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Numerical investigation of geometrically nonlinear clamped uniform rods and rods with sections varying exponentially free vibration

E. Abdeddine *, A. Majid, N. Bouzida, Z. Beidouri, Kh. Zarbane

Laboratory of Advanced Research in Industrial and Logistic Engineering (LARILE),
National Higher School of Electricity and Mechanics, Hassan II University of Casablanca,
Km7 Route El Jadida, Casablanca, Morocco

* Corresponding e-mail address: e.abdeddine@ensem.ac.ma

ORCID identifier: https://orcid.org/0000-0002-0385-3928 (E.A.)

ABSTRACT

Purpose: The present paper is intended to investigate the problem of linear and non-linear longitudinal free vibration of uniform rods and rods whose cross-sections vary exponentially at large vibration amplitudes.

Design/methodology/approach: The method adopted consists in discretizing the energy term on linear k_{ij} and non-linear rigidity tensor b_{ijkl}, as well as the mass tensor m_{ij}. Therefore, the formulation of this structure is based on Lagrange equations and the harmonic balance method so as to obtain the nonlinear algebraic equations. These latter are solved numerically and analytically through the explicit and linearized method.

Findings: The response of Clamped-Clamped uniform and non-uniform rods on our structure are highlighted in the amplitude frequency and associated first three mode shapes. Moreover, this research leads to study the influence of the exponential slope on the maximum displacement, thus emphasizing the non-uniform bars usefulness. The obtained results are then compared with the available literature with a view to validating this theory.

Research limitations/implications: As a perspective, the method used in this paper would be pushed to study the FDM material, taking into account other parameters related to additive manufacturing, and later to be validated experimentally.

Practical implications: Longitudinal vibrations are important in mechanical structures; therefore, the determination of their dynamic behaviour needs to be understood. In the present study, the effect of the displacement amplitude on the exponential slope of the structure was analysed, which led to the determination of the reduction range of the vibration amplitude under resonance. However, this should be taken into account in the design process. Besides, the usefulness of the non-linearity geometric effects was demonstrated to examine these structures by considering all the parameters involved.

Originality/value: A linearized procedure is used to solve a nonlinear algebra equation. The use of this method leads to reduce calculation time contrary to iterative methods.

Keywords: Non-linear longitudinal vibration, Lagrange equations, Harmonic balance method, Linearized approach

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ANALYSIS AND MODELLING

1. Introduction

The longitudinal vibration analysis of rods is a critical concern in structural engineering applications such as high-rise buildings, long-span bridges, aerospace vehicles and many other industrial usages. In the literature, there are many studies that deal with vibration [1,2]. [3], for instance, used a new approximated analytical solution of the free vibration analysis to evaluate the natural frequencies of functionally graded rectangular sandwich plates with porosities. [4], for the same issue, were based on the classical plate theory. Other studies address the vibration in the case of sandwich beams with different core metals and thicknesses [5] and the case of a fully cracked concrete beam using a theoretical formulation of elasto-dynamics behaviour [6]. Longitudinal and torsional vibrations are significant in these structures and their natural frequencies have to be considered in the design process. Exact solutions and numerical techniques for linear longitudinal vibration of homogeneous rods can be found in the literature [7], [8], however, analytical studies on inhomogeneous rods are scarce. Eisenberger [9], Matsuda and al [10], Bapat [11], Abrate [12], Kumar and Sujith [13], Li and al [14], Guo and al [15], and Anil Raj and Sujith [16] have studied the linear longitudinal vibration for rods with the variable sectional area and have obtained exact solutions for certain functional forms of an involved parameter. The large vibration amplitude of rods used in Pressurized Water Reactors (PWR) has been investigated numerically and experimentally in [17] following the formulation using by Ferrari et al. [18], [19]. In fact, this phenomenon is caused by the external force generated by fretted fluid at the interface of the fuel rods as reported in reference [20]. The obtained numerical and experimental results showed the softening behaviour of rods. Xu et al. [21] developed an exact solution for the longitudinal vibration of a general variable cross-segment rod with arbitrary boundary conditions, using both the summation of standard Fourier series and supplementary polynomials to express the displacement function. Thus, this approach leads to an estimation of all subsidiaries all the more direct throughout the whole solving region. Besides, to demonstrate the effect of cross-section area variance on vibration characteristics of non-uniform rods, Xu et al. [21] have recently employed various boundary conditions such as clamped and elastic boundaries. The results inferred from the study of Xu et al. 2011 are compared with those obtained in the present study. Additionally, in the study conducted by Li et al. [22], a longitudinal free vibration of a one-step non-uniform bar equation was reduced to Bessel's equation by selecting reasonable articulations. This latter is utilized to get the mode shape functions of a multi-step bar with or without lumped masses and spring supports. As a result, they demonstrated that the determined natural frequencies and mode shapes of a high-rise structure were in acceptable concurrence with the relating experimental data. Therefore, the present work is devoted to investigating the non-linear longitudinal vibration of rods using a model based on Lagrange equations and harmonic balance method. This problem is reduced to a non-linear algebraic system, which can be solved by numerical (iterative or linearized) or explicit procedures. These latter have been adopted and used previously to examine the non-linear behaviour of continuous structures and discrete systems (transverse vibration of beams) [23-28], longitudinal and transverse vibration of 2-dof systems [29]. In our case, an application is made using the linearized procedure [23,30] leading to the fundamental non-linear longitudinal mode shape of C-C (Clamped-Clamped) rods. The second section treated an application of the linear case, which was represented by the uniform rods and rods with section varying exponentially. Indeed, the solution of the linear case was determined and included in the nonlinear solution case which was detailed in the third section. Lagrange equations and harmonic balance method were used to formulate the present problem, and numerical method was applied to solve the obtained non-linear algebraic equation in the latter section. Regarding the fourth section, it outlined the linearized method that was used as a direct solution of a modified eigenvalue problem. The employed method in this paper was adopted because it was validated by several studies. Additionally, this method was used to investigate the rectangle plate, and provided good arguments comparing with the iterative and explicit methods. Section five was devoted to reporting the obtained findings of the nonlinear longitudinal study for both uniform and non-uniform rods cases, that highlight the geometrically nonlinear effect. The conclusion was eventually reported in section six.

2. Linear model

In this section, an overview of the natural frequencies, mode shapes and displacements of C-C uniform rods formulas is provided. Afterwards, new findings are presented for rods, where the cross sections vary exponentially, are clamped at both ends.

2.1. Overview of mathematics formulation for uniform rods

The uniform rods having the following characteristics: *E*: Young's modulus, ρ : mass per unit volume, L : rod length, *S*: the cross-section area, *U(x,t)*: the longitudinal displacement assumed to be harmonic, i.e. *U(x,t)* = *u(x)*sin*(ωt)*, are well-known by the equation of longitudinal vibrations of rods:

$$\frac{\partial}{\partial x}\left(S\frac{\partial U}{\partial x}\right) = \frac{S}{c^2}\frac{\partial^2 U}{\partial t^2} \quad (1)$$

The propagation velocity of the displacement or stress waves in the rod is then equal to:

$$C = \sqrt{\frac{E}{\rho}} \quad (2)$$

It can be noticed that in the equation (2) , C does not depend on the section *S*. For a C-C rod, the displacement at the ends must be equal to zero: $u(0,t) = 0$ and $u(L,t) = 0$. Since these boundary conditions ought to be satisfied at any time *t*. The frequency of vibration is thus given by:

$$\omega_n = n\frac{\pi C}{L}$$

where n represents the mode order. The associated mode shape can be then written as:

$$u_n(x) = sin\frac{n\pi}{L}x$$

2.2. Rods with sections varying exponentially

The longitudinal vibrations of the rod with a variable section (Fig. 1), are considered taking into account the aforementioned characteristics. Hence, the section S is supposed to vary exponentially as the following equation (3):

$$S(x) = S_0\, exp(\delta x) \quad (3)$$

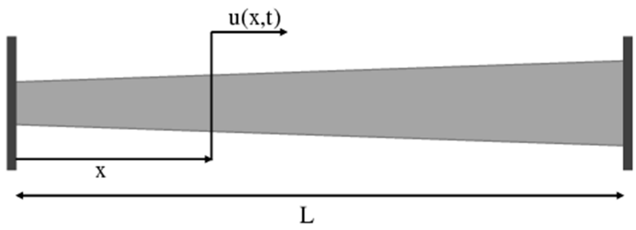

Fig. 1. Notation for longitudinal vibration of a rod with sections varying exponentially

By substituting the equation (3) into (1) and using the classical method of separation of variables, we obtain the following differential equation:

$$\frac{\partial^2 u}{\partial x^2} + \delta\frac{\partial u}{\partial x} + \frac{\omega^2}{C^2}u = 0 \quad (4)$$

Assuming a solution $u(x) = \exp(rx)$, which lead to the general solution of the equation (4) can be written as:

$$u(x) = e^{-\frac{\delta}{2}x}(A\,sin(\beta x) + B\,cos(\beta x))$$

with:

$$\beta = \sqrt{\left(\frac{\omega^2}{C^2} - \frac{\delta^2}{4}\right)}$$

Using the boundary conditions in the case of a C-C rod, the solution of the differential equation gives the following mode:

$$u_n(x) = \left(sin\frac{n\pi}{L}x\right)e^{-\frac{\delta}{2}x}$$

and the corresponding the frequencies as:

$$\omega_n = \frac{1}{2}\sqrt{\frac{E}{\rho}}\sqrt{\left(\frac{4\pi^2 n^2}{L^2} + \delta^2\right)}$$

with:

$$\beta_n = \frac{1}{2}\sqrt{\left(4\frac{\rho}{E}\omega_n^2 - \delta^2\right)}$$

3. Non-linear model

The main focus of the present paper is to determine the non-linear longitudinal mode shapes and natural frequencies of C-C uniform rods and rods with section varying exponentially. The section 3.1 looks at the formulation of the basic theory of geometrically non-linear longitudinal free vibration of the studied structures. The application of the method on the rods with a variable section and in the case of C-C ends free vibration will be represented in sections 4 and 5.

