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A PARTICULAR CASE OF A VARIATIONAL PROBLEM OF CONTROL IN AN ACTIVE AVIATION SYSTEM

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Abstract

It is theoretically considered one of the simplest particular cases of a variational problem of control in an active aviation system acting in conditions of multi-alternativeness and conflicts. With the help of the Euler-Lagrange equations it is obtained the canonical distributions of the subjective preferences for the given functional; differential equation of the second order for finding extremals of the controlled functions of the functional. The differential equation satisfies the conditions of existence and unique solution. Mathematical modeling is fulfilled with the help of the hybrid pseudo-entropy function. Plotted corresponding diagrams.

Key words: active system, active element, aviation transportation, multi-alternativeness, conflicts, safety in aviation, human factor, subjective preferences functions, subjective entropy, hybrid model of combined pseudo-entropy function of subjective preferences, preferences prevailing/dominating factor/index, variational principle, core of conflict.

INTRODUCTION

One of the peculiarities of an *active system* is its controllability through the *active element* and by the active element of the active system. In *aviation transportation* industry, uncertainty of operational situations may lead to *dangerous states* and the active element (making managing decisions person) is forced acting in conditions of *multi-alternativeness* and *conflicts* because of the necessity to act in the lack of resources (such as, for example, time, altitude, distance etc.).

Urgency of researches. It is undoubtedly that *safety in aviation* is the top priority. Thus, it is obviously that there is the grate urgency of researches in the field of *human factor* influence upon safety. This problem is an actual one and requires expressing an individual's preferences with the help of a certain function in a real view.

Analysis of the latest researches and publications. Systematic fundamental researches of multi-alternativeness in the frame of the problem-resource situation concept approach were apparently initiated in the series of the monographs [1-3]. There, [2, P. 120, (3.39)-(3.42)], it was considered the two general conjugated problems:

1. To find out the distributions of the active element's (individual's) *subjective preferences functions* of $\pi(\sigma_i)$; on the set of achievable for him alternatives σ_i, $i = 1, \ldots, N$, where N is the

number of the alternatives; giving the extremal value to the *subjective entropy* of H_π subject to the "isoperimetric constraints" of [2, P. 120, (3.39), (3.40)]:

$$\varepsilon(\pi, U, \ldots) = \varepsilon_0 \qquad P_\pi(\pi) = A_0 \qquad \pi_{extr}(\cdot) = \arg\underset{\pi(\cdot)\in\Pi}{extr} H_\pi \tag{1}$$

where ε – *subjective efficiency function*; U – *subjective utility function*; ε_0 – "isoperimetric constraint" for the subjective efficiency function of ε; $P_\pi(\pi)$ – a certain subjective "perimetric function" for the subjective preferences functions of $\pi(\sigma_i)$; A_0 – "isoperimetric constraint" for the subjective "perimetric function" of $P_\pi(\pi)$; Π– the class of the preferences functions, out of which the extremal distribution of $\pi_{extr}(\cdot)$ is being chosen.

2. To find out the distributions of the $\pi(\cdot)$, giving the extremal value to the subjective efficiency function of ε $(\pi, U, \ldots)$ subject to the "isoperimetric constraints" of [2, P. 120, (3.41), (3.42)]:

$$H_\pi = H_0 \qquad P_\pi(\pi) = A_0 \qquad \pi_{extr}(\cdot) = \arg\underset{\pi(\cdot)\in\Pi}{extr}\, \varepsilon(\pi, U, \ldots) \tag{2}$$

where H_0 – "isoperimetric constraint" for the subjective entropy of H.

Also, it is postulated that, for an active system control, the optimized functional can be taken in the quite general view of [2, P. 119, (3.38)]

$$\Phi_\pi = \alpha H_0 + \beta\varepsilon + \gamma N \tag{3}$$

where α, β, γ,– structural parameters (can be considered in different situations as *Lagrange coefficients* or *weight coefficients*), they reflect the *endogenous parameters of psych*;

γN – normalizing condition.

