because any functions of x and t are presented in them as an unknown but are formally required by the procedure.
Having now two sets of integrals (5.3), (5.4) and (5.7), (5.8) of this equation, respectively, in which the integrals may be considered as specific, it is possible to verify all integrals obtained in this study by determining the values which will be accepted by the aforementioned functions of x and t.
Let's start with the one-dimensional integrals of the continuity equation obtained using Schwarz' theorem, i.e. from (2.11), (2.12) and (4.11), (4.12), respectively.
Substituting these formulas, (5.7), (5.8) and (5.3), (5.4), we obtain:

$$h(x) \equiv 0 \tag{5.9}$$

$$g(t) \equiv 0 \tag{5.10}$$

and:

$$j(x) \equiv 1 \tag{5.11}$$

$$i(t) \equiv 0 \tag{5.12}$$

Let's now try to do the same with the one-dimensional integrals of the continuity equation obtained using the Cauchy-Lagrangian method, i.e. (1.9) and (3.9).
Substituting these formulas, (5.8) and (5.4) respectively, we then obtain:

$$g(t) \equiv 0 \tag{5.13}$$

$$i(t) \equiv 0 \tag{5.14}$$

of course, according to (5.10) and (5.12).

6. INTERPRETATION OF RESULTS

In light of the effects of the Cauchy-Lagrangian method for the integration of the one-dimensional continuity equation presented in par. 1, 3, and 5 of this paper, we can determine the assumptions (1.3) and (3.3) and their role in the procedure leading to the formation of the integrals. This determination of the suitability and role of these assumptions (as above) will also be related to the well-known assumptions made by Cauchy and Lagrange in their procedure for integrating the Euler equations of motion.
Since Cauchy and Lagrange have achieved indisputable historical precedence, let's now try to go a little further in our analysis of their course of thought presented in [1].
We can certainly assume that the assumptions of Cauchy and Lagrange:

$$u = \frac{\partial \varphi}{\partial x} \tag{6.1}$$

where u is the velocity of a fluid and φ is a potential of velocity, could not be conceived as a guess of one of the integrals[2] of the Euler equations of motion! Therefore (6.1) could be an expression that – when substituted for the integrated equation – would only allow us to obtain an integral.
Thus, the above assumption made by Cauchy and Lagrange is only a substitution that allowed the integration of the Euler equations of motion by the "substitution method". From here, the purpose of these assumptions it quite clearly revealed – both Cauchy and Lagrange utilized them for the integration of the Euler equations of motion, as well as (1.3) and (3.3) in this paper – for the integration of the one-dimensional continuity equation and their role in the procedure leading to the integrals.

At this point, when considering the method Cauchy and Lagrange applied to the integration of partial differential equations, it becomes possible to formulate comments and summarize that method.
Apart from this imperfection, which affects only one integral, let's take a closer look at the assumptions, which are only formally (technically) sufficient for the existence of such a method.
As a necessary condition to obtain the valued and real results of integration, it is an obvious requirement that these assumptions not be false.

[2] If so, then the expression (6.1) should include any function of x.

Let's examine the Cauchy and Lagrange assumption (6.1) for the integration of the Euler equation of motion, and further assumptions (1.3) and (3.3) for the integration of the one-dimensional continuity equation.
We can see that assumption (1.3) – in the Cauchy-Lagrangian procedure of integration for the one-dimensional continuity equation – is true, because it leads to the solution (1.9) in accordance with (2.12), and therefore with (5.8), being in line with the (2.11), and therefore with (5.7).
Whereas the assumption (3.3) – in the procedure above – is false because although it leads to the solution (3.9) in accordance with (4.12), and therefore with (5.4), it finds itself against the work demonstrated in (4.11), and therefore with (5.3).
Although a classification of the truth or falsity of the assumptions (1.3) and (3.3) was made using the obvious criteria, which were expressions of the integrals of the one-dimensional continuity equation known from [2], for the making of such classification of assumptions (6.1) – the assumptions made by Cauchy and Lagrange – we surely lack any of the necessary criteria.
Indeed, we are not quite sure whether assumption (6.1) – leading to the famous Euler integral equations of motion – is true or false.
It follows that the risk of adopting a false assumption for the integration of partial differential equations using the Cauchy-Lagrangian method could be avoided, just in case it were possible to guess one of the of integrals of the equation in its full form.

CONCLUSIONS

Par. 6 presented the basic shortcomings of the Cauchy-Lagrangian method of integration of partial differential equations.
However, in par. 2 and par. 4, the method that uses the Schwarz' theorem was applied to the integration of the one-dimensional continuity equation.

As you can see here, when comparing the two procedures, the method of integrating partial differential equations using Schwarz' theorem is free of the shortcomings of the Cauchy-Lagrangian method. The assumptions in (1.3) and (3.3) are simply not needed. Having to consider guessing one of the of integrals of the equation in its full form is not necessary either.

The integration of partial differential equations using Schwarz' theorem – in general case – allows you to obtain two integrals of the equation.

In this study, the result of the integration of the one-dimensional continuity equation using this method show two sets of two integrals for the two different functions of ξ and η.

REFERENCES

1. Dowkontt Szymon, Dowkontt Gerard, *The Cauchy and Lagrange integral of the Euler equation of motion.* TTS Technika Transportu Szynowego 2015, nr 12, p. 423-426.
2. Dowkontt Gerard, *Jednowymiarowy, izentropowy przepływ nieustalony, Próba nowego postawienia zadania*. Prace Instytutu Maszyn Przepływowych, Gdańsk, 1991.

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