Theoretical formulation

The nonlinear dynamic behaviour for a conservative system, which is obtained by the application of Lagrange equations, can be written as the following equation (5) when no forcing term is considered:

$$-\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_r}\right)+\frac{\partial T}{\partial q_r}-\frac{\partial V}{\partial q_r}=0, \quad r=1,\dots,n \tag{5}$$

where T is the kinetic energy of the rod expressed by:

$$T=\frac{1}{2}\int_0^L \rho S\left(\frac{\partial u}{\partial t}\right)^2 dx \tag{6}$$

V represent the total strain energy, which can be given as the sum of the linear strain energy V_l, and the nonlinear strain energy V_{nl} as follows:

$$V=V_l+V_{nl}$$

$$V_l=\frac{1}{2}\int_0^L ES\left(\frac{\partial u}{\partial x}\right)^2 dx \tag{7}$$

$$V_{nl}=\frac{1}{2}\left[\int_0^L ES\left(\frac{\partial u}{\partial x}\right)^3 dx+\frac{1}{4}\int_0^L ES\left(\frac{\partial u}{\partial x}\right)^4 dx\right]$$

Assuming that the motion is harmonic, the expanding longitudinal displacement (U) in the form of a finite series is defined as:

$$U(x,t)=q_i(t)u_i(x)=a_iu_i(x)\sin\omega t \tag{8}$$

where the usual summation convention for the repeated index i is used. $\{u_i, i=1,\dots,n\}$ is the set of the assumed series of n spatial trial functions. Substituting the function U in the energy equations (6)-(7) by the expression given in equation (8) and proceeding with discretization. Thus, the equations corresponding to the potential and kinetic energies are obtained as mentioned below:

$$T=\frac{1}{2}\dot{q}_i\dot{q}_jm_{ij}$$

$$V_l=\frac{1}{2}q_iq_jk_{ij}$$

$$V_{nl}=\frac{1}{2}q_iq_jq_kq_lb_{ijkl}+\frac{1}{2}q_iq_jq_kc_{ijk}$$

where k_{ij} denotes the classical rigidity tensor due to V_l. b_{ijkl} and c_{ijk}, represent the non-linearity tensors due to V_{nl} and m_{ij} stands for the mass tensor attributable to T. The expressions for the general terms of these tensors are given by:

$$m_{ij}=\int_0^L \rho Su_iu_jdx$$

$$k_{ij}=\int_0^L ES\frac{\partial u_i}{\partial x}\frac{\partial u_j}{\partial x}dx$$

$$c_{ijk}=\int_0^L ES\left(\frac{\partial u_i}{\partial x}\frac{\partial u_j}{\partial x}\frac{\partial u_k}{\partial x}\right)dx$$

$$b_{ijkl}=\int_0^L \frac{ES}{4}\left(\frac{\partial u_i}{\partial x}\frac{\partial u_j}{\partial x}\frac{\partial u_k}{\partial x}\frac{\partial u_l}{\partial x}\right)dx$$

It should be noted that the non-dimensional parameters are obtained from the study [24]:

$$u_i(x)=Lu_i^*\left(\frac{x}{L}\right)=Lu_i^*(x^*);\frac{\omega^2}{\omega^{*2}}=\frac{E}{\rho L^2};$$

$$\frac{m_{ij}}{m_{ij}^*}=\rho SL^3;\frac{k_{ij}}{k_{ij}^*}=ESL;\frac{b_{ijkl}}{b_{ijkl}^*}=ESL$$

The amplitudes a_i are the unknowns to be determined as well as the frequency ω. Inserting T and V in equation (5) by their expressions given previously. By applying the harmonic balance method leads to the following set of nonlinear algebraic equations:

$$2a_ik_{ir}^*+3a_ia_ja_kb_{ijkr}^*-2\omega^{*2}a_im_{ir}^*=0 \quad r=1,\dots,n \tag{9}$$

Which can be written in matrix form as:

$$2[K^*]\{A\}+3[B^*(\{A\})]\{A\}-2\omega^{*2}[M^*]\{A\}=0 \tag{10}$$

where ω^* is the non-dimensional response of the nonlinear frequency parameter. The expression of this later can be obtained by pre-multiplying equation (10) by $\{A\}^T$, as below:

$$\omega^2=\frac{\{A\}^T[K]\{A\}+\frac{3}{2}\{A\}^T[B(\{A\})]\{A\}}{\{A\}^T[M]\{A\}} \tag{11}$$

The system (9) can then be written as:

$$3a_i a_j a_k b^*_{ijkr} + 2a_i k^*_{ir}$$

$$-2\left(\frac{a_i a_j k^*_{ij} + \frac{3}{2} a_i a_j a_k a_1 b^*_{ijkl}}{a_i a_j m^*_{ij}}\right) a_i m^*_{ir} = 0; \quad r = 1, \dots, n \qquad (12)$$

Equation (12) is a non-linear algebraic system which can be solved using linearized procedure. The later equation has been established, in order to study the displacement dependence for the first, second and the third nonlinear longitudinal mode shapes of C-C uniform and non-uniform rods.

4. Solution procedure

Various non-linear vibration models provide a solution for a non-linear algebraic system, that is formally similar to equation (11) with the following formula:

$$\left([K_l] + \frac{3}{2}[B(A)]\right)\{A\} - \omega^2[M]\{A\} = \{0\} \qquad (13)$$

$$\left([K_l] + \frac{3}{2}[K_{nl}]\right)\{A\} = \omega^2[M]\{A\} \qquad (14)$$

Each term of the matrix $[K_{nl}]$ is a quadratic function of the column matrix of coefficients A, and is given by $(K_{nl})_{ij} = (3/2)a_k a_l b_{ijlk}$. It can be seen that when the non-linear term is neglected, the non-linear eigenvalue problem (14) is reduced to the classical linear eigenvalue problem:

$$[K_l]\{A\} = \omega^2[M]\{A\} \qquad (15)$$

The equation above represents the Rayleigh-Ritz formulation of the linear vibration problem. In the linear case, the eigenvalue equation (15) leads to a series of eigenvalues and corresponding eigenvectors. Regarding the non-linear case, the solution of Eq. (14) should provide a set of amplitude dependent eigenvectors with their associated amplitude dependent eigenvalues. However, to solve the non-linear eigenvalue problem (14), incremental-iterative methods are generally used. Furthermore, a simplified alternative method, which leads to explicit solutions, has been developed and applied to various vibration problems [31]. Indeed, this method intends to replace the iterative method applied to solve the non-linear eigenvalue system (14) by using a direct solution of a modified linear eigenvalue problem. Accordingly, in similar problems, the obtained numerical results indicate that the contribution coefficient a_c of the assigned basic function (or linear mode shape) remains predominant for a wide range of amplitudes compared to the other basic function (or linear mode shape contribution coefficients). Based on this finding, only the resonance neighbourhoods are taken into consideration. And therefore, the general term of the non-linear geometrical stiffness matrix is represented in the following simplified form:

$$(K_{nl})_{ij} = \frac{3}{2} a_c^2 b_{ccij}$$

The non-linear eigen value problem (15) becomes

$$[K]\{A\} = \omega^2[M]\{A\}$$

where [K] is the modified global stiffness matrix of the structure which is defined as:

$$K_{ij} = (K_l)_{ij} + \frac{3}{2} a_c^2 b_{ccij} \qquad (16)$$

However, for a given value of the predominant contribution a_c of the considered mode, the modified rigidity matrix [K] is constant, and thus the eigenvalue problem (16) is a classical one. For a specific value of a_c, the direct solution of the linear eigenvalue problem (16); which is related to a fixed amplitude of a selected point vibration; leads to the eigenvalue ω^2 and the corresponding eigenvector A. These parameters are normalized in such a way that their c^{st} components are accurately the specified value of a_c assigned to each one at the beginning of the process. Through a previous study [32], the linearized method proved that the application of the geometrically non-linear free vibration of rectangular plates and discrete systems gave good results, and subsequently avoided the need for iterative methods. It can also be applied to the solution of any non-linear algebraic systems, after rewriting it in modal basis. In fact, using the iterative method allowed to solve the whole matrix of rigidity nonlinear (b_{ijkl}), which has a very considerable number of terms. For instance, 19 basic functions were used to examine the third mode shape, then 19^4=130 321 terms of the non-linear rigidity must be calculated. Conversely, 19^2=361 terms can be used to describe the present problem via the linearized method, which ended up with a modified global stiffness matrix (16).