As a kind of variational problem generalization concerning the functional of (3), it was made an attempt to in the newly publication of [4]. There, the functional of the type of the (3), in the sense of the problem setting of (1), was taken in the more generalized view of

$$\Phi_\pi = \int_{t_0}^{t_1} \left(-\sum_{i=1}^{N} \pi_i(t) \ln \pi_i(t) + \beta \sum_{i=1}^{N} \pi_i(t) F_i + \gamma \left[\sum_{i=1}^{N} \pi_i(t) - 1 \right] \right) dt, \tag{4}$$

where

t – time

$-\sum_{i=1}^{N} \pi_i(t) \ln \pi_i(t)$ – entropy of subjective preferences of $\pi_i(t)$

N – number of the achievable alternatives

β, γ – structural parameters;

F_i – subjective efficiency function of the i^{th} alternative;

$\sum_{i=1}^{N} \pi_i(t) - 1$ – normalizing condition.

For the simplest problem setting, there were considered $x(t)$ and $\dot{x}(t)$ as the subjective efficiency functions of the two achievable alternatives with the corresponding preferences of

$\pi_1(t)$, $\pi_2(t)$. With respect to particular combinations of $x(t)$, $\dot{x}(t)$, $x(t)\dot{x}(t)$, and $\frac{\dot{x}(t)}{x(t)}$, there were got eleven variants of the

$$\Phi_\pi = \int_{t_0}^{t_1} \left(-\sum_{i=1}^{N=4} \pi_i(t) \ln \pi_i(t) + \beta \left[\pi_1(t) x(t) + \alpha_2 \pi_2(t) \dot{x}(t) + \alpha_3 \pi_3(t) x(t) \dot{x}(t) + \right.\right.$$

$$\left.\left. \alpha_4 \pi_4(t) \frac{\dot{x}(t)}{x(t)} \right] + \gamma \left[\sum_{i=1}^{N=4} \pi_i(t) - 1 \right] \right) dt$$

where α_2, α_3, α_4, – coefficients that consider the differences in the measurement units.

In the articles of [5], [6], there were proposed the measure of certainty in the view of the *hybrid model* of the *combined pseudo-entropy function of the subjective preferences* [6, P. 123, (21)], [5, P. 64, (19)]:

$$H_{\max - \frac{\Delta\pi}{|\Delta\pi|}} = \frac{H_{\max} - H_\pi}{H_{\max}} \frac{\Delta\pi}{|\Delta\pi|} = \frac{H_{\max} + \sum_{i=1}^{N} \pi(\sigma_i) \ln \pi(\sigma_i)}{H_{\max}} \frac{\left[\sum_{j=1}^{M} \pi(\sigma_j^+) - \sum_{k=1}^{L} \pi(\sigma_k^-) \right]}{\left| \sum_{j=1}^{M} \pi(\sigma_j^+) - \sum_{k=1}^{L} \pi(\sigma_k^-) \right|} \tag{5}$$

where $H_{\max}$ – the maximal value of the entropy, in the problems formulated above (1)-(4) in the view of [2, P. 100]:

$$H_{\max} = \ln N \tag{6}$$

$\Delta\pi$ – *preferences prevailing/dominating factor/index* [5, P. 62, (11)]:

$$\Delta\pi = \sum_{j=1}^{M} \pi(\sigma_j) - \sum_{k=1}^{L} \pi(\sigma_k) \tag{7}$$

where σ_j^+ – positive and σ_k^- – negative alternatives correspondingly; M – the number of the positive alternatives; L – the number of the negative alternatives correspondingly:

$$M+L=N \tag{8}$$

But the results of generalization [4] obtained from the functional in the view of (4) were without specific numeral solutions.

Thus, there is a necessity of mathematical modeling for some particular case of (4) applicable to some operational process.

The task setting. The purpose of this paper is to consider theoretically one particular case of a variational problem of control in an active aviation system.

The main content (material). For one of the simplest cases, we solve a variational problem, consider the conditions of existence and unique solution for the differential equation of the second order obtained for the extremals, analyze conflictability in the multi-alternative situation.

