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The Caputo vs. Caputo-Fabrizio operators in modeling of heat transfer process

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Abstract. In the paper two non-integer order, state space models of heat transfer process are compared. The first uses a known Caputo operator and the second – a new operator proposed by Caputo and Fabrizio in 2015. Both discussed models are modifications of a known, integer order, state space, semigroup model of heat transfer process. Parameters of both models were identified by means of optimization of MSE cost function with the use of simplex method, available in MATLAB. Both proposed models have been compared in the aspect of accuracy and convergence. Analytical and numerical results show that the Caputo-Fabrizio model is faster convergent and easier to implement than the Caputo model. However, its accuracy in the sense of MSE cost function is worse.

Key words: fractional order systems, Caputo operator, Caputo-Fabrizio operator, non singular kernel operator, heat transfer, simplex method.

1. Introduction

The non-integer order calculus is often employed in modeling of processes and phenomena hard to describe using other approaches. For many physical problems non-integer models have been presented by many authors, for example [1–6]. The analysis of anomalous diffusion problem with the use of fractional order approach and semigroup theory was presented for example in [7].

Heat transfer is also possible to describe with the use of non-integer order models. These models can take different forms, they can be both transfer functions and state equations. It has been discussed for example in [8] or [9]. The use of biologically inspired optimization methods in fractional order (FO) modeling has been presented for example in [10, 11]. This is one of the interesting directions in the analysis of fractional order systems.

This paper is intented to compare two fractional order models of heat transfer process. The first uses Caputo operator (it will be called the C model). It was already presented in papers [12–14]. The second applies Caputo-Fabrizio operator, it is described in paper [15] and it will be called CF model. Both models have been compared in the sense of accuracy and the rate of convergence. Analytical results are illustrated by simulations done with the use of MATLAB and experimental results. A similar problem: the comparison of Atangana-Baleanu and Caputo-Fabrizio operators employed in partial differential equation has been presented in [16], the use of CF operator in partial differential equations is discussed in [17], the use of CF operator in modeling of heat transfer processes is given in [18].

The paper is organized as follows: at the beginning any elementary ideas from non-integer order calculus are recalled, both fractional order operators: Caputo (C) and Caputo-Fabrizio (CF) are presented as well. Next, the considered experimental heat plant and its FO state space models employing C and CF operators are recalled. Furthermore, analysis of the rate of convergence and the accuracy for both models is given. Finally, the numerical comparison of the CF vs C model is presented and discussed.

2. Preliminaries

At the beginning an idea of a non-integer order, integro-differential operator needs to be presented. This operator can be defined as follows (see for example [19]):

Definition 1. (The non-integer order integro-differential operator)

$$ {}_aD_t^{\alpha} f(t) = \begin{cases} \dfrac{d^{\alpha} f(t)}{dt^{\alpha}} & \alpha > 0, \\ 1 & \alpha = 0, \\ \displaystyle\int_a^t f(\tau)(d\tau)^{-\alpha} & \alpha < 0, \end{cases} \tag{1} $$

where a and t denote time limits to operator calculation, $\alpha \in \mathbb{R}$ denotes the non-integer order of the operation.

An idea of Mittag-Leffler function should be given next. It is a non-integer order generalization of the exponential function $e^{\lambda t}$ and it plays a crucial role in the solution of FO state equation. One parameter Mittag-Leffler function is defined as underneath:

Definition 2. One parameter Mittag-Leffler function

$$ E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + 1)}. \tag{2} $$

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The fractional-order, integro-differential operator (1) can be described by different definitions, given by Grünvald and Letnikov (GL definition), Riemann and Liouville (RL definition) and Caputo (C definition). In this paper the Caputo and Caputo-Fabrizio definitions will be applied.