The selection of the basic functions number is based on the assignment contribution coefficients of the other mode shapes. Figures 2a,b,c depict the contribution coefficients value of the first, second and third longitudinal mode shapes.

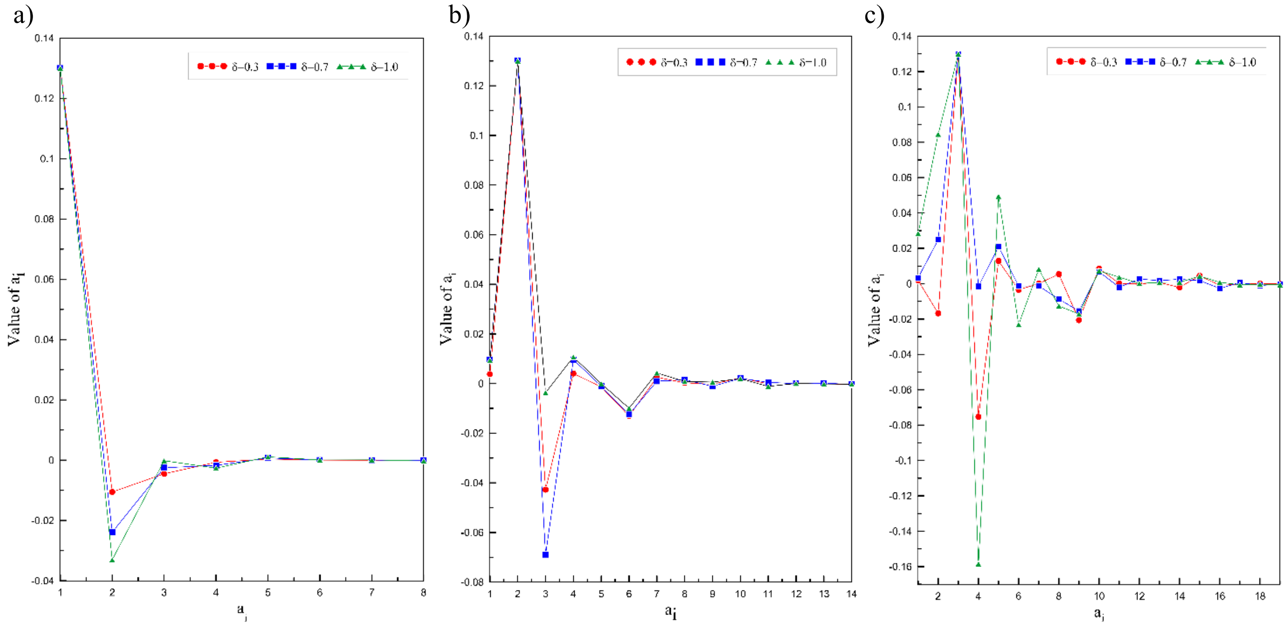

Fig. 2. Contribution coefficients value of mode shapes for various values of δ for the first three mode shapes: a) First mode shape, b) Second mode shape, c) Third mode shape

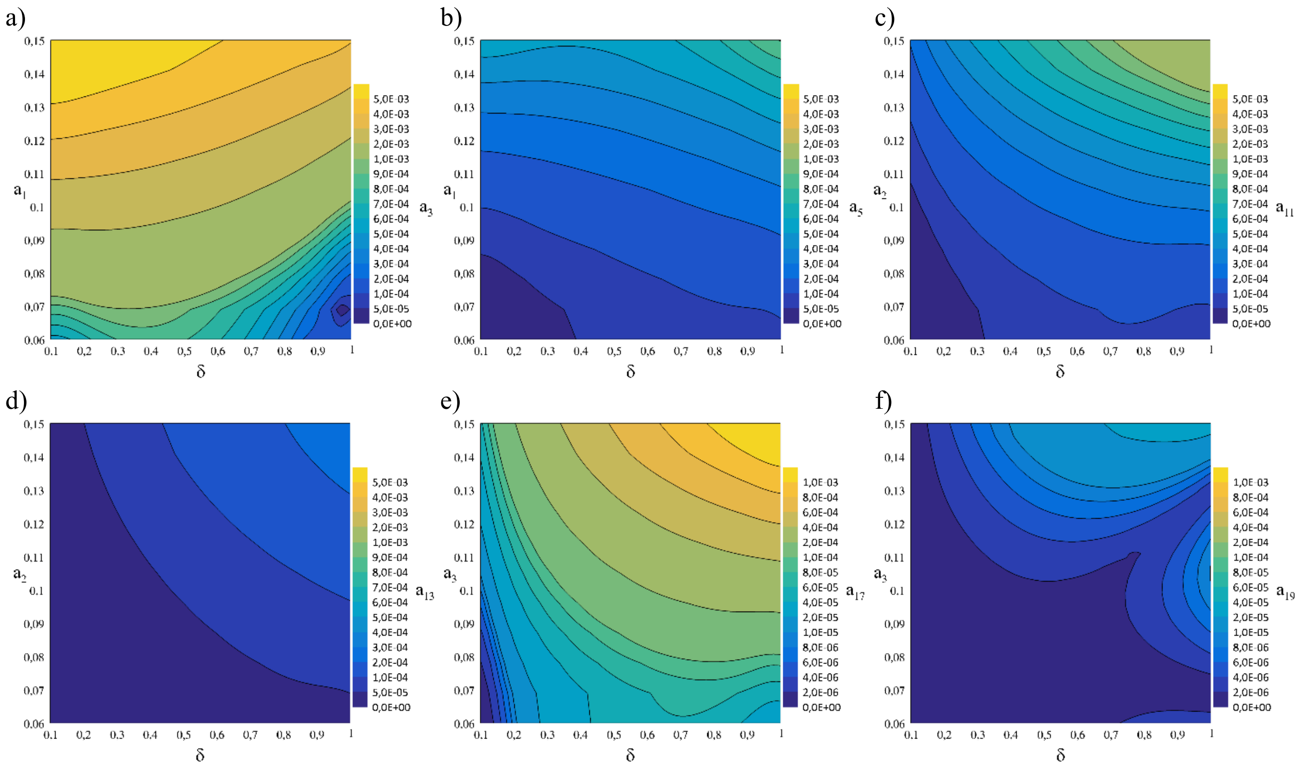

Fig. 3. Contribution coefficients values for the first three mode shapes: a) 3rd a_c values (Fist mode shape), b) 5th a_c values (Fist mode shape), c) 11th a_c values (Second mode shape), d) 13th a_c values (Second mode shape), e) 17th a_c values (Third mode shape), f) 19th a_c values (Third mode shape)

According to these figures, it is clear that the δ values increase with increasing the contribution coefficients. Meanwhile, the number of the basic functions that contributed significantly to the first, second, and the third mode shapes are 5, 13 and 19, respectively.

Figure 3 shows the influence of the exponential slope and the predominant contribution for a range of δ [0.1,1] and a_c [0.06,0.15] on different contribution coefficients value. Generally, whether the investigated mode, the contributions which are close to zero are compared with the aforementioned parameters (δ, a_c), in order to obtain the converge solution.

For the first mode shape, the third contribution (a_3) can only be taken into account for high value of δ and for low value of the predominant contribution a_1. However, the value of the contribution a_5 does not surpass 9.0E-04 no matter the value of δ and a_1. Hence, for the first mode shape, the number of contributions should be 5 so that the solution converges. Regarding the second mode shape, the choice number of the contribution coefficients goes to 13^{th}. The later does not exceed 3.0*E-04,* while the 11^{th} contribution value reaches 2.0E-03 for the above-mentioned range of δ and a_c. Hence, the convergence solution of the second mode shape requires the 13's first contributions. Parallelly, the third mode shape is obtained for 19's first contributions that provide the appropriate solution. The last contribution coefficient is smaller than 4.0E-05, whereas the 17^{th} can achieve 1.0E-03.