The problem formulation. Let us consider one of the simplest but typical cases in the flight safety.

The postulated functional of (4) is

$$\Phi_\pi = \int_{t_0}^{t_1}\left(-\sum_{i=1}^{N=2}\pi_i(t)\ln\pi_i(t)+\beta[\pi_1(t)x(t)+\alpha\pi_2(t)\dot{x}(t)]+\gamma\left[\sum_{i=1}^{N=2}\pi_i(t)-1\right]\right)dt \quad (9)$$

where α – coefficient that considers the differences in the measurement units.

This relates, for instance, to the situation with the set of the two achievable alternatives such as the flight altitude and the rate of its change with respect to time; the same to the approach to some danger (hurricane, thunderstorm etc.); and so on.

The corresponding subjective efficiency functions are $x(t)$ and $\dot{x}(t)$ (control); the corresponding individual's preferences functions are $\pi_1(t)$, $\pi_2(t)$.

We name the member of

$$\beta\left[\pi_1(t)x(t)+\alpha\pi_2(t)\dot{x}(t)\right] \quad (10)$$

a *cognitive function* in the functional of (9).

At some specific conditions the given multi-alternativeness with a background of psychic stress may lead to a conflict (in a various kinds of sense). A conflict; like in the sociology, game theory or optimal control theory; is not necessarily a bad thing. It may have negative versus positive consequences. The pseudo-entropy function of (5) on conditions of (6)-(8) with respect to correlations [2] is a useful tool for estimation the entropy thresholds of the conflicts.

In an analogous way to the definition that conflict is the core of a progress, system, process, game or control dynamic development, we define the expression in the view of

$$-\sum_{i=1}^{N=2}\pi_i(t)\ln\pi_i(t)+\beta\left[\pi_1(t)x(t)+\alpha\pi_2(t)\dot{x}(t)\right] \quad (11)$$

as the *core of the conflict*.

That is the expression of the type of (11) is a combination of the entropy or pseudo-entropy function and the cognitive function of (10); measure of uncertainty or certainty with respect to endogenous evaluation of effectiveness.

The problem solution. Applying the necessary conditions for extremums in the view of Euler-Lagrange equations

$$\frac{\partial R^*}{\partial\pi_i}-\frac{d}{dt}\left(\frac{\partial R^*}{\partial\dot{\pi}_i}\right)=0, \qquad \frac{\partial R^*}{\partial x}-\frac{d}{dt}\left(\frac{\partial R^*}{\partial\dot{x}}\right)=0, \quad (12)$$

where R^* – the under-integral function of the corresponding integral (9), we get the corresponding expressions of canonical distributions of the preferences

$$\pi_1 = \frac{e^{\beta\cdot x}}{e^{\beta\cdot x} + e^{\alpha\cdot\beta\cdot\dot{x}}}, \qquad \pi_2 = \frac{e^{\alpha\cdot\beta\cdot\dot{x}}}{e^{\beta\cdot x} + e^{\alpha\cdot\beta\cdot\dot{x}}}. \tag{13}$$

From (9) upon the condition of (12)

$$\pi_1 = \alpha\dot{\pi}_2. \tag{14}$$

The differential equation for determination the exremals

$$\ddot{x} - \frac{\dot{x}}{\alpha} - \frac{1+e^{\beta(x-\alpha\dot{x})}}{\alpha^2\beta} = 0. \tag{15}$$

Let us consider the conditions of existence and unique solution of the differential equation of (15). Denoting

$$x = y_0 \tag{16}$$

for the equation of (15), the corresponding system of ordinary differential equations of the first order:

$$\begin{cases} \dot{x} = \dfrac{dy_0}{dt} = y_1 \\ \ddot{x} = \dfrac{dy_1}{dt} = \dfrac{y_1}{\alpha} + \dfrac{1+e^{\beta(y_0-\alpha y_1)}}{\alpha^2\beta}. \end{cases} \tag{17}$$