Definition 3. (The Caputo definition of the FO operator)

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(M-\alpha)} \int_0^\infty \frac{f^{(M)}(\tau)}{(t-\tau)^{\alpha+1-M}} d\tau. \tag{3} $$

In (3) M is a limiter of the non-integer order: $M-1 \leq \alpha < M$. If $M=1$ then consequently $0 \leq \alpha < 1$ is considered and the definition (3) takes the form:

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \frac{\dot{f}(\tau)}{(t-\tau)^{\alpha}} d\tau. \tag{4} $$

The Laplace transform for the above definition is a generalization of Laplace transform for integer order case (see for example [19]). It takes the form as underneath:

Definition 4. (Laplace transform for Caputo operator)

$$ \begin{gathered} \mathscr{L}({}_0^C D_t^\alpha f(t)) = s^\alpha F(s), \qquad \alpha < 0, \\ \mathscr{L}({}_0^C D_t^\alpha f(t)) = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} {}_0D_t^k f(0), \\ \alpha > 0, \qquad n-1 < \alpha \leq n \in N. \end{gathered} \tag{5} $$

Consequently an inverse Laplace transform can be given as follows (see for example [20], p. 29):

$$ \begin{gathered} \mathscr{L}^{-1}[s^\alpha F(s)] = {}_0D_t^\alpha f(t) + \sum_{k=0}^{n-1} \frac{t^{k-1}}{\Gamma(k-\alpha+1)} f^{(k)}(0^+) \\ n-1 < \alpha < n, \qquad n \in \mathbb{Z}. \end{gathered} \tag{6} $$

Next, the Caputo-Fabrizio derivative needs to be presented. It was given firstly in [21], it is also discussed in [22]. It derives from (3). After replacing the kernel $(t-\tau)^{-\alpha}$ by the exponential function in the form $exp\left(\frac{-\alpha}{1-\alpha}t\right)$ we obtain [21]:

Definition 5. (The Caputo-Fabrizio definition of the FO operator)

$$ {}_0^{CF} D_t^\alpha f(t) = \frac{M_n(\alpha)}{(1-\alpha)} \int_0^\infty \dot{f}(\tau) \exp\left(\frac{\alpha(t-\tau)}{1-\alpha}\right) d\tau, \tag{7} $$

where $M_n(0) = M_n(1) = 1$ is a normalization function. For the operator defined by (7) the Laplace transform is also defined (see [21]). For $0 \leq \alpha < 1$ it takes the following form:

Definition 6. (The Laplace transform of Caputo-Fabrizio operator)

$$ \mathscr{L}({}_0^{CF} D_t^\alpha f(t)) = \frac{s\mathscr{L}(f(t)-f(0))}{s+\alpha(1-s)}. \tag{8} $$

The Laplace transform of CF operator (8) also allows to define a transfer function. It takes the following form:

$$ G(s) = \frac{s}{(1-\alpha)s+\alpha}. \tag{9} $$

Notice, that the above transfer function does not require to use any approximation in MATLAB. On the other hand, it has the form of a transfer function for a real derivative plant. The CF operator can also be employed to define a FO state equation (see for example [22]):

Definition 7. (FO state equation using CF operator)

$$ \begin{cases} {}_0^{CF} D_t^\alpha x(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{10} $$

where $x(t)$, $u(t)$ and $y(t)$ are state, control and output of the system, A, B and C are the state, control and output matrices respectively. The solution of the state equation (10) takes the form as underneath:

$$ \begin{gathered} \widehat{x}(t) = e^{\widehat{A}t}\left(\widehat{x}(0) + \widehat{B}u(0)\right) \\ + \int_0^t e^{\widehat{A}(t-\tau)} B\left[\gamma u(\tau) + \dot{u}(\tau)\right] d\tau. \end{gathered} \tag{11} $$

where

$$ \begin{gathered} \widehat{A} = \alpha\left[I-(1-\alpha)A\right]^{-1} A, \\ \widehat{B} = (1-\alpha)\left[I-(1-\alpha)A\right]^{-1} B, \\ \widehat{x_0} = \left[I-(1-\alpha)A\right]^{-1} x_0, \\ e^{\widehat{A}t} = \mathscr{L}^{-1}\left\{\left[sI-\widehat{A}\right]^{-1}\right\}, \\ \gamma = \frac{\alpha}{1-\alpha}. \end{gathered} \tag{12} $$