Tables 1, 2 and 3 reveal that the functions value with significant contributions are those with high values of the predominant contributions a_1, a_2 and a_3 for the form of the first, second and third mode, respectively. These functions correspond to the maximum displacement and frequency.

5. Results and discussion

The calculation of linear and nonlinear parameters has been made using the software "MATLAB". The geometrical non-linearity that has been obtained from equation (16), amplitude frequency dependence, and the influence of different parameters at the maximum displacement will be discussed in the following.

5.1. Uniform rods

Numerical results have been obtained in the case of C-C uniform rods.

Table 1.
Contribution coefficients to the first non-linear mode shape of a CC non-uniform rods, δ = 0.3

U_{max}	ω^*_{nl}/ω^*_l	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
0.0711	1.0271	0.05	-2.59E-03	-2.55E-04	-1.67E-04	1.33E-05	-2.96E-05	2.52E-06	-1.46E-05
0.1448	1.1019	0.10	-6.87E-03	-2.18E-03	-4.81E-04	1.38E-04	-2.44E-05	3.50E-06	-4.16E-05
0.2230	1.2109	0.15	-1.34E-02	-6.58E-03	-7.87E-04	6.91E-04	8.47E-05	-3.59E-05	-1.05E-04
0.3061	1.3417	0.20	-2.19E-02	-1.35E-02	-8.34E-04	2.02E-03	3.29E-04	-2.24E-04	-2.40E-04
0.3942	1.4861	0.25	-3.19E-02	-2.26E-02	-4.84E-04	4.34E-03	6.85E-04	-6.89E-04	-4.65E-04
0.4868	1.6391	0.30	-4.28E-02	-3.34E-02	2.80E-04	7.69E-03	1.11E-03	-1.51E-03	-7.76E-04
0.5833	1.7975	0.35	-5.44E-02	-4.57E-02	1.41E-03	1.20E-02	1.57E-03	-2.72E-03	-1.15E-03
0.6830	1.9598	0.40	-6.65E-02	-5.90E-02	2.84E-03	1.71E-02	2.04E-03	-4.28E-03	-1.58E-03
0.7851	2.1249	0.45	-7.88E-02	-7.32E-02	4.49E-03	2.28E-02	2.51E-03	-6.14E-03	-2.04E-03
0.8890	2.2921	0.50	-9.13E-02	-8.79E-02	6.32E-03	2.90E-02	2.96E-03	-8.24E-03	-2.51E-03
0.9943	2.4612	0.55	-1.04E-01	-1.03E-01	8.27E-03	3.56E-02	3.39E-03	-1.05E-02	-2.99E-03
0.2072	2.8040	0.65	-1.29E-01	-1.34E-01	1.24E-02	4.95E-02	4.22E-03	-1.55E-02	-3.95E-03
0.4218	3.1521	0.75	-1.55E-01	-1.65E-01	1.67E-02	6.39E-02	4.99E-03	-2.08E-02	-4.90E-03
0.6367	3.5044	0.85	-1.80E-01	-1.96E-01	2.10E-02	7.85E-02	5.72E-03	-2.61E-02	-5.84E-03
0.8515	3.8604	0.95	-2.05E-01	-2.27E-01	2.53E-02	9.30E-02	6.42E-03	-3.15E-02	-6.75E-03
0.9587	4.0396	1.00	-2.18E-01	-2.42E-01	2.75E-02	1.00E-01	6.77E-03	-3.42E-02	-7.21E-03
0.0230	5.8614	1.50	-3.41E-01	-3.93E-01	4.83E-02	1.71E-01	1.01E-02	-6.05E-02	-1.16E-02
0.0762	7.7149	2.00	-4.62E-01	-5.40E-01	6.83E-02	2.39E-01	1.33E-02	-8.58E-02	-1.58E-02

Table 2.
Contribution coefficients to the second non-linear mode shape of a CC non-uniform rods, $\delta = 0.3$

U_{max}	ω_{nl}^*/ω_l^*	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_{10}
0.0740	1.1024	1.81E-03	0.05	-3.06E-03	8.26E-05	-2.83E-04	-1.19E-03	1.45E-04	5.06E-05
0.1586	1.3471	2.73E-03	0.10	-2.77E-03	-1.29E-04	-1.03E-03	-7.31E-03	6.69E-04	8.40E-04
0.2589	1.6530	4.12E-03	0.15	1.62E-03	-5.52E-04	-2.55E-03	-1.81E-02	6.37E-04	3.19E-03
0.3735	1.9840	6.57E-03	0.20	9.76E-03	-1.05E-03	-4.97E-03	-3.18E-02	-6.49E-04	6.89E-03
0.4990	2.3278	1.00E-02	0.25	2.10E-02	-1.53E-03	-8.22E-03	-4.68E-02	-3.28E-03	1.14E-02
0.6320	2.6798	1.43E-02	0.30	3.44E-02	-1.95E-03	-1.21E-02	-6.25E-02	-7.00E-03	1.64E-02
0.7696	3.0380	1.92E-02	0.35	4.92E-02	-2.30E-03	-1.65E-02	-7.83E-02	-1.15E-02	2.15E-02
0.9096	3.4011	2.43E-02	0.40	6.47E-02	-2.57E-03	-2.11E-02	-9.42E-02	-1.64E-02	2.67E-02
0.0508	3.7681	2.97E-02	0.45	8.07E-02	-2.79E-03	-2.59E-02	-1.10E-01	-2.16E-02	3.18E-02
0.1923	4.1383	3.52E-02	0.50	9.67E-02	-2.98E-03	-3.07E-02	-1.26E-01	-2.69E-02	3.70E-02
0.3337	4.5112	4.06E-02	0.55	1.13E-01	-3.14E-03	-3.56E-02	-1.41E-01	-3.23E-02	4.21E-02
0.6155	5.2634	5.16E-02	0.65	1.44E-01	-3.42E-03	-4.53E-02	-1.72E-01	-4.30E-02	5.21E-02
0.8954	6.0217	6.24E-02	0.75	1.76E-01	-3.67E-03	-5.49E-02	-2.02E-01	-5.35E-02	6.20E-02
0.1734	6.7846	7.31E-02	0.85	2.06E-01	-3.91E-03	-6.44E-02	-2.32E-01	-6.38E-02	7.18E-02
0.4499	7.5509	8.37E-02	0.95	2.37E-01	-4.16E-03	-7.38E-02	-2.62E-01	-7.40E-02	8.15E-02
0.5876	7.9350	8.89E-02	1.00	2.52E-01	-4.28E-03	-7.84E-02	-2.77E-01	-7.90E-02	8.63E-02
0.9524	11.7975	1.40E-01	1.50	3.98E-01	-5.63E-03	-1.24E-01	-4.23E-01	-1.28E-01	1.33E-01
0.3048	15.6799	1.90E-01	2.00	5.40E-01	-7.08E-03	-1.68E-01	-5.68E-01	-1.75E-01	1.80E-01

Table 3.
Contribution coefficients to the third non-linear mode shape of a CC non-uniform rods, $\delta = 0.3$

U_{max}	ω_{nl}^*/ω_l^*	a_2	a_3	a_4	a_5	a_6	a_9	a_{10}	a_{16}
0.0856	1.1937	2.69E-03	0.05	-1.42E-02	9.39E-04	-5.80E-04	-2.41E-03	5.41E-05	3.85E-05
0.2348	1.5594	-6.09E-03	0.10	-5.03E-02	6.82E-03	-2.12E-03	-1.23E-02	3.87E-03	-1.84E-04
0.4299	1.9696	-2.50E-02	0.15	-9.22E-02	1.76E-02	-4.45E-03	-2.66E-02	1.23E-02	-1.60E-03
0.6441	2.4043	-4.75E-02	0.20	-1.34E-01	3.08E-02	-7.24E-03	-4.25E-02	2.30E-02	-3.93E-03
0.8643	2.8561	-7.07E-02	0.25	-1.76E-01	4.50E-02	-1.03E-02	-5.87E-02	3.43E-02	-6.66E-03
0.0852	3.3203	-9.37E-02	0.30	-2.17E-01	5.95E-02	-1.34E-02	-7.49E-02	4.57E-02	-9.53E-03
0.3049	3.7936	-1.16E-01	0.35	-2.58E-01	7.40E-02	-1.65E-02	-9.08E-02	5.70E-02	-1.24E-02
0.5229	4.2735	-1.38E-01	0.40	-2.98E-01	8.83E-02	-1.96E-02	-1.06E-01	6.82E-02	-1.53E-02
0.7393	4.7583	-1.60E-01	0.45	-3.38E-01	1.02E-01	-2.27E-02	-1.22E-01	7.92E-02	-1.81E-02
0.9542	5.2469	-1.82E-01	0.50	-3.78E-01	1.17E-01	-2.58E-02	-1.37E-01	9.00E-02	-2.08E-02
0.1678	5.7385	-2.03E-01	0.55	-4.18E-01	1.30E-01	-2.88E-02	-1.53E-01	1.01E-01	-2.36E-02
0.5922	6.7279	-2.45E-01	0.65	-4.97E-01	1.58E-01	-3.48E-02	-1.83E-01	1.22E-01	-2.89E-02
0.0137	7.7232	-2.87E-01	0.75	-5.76E-01	1.85E-01	-4.08E-02	-2.13E-01	1.43E-01	-3.42E-02
0.4330	8.7226	-3.28E-01	0.85	-6.55E-01	2.12E-01	-4.67E-02	-2.43E-01	1.64E-01	-3.94E-02
0.8508	9.7248	-3.69E-01	0.95	-7.33E-01	2.38E-01	-5.26E-02	-2.73E-01	1.84E-01	-4.46E-02
0.0593	10.2267	-3.89E-01	1.00	-7.72E-01	2.52E-01	-5.55E-02	-2.87E-01	1.94E-01	-4.71E-02
0.1344	15.2628	-5.91E-01	1.50	-1.16E+00	3.83E-01	-8.44E-02	-4.35E-01	2.96E-01	-7.23E-02
0.2007	20.3140	-7.91E-01	2.00	-1.55E+00	5.14E-01	-1.13E-01	-5.82E-01	3.97E-01	-9.73E-02