For the system of equations (17), accordingly to [7, P. 47-49], at:

$$y_0(t_1) = y_{0_0}; \qquad y_1(t_1) = y_{1_0} \tag{18}$$

we get the system of the equivalent integral equations:

$$\begin{cases} y_0 = y_{0_0} + \displaystyle\int_{t_1}^{t} y_1 dt \\ y_1 = y_{1_0} + \displaystyle\int_{t_1}^{t} \left(\dfrac{y_1}{\alpha} + \dfrac{1+e^{\beta(y_0-\alpha y_1)}}{\alpha^2\beta} \right) dt. \end{cases} \tag{19}$$

In the region D, defined with the inequalities of:

$$t_1 - a \le t \le t_1 + a, \qquad y_{0_0} - b_0 \le y_0 \le y_{0_0} + b_0, \qquad y_{1_0} - b_1 \le y_1 \le y_{1_0} + b_1, \tag{20}$$

the right-hand parts of the equations of (17) satisfy the conditions of:

1) functions:

$$y_1 \quad \text{and} \quad \frac{y_1}{\alpha} + \frac{1 + e^{\beta(y_0 - \alpha y_1)}}{\alpha^2 \beta} \tag{21}$$

are unbreakable, and therefore restricted:

$$|y_1| \le M \quad \text{and} \quad \left| \frac{y_1}{\alpha} + \frac{1 + e^{\beta(y_0 - \alpha y_1)}}{\alpha^2 \beta} \right| \le M, \tag{22}$$

where M – number, equal to the maximum value of the absolute values of the functions of (21) in the region of D [7, P. 29];

2) the functions of (21) satisfy the condition by Lipchitz [7, P. 29]:

$$\left| \left(\frac{y_1}{\alpha} + \frac{1 + e^{\beta(y_0 - \alpha y_1)}}{\alpha^2 \beta} \right) - \left(\frac{z_1}{\alpha} + \frac{1 + e^{\beta(z_0 - \alpha z_1)}}{\alpha^2 \beta} \right) \right| \le N \left(|y_0 - z_0| + |y_1 - z_1| \right), \tag{23}$$

where N – constant [7, P. 35].

Now, the point of the complete metric space of C; the points of which are all possible unbreakable functions of $y(t)$, determined at the segment of (20), the diagrams of which are lying in the region of D, i.e. the .. space of the uniform convergence; will be the system of two unbreakable functions of:

$$(y_0, y_1), \tag{24}$$

that is the two-dimension vector-function:

$$Y(t) \tag{25}$$

with the coordinates of:

$$y_0(t), \qquad y_1(t), \tag{26}$$

determined at the segment of:

$$t_1 - h_1 \le t \le t_1 + h_1 \tag{27}$$

where:

$$h_1 \le \min\left(a, \frac{b_0}{M}, \frac{b_1}{M} \right) \tag{28}$$

and it will be further chosen more precisely. The distance in the space of C is defined with the equality of:

$$\rho(Y(t), Z(t)) = \max|y_0 - z_0| + \max|y_1 - z_1| \tag{29}$$

where z_0, z_1, – coordinates of the vector-function of $Z(t)$.

It is easy to check that at such a definition of the distance, the set of C of the two-dimension vector-functions of $Y(t)$ turns into the complete metric space. The operator of A is defined with the equality of:

$$A[Y] = \left(y_{0_0} + \int_{t_1}^{t} y_1 dt,\ y_{1_0} + \int_{t_1}^{t} \left(\frac{y_1}{\alpha} + \frac{1+e^{\beta(y_0 - \alpha y_1)}}{\alpha^2 \beta} \right) dt \right) \tag{30}$$

i.e. at the operator of A action upon the point of (24), we get a point of the same space of C with the coordinates equal to the right-hand parts of the system of (19).

The point of $A[Y]$ belongs to the space of C because all of its coordinates are unbreakable functions, not leaving the region of D if the coordinates of the vector-function of Y have not left the region of D.