3. The considered heat plant and its non-integer order, state space models

3.1. The experimental system. Let us consider the experimental heat plant shown in Fig. 1. It has the form of a thin copper rod heated with an electric heater of length Δx_u localized at one end of the rod. The output temperature is measured with the use of three RTD sensors Δx long, located at points 0.3, 0.5 and 0.7 of the rod length. For further considerations it will be assumed, that the length of a rod is equal 1.0. The construction of the plant is given with details in papers [13] and [15]. The temperature distribution with respect to time and length is presented in Fig. 2. This distribution will be applied as a standard during convergence and accuracy estimating for both compared models.

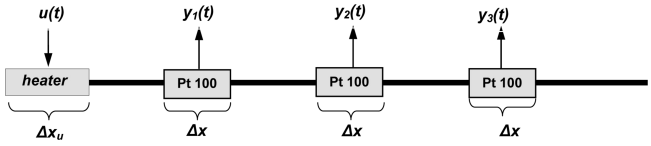

Fig. 1. The experimental system

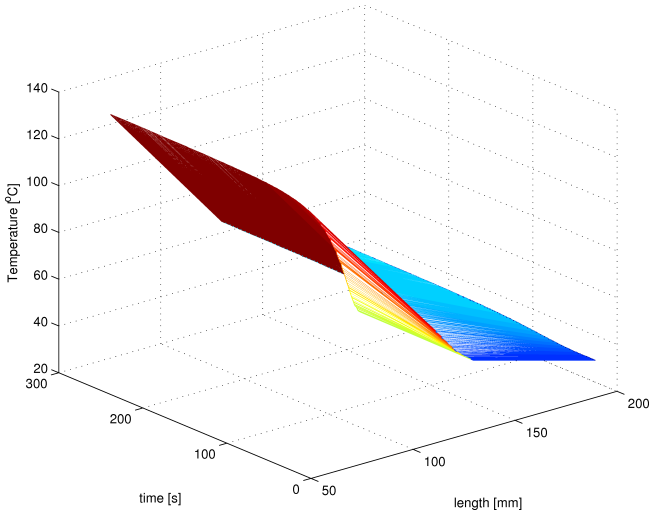

Fig. 2. The spatial-time temperature distribution in the plant

The fundamental mathematical model describing the heat conduction in the plant is the partial differential equation of the parabolic type with the homogeneous Neumann boundary conditions at the ends, the homogeneous initial condition, the heat exchange along the length of a rod and distributed control and observation. This equation with integer orders of both differentiations was presented in [12–14].

3.2. The non-integer order model of the heat plant using the Caputo operator. The non-integer order model presented in this section, with respect to both time and space coordinates has been discussed in [13], the non-integer order model with respect to time has been given in [14]. Its form is motivated by the fact that the non-integer order differentiation is expected to better describe the heat processes than the integer order model. Let us assume that non-integer order difference with respect to time is described by Caputo definition and non-integer order difference with respect to length is described by the Riesz definition. Then the non-integer order, state space heat transfer equation takes the following form:

$$\begin{cases} {}_0^C D_t^{\alpha} Q(t) = AQ(t) + Bu(t), \\ Q(0) = 0, \\ y(t) = CQ(t), \end{cases} \tag{13}$$

where $Q(t)$ is the temperature in time moment t and point x, $0 < \alpha < 1$ is the non-integer order of the model with respect to time

$$\begin{cases} AQ = a\dfrac{\partial^{\beta} Q(x)}{\partial x^{\beta}} - R_a Q, \\ D(A) = \{Q \in H^2(0,1) : Q'(0) = 0, Q'(1) = 0\}, \\ a, R_a > 0, \\ H^2(0,1) = \{u \in L^2(0,1) : u', u'' \in L^2(0,1)\}, \\ CQ(t) = \langle c(x), Q(t)\rangle, Bu(t) = b(x)u(t). \end{cases} \tag{14}$$

In (14) $\beta > 1.0$ is the non-integer order of spatial derivative, R_a and a denote coefficients of heat conduction and heat exchange, $b(x)$ denotes the heater (control) function, expressed by (19), $c(x)$ is the sensor (observation) function, described by (22), $< .. >$ is the scalar product.