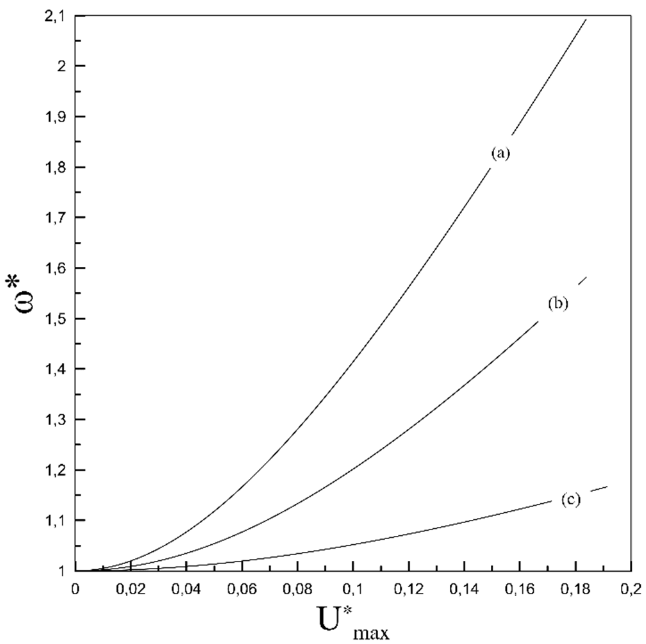

Fig. 4. Backbone curves corresponding to the first three nonlinear longitudinal mode shapes of a C-C uniform rod: (a) The first nonlinear longitudinal mode shape (b) The second nonlinear longitudinal mode shape (c) The third nonlinear longitudinal mode shape

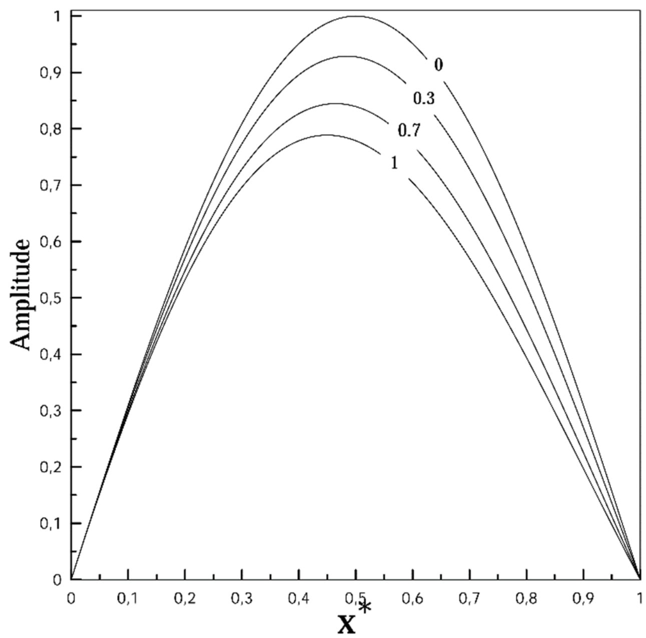

Fig. 5. First linear mode shape of a C-C rods with a section varying exponentially for various values of δ

The backbone curves corresponding to this structure are summarized in Figure 4 for the first three nonlinear longitudinal mode shapes. The later indicate a hardening type of nonlinear behaviour with the increase of the nonlinear frequency parameter by 12%, 46% and 88%, respectively for the first, second and third nonlinear mode shapes which corresponding to the maximum non dimensional vibration displacement equal to 0.16.

5.2. Rods with section varying exponentially

The first three linear longitudinal mode shapes are obtained by varying the parameter δ and are illustrated in Figure 5, with δ= 0.3,0.7,1. The later are compared with δ=0 which corresponds to the uniform case. These linear modes correspond to the basic functions that are used in each case within the nonlinear theory.

Figure 6 shows the effect of the maximum displacement (U_{max}) on the exponential slope (δ) for the first, second and third mode shape. In the case of the first mode shape, the decrease of the maximum displacement on high exponential slope value was shown. For the second and third mode shapes, we notice a global increase of the maximum displacements for high exponential slope value. In addition, the later mode shapes know respectively a brutal increase starting from 2.1 and 1.2 δ value. So, we can conclude for the best choice of δ should not cross 1.2 for the fact that the displacement increases for the second and third mode shape. The optimal choice will depend on other factors, such as a construction limitation, material optimization, etc.

Figures 7, 8 and 9 illustrate the backbone curves of a C-C rods with a section varying exponentially for the first three nonlinear longitudinal mode shapes. In addition, the effect of large vibrations (geometric nonlinearity effect) shows a hardening response of our structure. Thus, for the first, second, and third nonlinear mode shapes corresponding to the maximum non dimensional vibration amplitude are equal, respectively, to 0.17, 0.16 and 0.21. For the first mode shape the nonlinear frequencies increase with 13.2%, 13.4% and 13.6% respectively, in which delta equals to 1, 0.7 and 0.3. In the case of the second nonlinear mode shape, the nonlinear frequencies increase with 37.5%, 43.5% and 47% respectively for delta equals to 1, 0.7 and 0.3. And finally, for the third nonlinear mode shape, the nonlinear frequencies also increase by 51%, 61% and 87% respectively for delta equals to 1, 0.7 and 0.3.

In Figures 10, 11 and 12 depict the amplitude dependency of the first three longitudinal mode shapes. The dependency is obtained by varying the amplitude a_c for the values 0.001, 0.044, 0.087, and 0.130, in order to illustrate the effect of geometrical non-linearity free vibration amplitude.

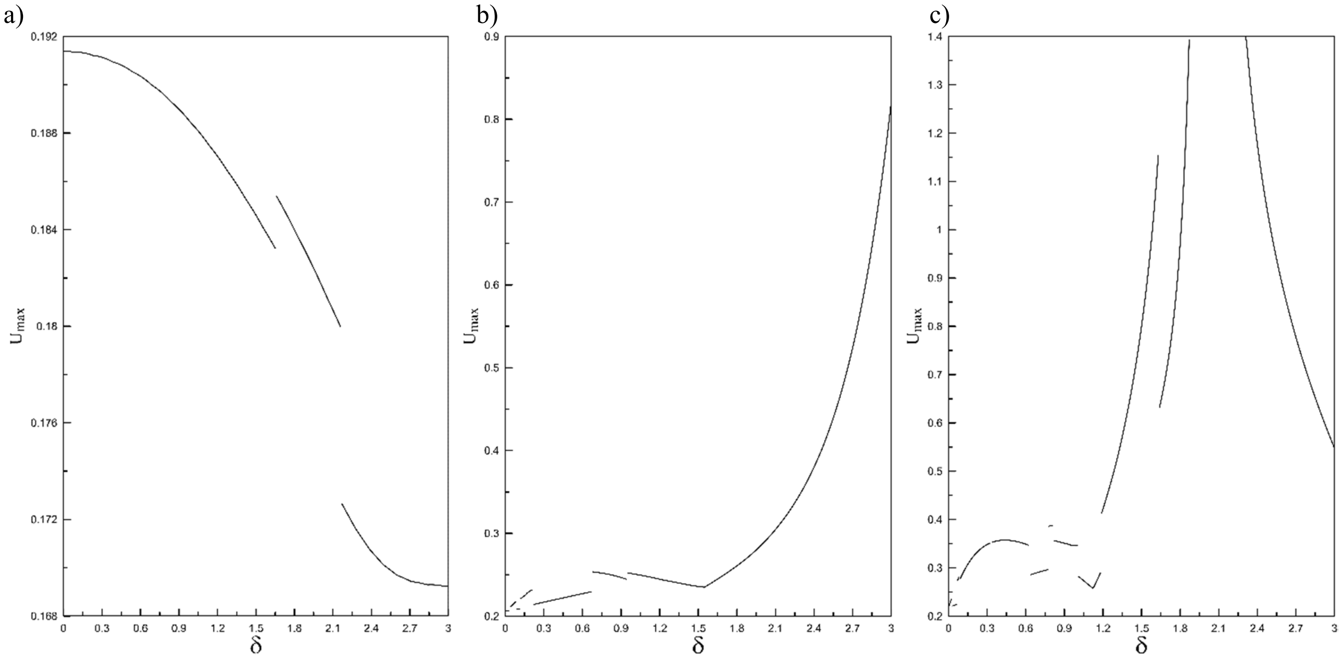

Fig. 6. Maximum amplitude versus the slope of exponential function coefficient for the first three mode shapes: a) First mode shape, b) Second mode shape, c) Third mode shape