Indeed,

$$\left| \int_{t_1}^{t} y_1 dt \right| \le M \left| \int_{t_1}^{t} dt \right| \le M h_1 \le b_0 \tag{31}$$

$$\left| \int_{t_1}^{t} \left(\frac{y_1}{\alpha} + \frac{1+e^{\beta(y_0 - \alpha y_1)}}{\alpha^2 \beta} \right) dt \right| \le M \left| \int_{t_1}^{t} dt \right| \le M h_1 \le b_1 \tag{32}$$

hence,

$$\left| y_0 - y_{0_0} \right| \le b_0, \qquad \left| y_1 - y_{1_0} \right| \le b_1 . \tag{33}$$

It remains to check the fulfilment of the condition of 2) of the principle of contracted reflections:

$$\rho(A(Y), A(Z)) = \max \left| \int_{t_1}^{t} (y_1 - z_1) dt \right| + \max \left| \int_{t_1}^{t} \left[\left(\frac{y_1}{\alpha} + \frac{1+e^{\beta(y_0 - \alpha y_1)}}{\alpha^2 \beta} \right) - \left(\frac{z_1}{\alpha} + \frac{1+e^{\beta(z_0 - \alpha z_1)}}{\alpha^2 \beta} \right) \right] dt \right| \le$$

$$\le \max \left| \int_{t_1}^{t} |y_1 - z_1| dt \right| + \max \left| \int_{t_1}^{t} \left| \left(\frac{y_1}{\alpha} + \frac{1+e^{\beta(y_0 - \alpha y_1)}}{\alpha^2 \beta} \right) - \left(\frac{z_1}{\alpha} + \frac{1+e^{\beta(z_0 - \alpha z_1)}}{\alpha^2 \beta} \right) \right| dt \right| \le$$

$$\le N \left[2 \max \left| \int_{t_1}^{t} \left(|y_0 - z_0| + |y_1 - z_1| \right) dt \right| \right] \le N \left(\max |y_0 - z_0| + \max |y_1 - z_1| \right) 2 \max \left| \int_{t_1}^{t} dt \right| =$$

$$N 2 h_1 \rho(Y, Z). \tag{34}$$

Hence, if we chose:

$$h_1 \le \frac{\alpha}{2N} \tag{35}$$

where:

$$0 < \alpha < 1, \qquad \text{or} \qquad N 2 h_1 \le \alpha < 1 \tag{36}$$

then the condition of 2) of the principle of contracted reflections will be satisfied and it will exist the only immovable point of $\overline{Y}$, being able to be found by the method of consequential approaching. But the condition of:

$$\overline{Y} = A\left(\overline{Y}\right) \tag{37}$$

by the definition of the operator of *A* is equivalent to the identities of:

$$\begin{cases} \overline{y}_0 = y_{0_0} + \int\limits_{t_0}^{t} \overline{y}_1 dt \\ \overline{y}_1 = y_{1_0} + \int\limits_{t_0}^{t} \left(\dfrac{\overline{y}_1}{\alpha} + \dfrac{1 + e^{\beta(\overline{y}_0 - \alpha\overline{y}_1)}}{\alpha^2 \beta} \right) dt, \end{cases} \tag{38}$$

where $\overline{y}_0$, $\overline{y}_1$ – coordinates of the vector-function of $\overline{Y}$, i.e $\overline{Y}$ is the only solution of the system of (19).

Practical application of the problem solution. Let us consider the fragmental part of a flight, namely climbing from the altitude of 1000�m1000 m.

The initial conditions are: $x_0 = 1\cdot10^3$ m; $\dot{x}_0 = 1\cdot10^1$ m/s. Other conditions are: $\alpha = 95$; $\beta = 0.1$; $t_0 = 0$. The solution in the context of (15)-(38) is illustrated in the fig. 1. The canonical distributions of the preferences accordingly to the concept of (1)-(14) will give the result shown in the fig. 2.