The following set of the eigenvectors for the state operator A creates the orthonormal basis of the state space:

$$h_i = \begin{cases} 0, & i = 0, \\ \sqrt{2}\cos(i\pi x), & i = 1, 2, ... \end{cases} \tag{15}$$

Eigenvalues of the state operator are expressed as underneath:

$$\lambda_i = -a\pi^{\beta} i^{\beta} - R_a, \qquad i = 0, 1, 2, ... \tag{16}$$

and consequently the state operator has the form:

$$A = \text{diag}\{\lambda_0, \lambda_1, \lambda_2, ...\}. \tag{17}$$

The input operator B has the following form:

$$B = [b_0, b_1, b_2, ...]^T, \tag{18}$$

where $b_i = \langle b, h_i\rangle$, $b(x)$ denotes the control function:

$$b(x) = \begin{cases} 1, & x \in [0, x_0], \\ 0, & x \notin [0, x_0]. \end{cases} \tag{19}$$

The output operator C is defined as underneath:

$$C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}. \tag{20}$$

Each row of output operator $C_{1,2,3}$ is associated to one RTD sensor and it has the following form:

$$C_j = [c_{j0}, c_{j1}, c_{j2}, ...] \qquad j = 1, 2, 3, \tag{21}$$

where $c_{ji} = \langle c, h_i\rangle$, $c(x)$ denotes the sensor function:

$$c(x) = \begin{cases} 1, & x \in [x_1, x_2], \\ 0, & x \notin [x_1, x_2]. \end{cases} \tag{22}$$

Coordinates x_1 and x_2 depend on sensor location on the rod and they are equal to:

$$\begin{cases} j = 1: \quad x = 0.29: \quad x_1 = 0.26, \ x_2 = 0.32, \\ j = 2: \quad x = 0.50: \quad x_1 = 0.47, \ x_2 = 0.53, \\ j = 3: \quad x = 0.73: \quad x_1 = 0.70, \ x_2 = 0.76. \end{cases}$$

According to (19) and (22) the control function $b(x)$ and output function $c(x)$ are the interval constant functions.

The solution of state equation (13) can be calculated with the use of Laplace transform for Caputo operator with assumptions that the initial condition is equal to zero: $Q(x,0)=0$, $0 \leq x \leq 1$ and state and control operators are described by (17)–(19). If we assume that the control signal has the form of the Heaviside'a function $u(t)=1(t)$ and consider the finite number of modes for infinite dimensional equation (24), then we obtain the solution for j-th output, $j=1,2,3$ as the following finite sum:

$$y_j(t)=\sum_{i=0}^{N}\frac{(E_\alpha(\lambda_i t^\alpha)-1(t))}{\lambda_i}c_{ji}b_i. \tag{23}$$

In (23) $E_\alpha(..)$ is the Mittag Leffler function (2), λ_i, c_{ji} and b_i are described by (16), (22) and (19) respectively, N denotes the order of finite approximation. Correctness of estimation N is a crucial problem during the use of the presented model. Numerical analysis of accuracy and convergence for this model is given in [13].