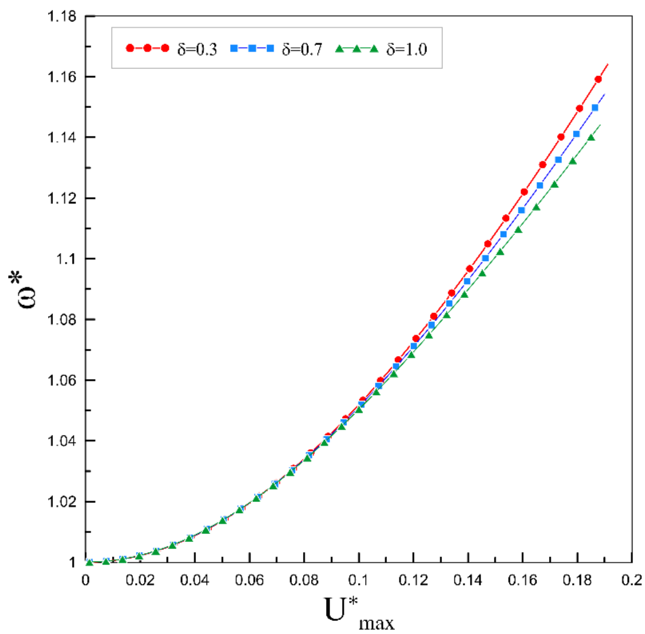

Fig. 7. Backbone curves of the first nonlinear longitudinal mode shape a C-C rod with a section varying exponentially for various values of δ

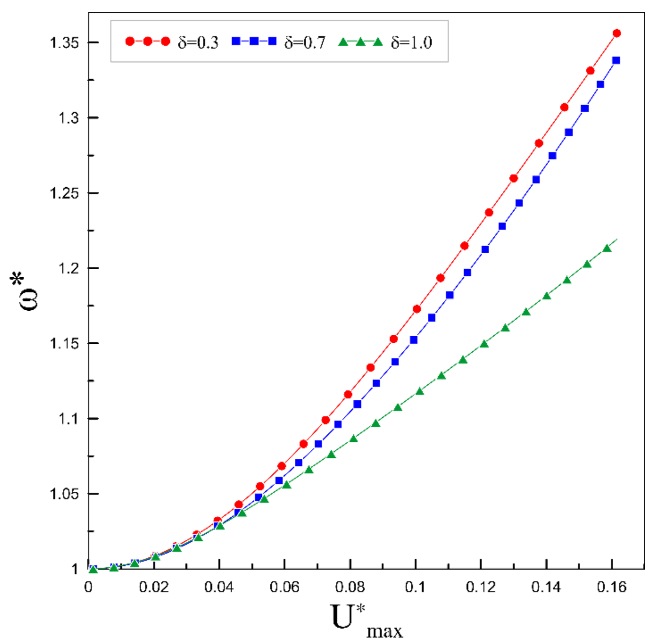

Fig. 8. Backbone curves of the second nonlinear longitudinal mode shape a C-C rod with a section varying exponentially for various values of δ

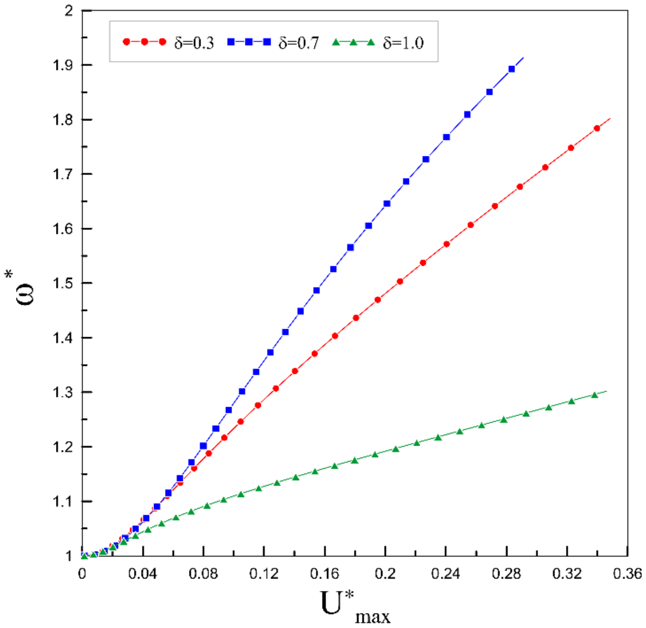

Fig. 9. Backbone curves of the third nonlinear longitudinal mode shape a C-C rod with a section varying exponentially for various values of δ

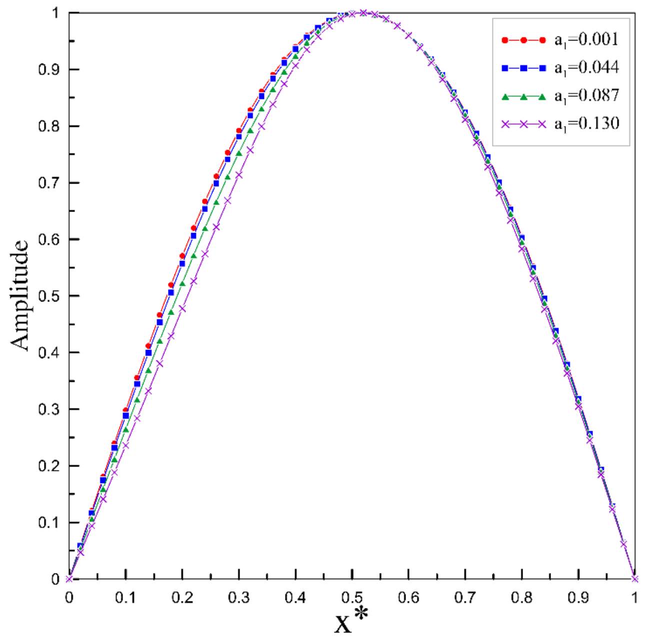

Fig. 10. Amplitude dependence of the first nonlinear longitudinal mode shapes of C-C rod with a section varying exponentially for $\delta = 0.3$

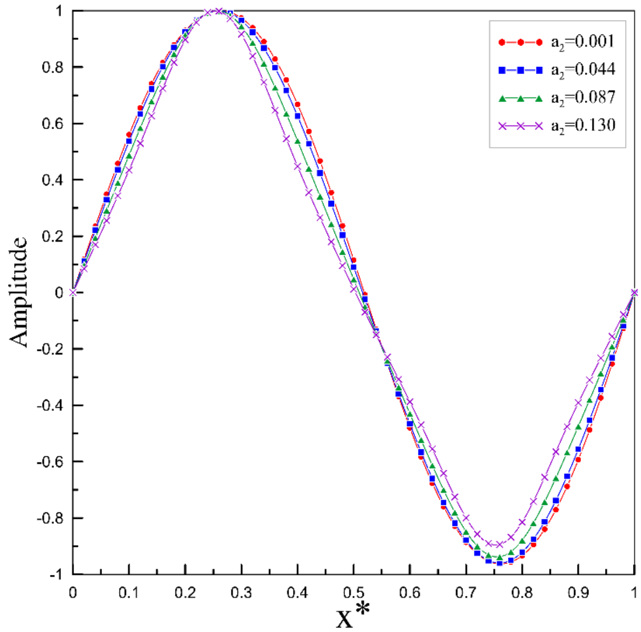

Fig. 11. Amplitude dependence of the second nonlinear longitudinal mode shapes of C-C rod with a section varying exponentially for $\delta = 0.3$