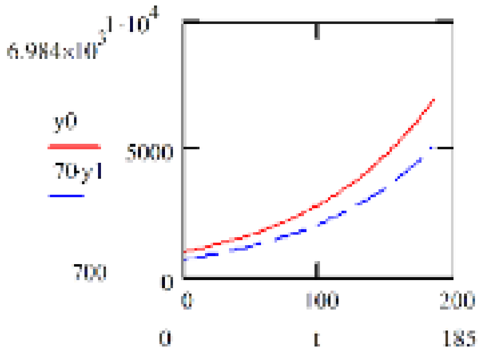

Fig. 1. – The extremals of the altitude and rate of its change with respect to time

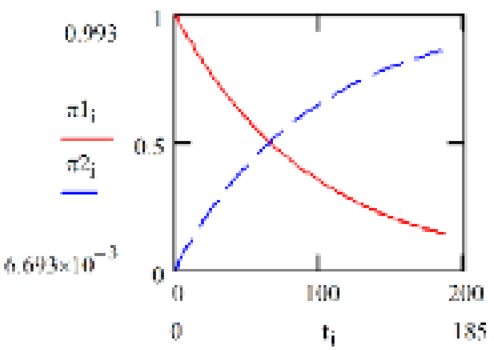

Fig. 2. – The individual's preferences functions

The subjective entropy is shown in the fig. 3. The hybrid model of the combined pseudo-entropy function; on condition that the first alternative is positive and the second one is negative; is illustrated with its plot in the fig. 4.

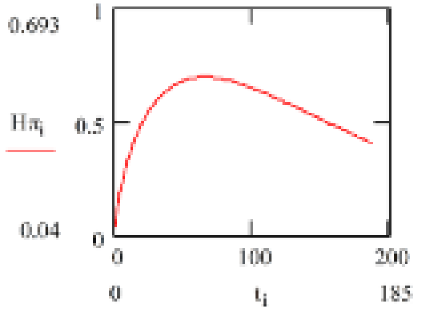

Fig. 3. – The subjective entropy

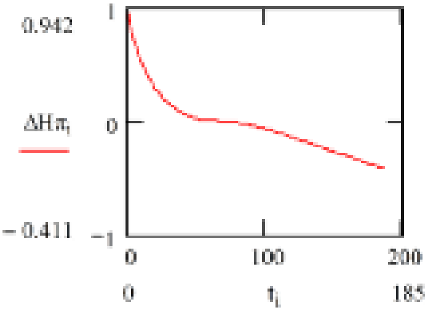

Fig. 4. – The combined pseudo-entropy function

The traditional entropy, hybrid model of the combined pseudo-entropy function, and preferences prevailing/dominating factor/index are plotted together in the fig. 5.

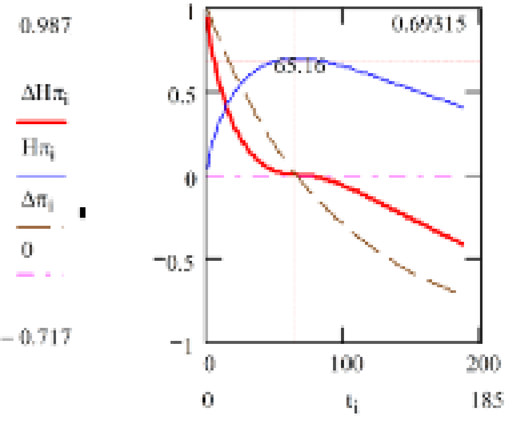

Fig. 5 – The subjective dependencies of the active aviation system

The researches results. The obtained results reflect the simple fact that the active element; when at the beginning he prefers the first alternative (let us say the altitude rather than its rate of change with respect to time) on conditions of no limitations in the corresponding to the alternatives effectiveness functions and no connections between them; will undoubtedly fall into conflict because neither climbing itself nor its rate can be endless. That testifies the necessity to have a corresponding member at the core of the conflict.

Conclusions. The inevitable conflict is the mere fact that "the appetite comes during eating". The combined pseudo-entropy function unlike the traditional entropy shows positive of negative sides of the control process. After its extremum the entropy of the traditional view starts falling down (the system is getting closer to the state of certainty) but the hybrid model entropy shows that this is a negative thing. Zones of conflicts have threshold entropies borders.

Prospects of further researches. The next step of the researches is to explore other modifications of the functional of (4) including the ones subject to an effectiveness functions constraint in the core of its conflict.

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