3.3. The non-integer order model of the heat plant using the Caputo-Fabrizio operator. The CF model is a natural consequence of the finite dimensional model using Caputo operator presented above. It takes the following form:

$$\begin{cases} {}_0^{CF}D_t^\alpha Q(t)=AQ(t)+Bu(t), \\ Q(0)=0, \\ y(t)=CQ(t), \end{cases} . \tag{24}$$

with all operators defined by (17)–(21) and solution in the form (11). The operators (12) are as follows:

$$\begin{gathered} \widehat{A}=\operatorname{diag}\left\{\frac{\alpha\lambda_0}{1+(\alpha-1)\lambda_0},...,\frac{\alpha\lambda_N}{1+(\alpha-1)\lambda_N}\right\}, \\ \widehat{B}=\left[\frac{(1-\alpha)b_1}{1+(\alpha-1)\lambda_0},...,\frac{(1-\alpha)b_N}{1+(\alpha-1)\lambda_N}\right]^T, \\ \widehat{C}=C, \\ \widehat{x_0}=\operatorname{diag}\left\{\frac{1}{1+(\alpha-1)\lambda_0},...,\frac{1}{1+(\alpha-1)\lambda_N},...\right\}x_0, \\ e^{\widehat{A}t}=\mathscr{L}^{-1}\left\{\left[s\mathbb{I}-\widehat{A}\right]^{-1}\right\}, \\ \gamma=\frac{\alpha}{1-\alpha}. \end{gathered} \tag{25}$$

Consequently the step response of the system with respect to (11) and (25) takes the form as underneath:

$$y(t)=\mathscr{L}^{-1}\left\{\frac{\widehat{C}\left[s\mathbb{I}-\widehat{A}\right]^{-1}\widehat{B}}{s}\right\}+\mathscr{L}^{-1}\left\{\widehat{C}\left[s\mathbb{I}-\widehat{A}\right]^{-1}\widehat{B}\right\}. \tag{26}$$

Using (25) in (26) we obtain the analytical step response for j-th output ($j=1,2,3$):

$$\begin{aligned} y_j(t)=\sum_{i=0}^{N}\Bigg(&\frac{1-\alpha}{\alpha\lambda_i}\left(e^{\frac{\alpha\lambda_i}{1+(\alpha-1)\lambda_i}t}-1(t)\right) \\ &+\frac{1-\alpha}{1+(\alpha-1)\lambda_i}e^{\frac{\alpha\lambda_i}{1+(\alpha-1)\lambda_i}t}\Bigg)c_{ji}b_i. \end{aligned} \tag{27}$$

The step response (26) can be calculated using MATLAB functions *step* and *impulse*, but the direct use of formula (27) is also possible.

4. Convergence and accuracy comparison

4.1. Convergence analysis. The convergence analysis for series of Mittag-Leffler functions was considered in [23], the convergence of FO operator described by Laguerre polynomials was discussed in [24], the convergence of approximations of CF operator is presented in [17]. The Rate of Convergence (ROC) as a function of time and order N for each considered model can be defined as underneath:

$$\mathrm{ROC}^{C,CF}(t,N)=\frac{|y_{j,N+1}^{C,CF}(t)-y_{ej}(t)|}{|y_{j,N}^{C,CF}(t)-y_{ej}(t)|}, \tag{28}$$

where $y_{ej}(t)$ is the experimental step response at the j-th ouput, $y_{j,N+1}^{C,CF}(t)$, $y_{j,N}^{C,CF}(t)$ are the step responses of the each model of order $N+1$ or N respectively.

Dependence between $y_{j,N+1}(t)$ and $y_{j,N}(t)$ for each model takes the following form:

$$y_{j,N+1}^{C,CF}(t)=y_{j,N}^{C,CF}(t)+\Delta y_{j,N+1}^{C,CF}(t), \tag{29}$$

where increment $\Delta y_{j,N+1}^{C,CF}$ is defined for C model as follows:

$$\Delta y_{j,N+1}^{C}(t)=\frac{(E_\alpha(\lambda_{N+1}t^\alpha)-1(t))}{\lambda_{N+1}}c_{j,N+1}b_{N+1} \tag{30}$$

and for CF model analogically:

$$\begin{aligned} \Delta y_{j,N+1}^{CF}(t)=\Bigg(&\frac{1-\alpha}{\alpha\lambda_{N+1}}\left(e^{\frac{\alpha\lambda_{N+1}}{1+(\alpha-1)\lambda_{N+1}}t}-1(t)\right) \\ &+\frac{1-\alpha}{1+(\alpha-1)\lambda_{N+1}}e^{\frac{\alpha\lambda_{N+1}}{1+(\alpha-1)\lambda_{N+1}}t}\Bigg)c_{j,N+1}b_{N+1}. \end{aligned} \tag{31}$$