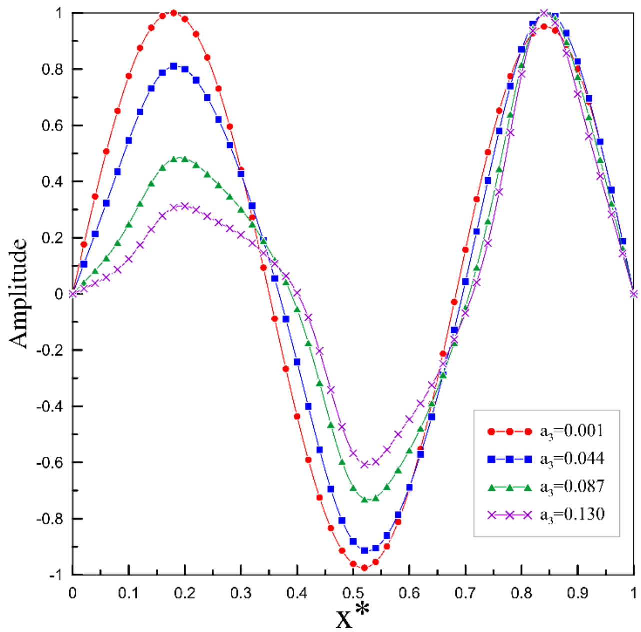

Fig. 12. Amplitude dependence of the third nonlinear longitudinal mode shapes of C-C rod with a section varying exponentially for $\delta = 0.3$

Table 4.
First mode shape value of C-C rod with a section varying exponentially for $\delta = 0.3$

X/L	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
Linear	0.000	0.328	0.614	0.833	0.964	0.999	0.936	0.784	0.561	0.291	0.000
Nonlinear Low amplitude	0.000	0.298	0.571	0.792	0.940	0.999	0.961	0.826	0.605	0.320	0.000
Relative difference %		9.92%	7.61%	5.16%	2.57%	0.04%	2.60%	5.04%	7.25%	9.22%	
Nonlinear High amplitude	0.000	0.236	0.478	0.714	0.907	0.997	0.960	0.811	0.583	0.305	0.000
Relative difference %		38.92%	28.56%	16.67%	6.37%	0.15%	2.47%	3.33%	3.72%	4.60%	

Table 5.
Second mode shape value of C-C rod with a section varying exponentially for $\delta = 0.3$

X/L	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
Linear	0.000	0.601	0.958	0.944	0.575	0.000	-0.558	-0.889	-0.876	-0.533	0.000
Nonlinear Low amplitude	0.000	0.560	0.930	0.974	0.668	0.115	-0.480	-0.886	-0.934	-0.593	0.000
Relative difference %		7.27%	3.04%	3.10%	13.98%		-16.21%	-0.35%	-6.19%	-10.13%	
Nonlinear High amplitude	0.000	0.435	0.897	0.917	0.448	0.012	-0.388	-0.800	-0.815	-0.391	0.000
Relative difference %		38.34%	6.82%	2.93%	28.30%		-43.91%	-11.18%	-7.45%	-36.35%	

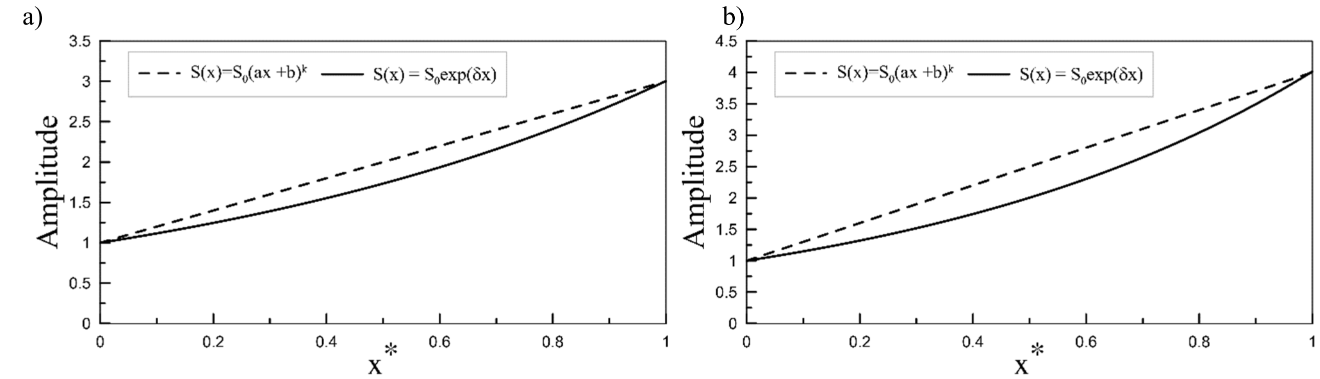

Fig. 13. Comparison of non-uniform rods: a) $a = 2$; $\delta = 1.1$, b) $a = 3$; $\delta = 1.39$

Furthermore, Tables 4 and 5 summary the difference between the linear and nonlinear longitudinal mode shape for the over entire length of our structure. These differences are investigated at low and high amplitudes a_c respectively, for 0.001 and 0.130. For the first mode shape, we notice that the non-linear effect has a high impact in our solution. The large amplitude case ($a_c = 0.130$) gives a relative difference up to 38.92% and the impact of low amplitude ($a_c = 0.001$) hardly crosses 9.92% comparing with the linear case. For the second mode shape, the relative difference reaches 43.91% value in the case of high amplitude. While the low amplitude goes to 16.21%. From these differences, we can conclude the important of the nonlinear theory in order to investigate these structures by tacking into account the parameter that involves the non-linearity geometric.

In order to validate our results, the non-dimensional frequencies obtained are compared with those calculated via Fourier series solution for the free longitudinal vibration of general non-uniform rods [18] by varying the stiffness of spring. The non-dimensional frequencies are obtained for a section $S(x) = S_0(ax + b)^k$ which are presented in the reference [18] with the elastic boundary restraints at both ends. The later structures are compared with those obtained for δ which has specific values, in order to get a close form to the later section. These are presented in Figure 13. When the stiffness spring approaches to the infinity such as high value (10^5), the frequencies obtained will be very close to those of uniform rod with clamped–clamped boundary conditions. The section's parameter a takes the values of 2 and 3 with $b = 1$ and $k = 1$. This section with the cited values, is closed to ours where δ equals to 1.1 and 1.39, respectively. Tables 5 and 6 summarize that our method gives acceptable results due to the section difference in the case of high stiffness value. The results and comparisons that are listed in tables give a good agreement for high stiffness value with those obtained via the adopted method in this paper. Moreover, the slight difference between the later values is due to the section difference.

Table 6.
Non-dimensional frequency comparison for non-uniform C-C rods

	Xu et al. [21]			Present
Spring stiffness	1	10	10^5	
$a = 2$; $\delta = 1.1$	0.9484	2.2290	3.0968	3.0595
$a = 3$; $\delta = 1.39$	0.8520	2.0745	3.0575	3.0731

6. Conclusions

The model based on Lagrange equations and harmonic balance method, is developed to determine the nonlinear mode shapes of transverse vibration of different structures. These later (beams, circular and rectangular plates, shells and rings), have been adapted to the case of nonlinear longitudinal vibration of uniform and non-uniform rods. Furthermore, the numerical formulation has been presented and the expressions for the linear and nonlinear rigidity tensors have been developed. The effect of the exponential slope on the maximum displacement is shown, this parameter supports the utility to lessen the amplitude vibration under resonance of the first mode shape. A comparison between the linear and the nonlinear theory was made at low and high amplitudes, respectively for $a_c = 0.001$ and $a_c = 0.130$. The obtained results highlight the important of the nonlinear theory to investigate these structures by taking into account all the parameters that involve the non-linearity geometric. The linearized procedure has been used to solve the obtained nonlinear algebraic equations. Our investigation has been applied on the C-C uniform and non-uniform rods. The Numerical results corresponding to the fundamental, the second and the third nonlinear modes shape have been presented showing qualitative consistence of the method and a hardening type behaviour are shown. In addition, these results have been compared with those obtained by Fourier series. In conclusion, the amplitude dependence is obtained for the first three nonlinear mode shapes for various values of δ.

The next stage of our research will be an experimental confirmation of our theory in order to validate the nonlinear geometrical effect and the impact of the non-uniform rods on the maximum amplitude vibration.

References

[1] V. Verma, K. Nallasivam, Static response of curved steel thin-walled box-girder bridge subjected to Indian railway loading, Journal of Achievements in Materials and Manufacturing Engineering 108/2 (2021) 63-74. DOI: https://doi.org/10.5604/01.3001.0015.5065

[2] J.H. Mohmmed, M.A. Tawfik, Q.A. Atiyah, Natural frequency and critical velocities of heated inclined pinned pp-r pipe conveying fluid, Journal of Achievements in Materials and Manufacturing Engineering 107/1 (2021) 15-27. DOI: https://doi.org/10.5604/01.3001.0015.2453

[3] E.K. Njim, S.H. Bakhy, M. Al-Waily, Analytical and numerical free vibration analysis of porous functionally graded materials (FGPMs) sandwich plate using Rayleigh-Ritz method, Archives of Materials Science and Engineering 110/1 (2021) 27-41. DOI: https://doi.org/10.5604/01.3001.0015.3593

[4] E.K. Njim, S.H. Bakhy, M. Al-Waily, Free vibration analysis of imperfect functionally graded sandwich plates: Analytical and experimental investigation, Archives of Materials Science and Engineering 111/2 (2021) 49-65. DOI: https://doi.org/10.5604/01.3001.0015.5805

[5] S.H. Bakhy, M. Al-Waily, M.A. Al-Shammari, Analytical and numerical investigation of the free vibration of functionally graded materials sandwich beams, Archives of Materials Science and Engineering 110/2 (2021) 72-85. DOI: https://doi.org/10.5604/01.3001.0015.4314

[6] G. Mohan, U.K. Pandey, Nonlinear homogeneous dynamical system of fully cracked concrete beam, Journal of Achievements in Materials and Manufacturing Engineering 106/1 (2021) 5-19. DOI: https://doi.org/10.5604/01.3001.0015.0525

[7] P.M. Morse, K. Uno Ingard, Theoretical Acoustics, Princeton University Press, Princeton, 1992.

[8] L. Meirovitch, Elements of Vibration Analysis, Second Edition, McGraw-Hill, New York, 1986.

[9] M. Eisenberger, Exact longitudinal vibration frequencies of a variable cross-section rod, Applied Acoustics 34/2 (1991) 123-130. DOI: https://doi.org/10.1016/0003-682X(91)90027-C

[10] H. Matsuda, T. Sakiyama, C. Morita, M. Kawakami, Longitudinal impulsive response analysis of variable cross-section bars, Journal of Sound and Vibration 181/3 (1995) 541-551. DOI: https://doi.org/10.1006/jsvi.1995.0156

[11] C.N. Bapat, Vibration of rods with uniformly tapered sections, Journal of Sound and Vibration 185/1 (1995) 185-189. DOI: https://doi.org/10.1006/jsvi.1995.0371
[12] S. Abrate, Vibration of non-uniform rods and beams, Journal of Sound and Vibration 185/4 (1995) 703-716. DOI: https://doi.org/10.1006/jsvi.1995.0410
[13] B.M. Kumar, R.I. Sujith, Exact solutions for the longitudinal vibration of non-uniform rods, Journal of Sound and Vibration 207/5 (1997) 721-729. DOI: https://doi.org/10.1006/jsvi.1997.1146
[14] J. Li, Y. Li, F. Zhang, Y. Feng, Nonlinear Analysis of Rod Fastened Rotor under Nonuniform Contact Stiffness, Shock and Vibration 2020 (2020) 8851996. DOI: https://doi.org/10.1155/2020/8851996
[15] Y. Guo, D. Zhang, X. Zhang, S. Wang, W. Ma, Experimental Study on the Nonlinear Dynamic Characteristics of Wire Rope under Periodic Excitation in a Friction Hoist, Shock and Vibration 2020 (2020) 8506016. DOI: https://doi.org/10.1155/2020/8506016
[16] A. Raj, R.I. Sujith, Closed-form solutions for the free longitudinal vibration of inhomogeneous rods, Journal of Sound and Vibration 283/3-5 (2005) 1015-1030. DOI: https://doi.org/10.1016/j.jsv.2004.06.003
[17] G. Ferrari, G. Franchini, L. Faedo, F. Giovanniello, S. Le Guisquet, P. Balasubramanian, K. Karazis, M. Amabili, Nonlinear vibrations of a 3 x 3 reduced scale PWR fuel assembly supported by spacer grids, Nuclear Engineering and Design 364 (2020) 110674. DOI: https://doi.org/10.1016/j.nucengdes.2020.110674
[18] G. Ferrari, P. Balasubramanian, S. Le Guisquet, L. Piccagli, K. Karazis, B. Painter, M. Amabili, Non-linear vibrations of nuclear fuel rods, Nuclear Engineering and Design 338 (2018) 269-283. DOI: https://doi.org/10.1016/j.nucengdes.2018.08.013
[19] S. Ferrari, C. Libanati, C.J.F. Lin, J.P. Brown, F. Cosman, E. Czerwiński, L.H. de Gregório, J. Malouf-Sierra, J.-Y. Reginster, A. Wang, R.B. Wagman, E.M. Lewiecki, Relationship Between Bone Mineral Density T-Score and Nonvertebral Fracture Risk Over 10 Years of Denosumab Treatment, Journal of Bone and Mineral Research 34/6 (2019) 1033-1040. DOI: https://doi.org/10.1002/jbmr.3722
[20] K.-T. Kim, The study on grid-to-rod fretting wear models for PWR fuel, Nuclear Engineering and Design 239/12 (2009) 2820-2824. DOI: https://doi.org/10.1016/j.nucengdes.2009.08.018
[21] D. Xu, J. Du, Z. Liu, An accurate and efficient series solution for the longitudinal vibration of elastically restrained rods with arbitrarily variable cross sections, Journal of Low Frequency Noise Vibration and Active Control 38/2 (2019) 403-414. DOI: https://doi.org/10.1177/1461348419825913
[22] Q.S. Li, J.R. Wu, J. Xu, Longitudinal vibration of multi-step non-uniform structures with lumped masses and spring supports, Applied Acoustics 63/3 (2002) 333-350. DOI: https://doi.org/10.1016/S0003-682X(01)00034-2
[23] R. Benamar, M.M.K. Bennouna, R.G. White, The effects of large vibration amplitudes on the mode shapes and natural frequencies of thin elastic structures part I: Simply supported and clamped-clamped beams, Journal of Sound and Vibration 149/2 (1991) 179-195. DOI: https://doi.org/10.1016/0022-460X(91)90630-3
[24] Z. Beidouri, Contribution to a Theory of Non-Linear Modal Analysis. Application to Continuous Structures and Discrete Systems with Localized Non-Linearities, PhD Thesis, EMI, Rabat, 2006 (in French).
[25] R. Benamar, M.M.K. Bennouna, R.G. White, The effects of large vibration amplitudes on the mode shapes and natural frequencies of thin elastic structures, part II: fully clamped rectangular isotropic plates, Journal of Sound and Vibration 164/2 (1993) 295-316. DOI: https://doi.org/10.1006/jsvi.1993.1215
[26] R. Benamar, M.M.K. Bennouna, R.G. White, The effects of large vibration amplitudes on the mode shapes and natural frequencies of thin elastic structures, part III: fully clamped rectangular isotropic plates—measurements of the mode shape amplitude dependence and the spatial distribution of harmonic distortion, Journal of Sound and Vibration 175/3 (1994) 377-395. DOI: https://doi.org/10.1006/jsvi.1994.1335
[27] A. Majid, E.M. Abdeddine, K. Zarbane, Z. Beidouri, Geometrically non-linear free and forced vibration of C-F-C-F rectangular plate at large transverse amplitudes, Proceedings of the XI International Conference on Structural Dynamics "EURODYN 2020", Athens, Greece, 2020, 225-238. DOI: https://doi.org/10.47964/1120.9018.19213
[28] A. Majid, E. Abdeddine, K. Zarbane, Z. Beidouri, Geometrically Nonlinear Forced Transverse Vibrations of C-S-C-S Rectangular Plate: Numerical and Experimental Investigations, Journal of Applied Nonlinear Dynamics 10/4 (2021) 739-757. DOI: https://doi.org/10.5890/JAND.2021.12.012
[29] Z. Beidouri, A. Eddanguir, R. Benamar, Geometrically nonlinear free transverse vibration of 2-dof systems with cubic nonlinearities, Proceedings of the VII European Conference on Structural Dynamics "EURODYN 2008", Southampton, United Kingdom, 2008.
[30] E.M. Abdeddine, A. Majid, Z. Beidouri, K. Zarbane, Nonlinear longitudinal free vibration of uniform rods

and rods with sections varying exponentially, Proceedings of the XI International Conference on Structural Dynamics "EURODYN 2020", Athens, Greece, 2020, 239-251.
DOI: https://doi.org/10.47964/1120.9019.19214
[31] C.L. Lou, D.L. Sikarskie, Nonlinear Vibration of Beams Using a Form-Function Approximation, Journal of Applied Mechanics 42/1 (1975) 209-214. DOI: https://doi.org/10.1115/1.3423520
[32] Z. Beidouri, R. Benamar, M. El Kadiri, Geometrically non-linear transverse vibrations of C-S-S-S and C-S-C-S, International Journal of Non-Linear Mechanics 41/1 (2006) 57-77.
DOI: https://doi.org/10.1016/j.ijnonlinmec.2005.06.002