The ROC (28) can be estimated using triagle inequality. This yields:

$$\mathrm{ROC}^{C,CF}(t,N)\leq 1+\frac{|\Delta y_{j,N+1}^{C,CF}(t)|}{|y_{j,N+1}^{C,CF}(t)-y_{e,j}(t)|}. \tag{32}$$

From (30) and (31) it can be noted that increment $\Delta y_{N+1}^{C}(t)$ as a function of time decreases with Mittag-Leffler function and increment $\Delta y_{N+1}^{CF}(t)$ decreases with exponential function. It is worth remembering that a stable exponential function with a certain, negative damping coefficient goes to zero faster than Mittag-Leffler function with the same, negative damping coefficient. In other words, the Mittag-Leffler function cannot be limited by exponential function at infinite time interval.

Next, the step response of the CF model is a sum of exponential modes in contrast to C model, where the step response is a sum of modes described by Mittag-Leffler function. For increasing order N and the same damping coefficient consecutive exponential modes of solution go to zero faster than modes expressed by Mittag-Leffler function. This will be also observed for the steady state.

The above consideration allows us to notice that the step response of CF model faster converges to its steady-state value than C model with the same spectrum.

The analogical reasoning can be run for the steady-state too. Then the value of ROC (28) can be calculated using the known final value theorem (FVT). It is equal to:

$$\mathrm{ROC}_{ss}^{C,CF}(N) = \frac{|y_{ss,j}^{C,CF}(N+1) - y_{sse,j}|}{|y_{ss,j}^{C,CF}(N) - y_{sse,j}|}, \tag{33}$$

where $y_{ss,j}^{C,CF}$ is the steady-state value of step response for each model and j-th output, $y_{sse,j}$ is the steady-state value of experimental response at j-th output. The increments $\Delta_{N+1}^{C,CF}$ can be also given for steady state analogically as (30) and (31). They are equal to:

$$\Delta y_{ss,j,N+1}^{C} = \frac{c_{j,N+1}b_{N+1}}{\lambda_{N+1}}, \tag{34}$$

$$\Delta y_{ss,j,N+1}^{CF} = \frac{1-\alpha}{\alpha}\frac{c_{j,N+1}b_{N+1}}{\lambda_{N+1}}. \tag{35}$$

From (34) and (35) it can be noted that for $0.5 < \alpha < 1.0$ and the same order of model N the increment of solution using CF operator, described by (35) is $\frac{1-\alpha}{\alpha}$ times smaller than analogical increment of solution employing C operator, expressed by (34).

From the whole above discussion it can be concluded that the CF model is faster convergent than the model employing C operator.

4.2. Accuracy analysis. The accuracy of both models can be estimated using their steady-state errors as a function of order N. Let us denote steady-state errors by $\epsilon^{C}(N)$ and $\epsilon^{CF}(N)$ for both considered models respectively. They are equal to:

$$\epsilon_j^{C}(N) = |y_{ej}(K_s) - C_{sj}A^{-1}B|, \tag{36}$$

$$\epsilon_j^{CF}(N) = |y_{ej}(K_s) - \frac{1-\alpha}{\alpha}C_{sj}A^{-1}B|. \tag{37}$$

In (36) and (37) $y_{ej}(K_s)$ denotes the steady-state value of experimental step response at j output, A, B and C_{sj} are described by (17), (18) and (21) respectively. It can be noted at once that for $0.5 < \alpha < 1.0$ and the same order N $\epsilon_j^{C}(N) < \epsilon_j^{CF}(N)$. This allows to expect that the accuracy of the CF model will be worse than the accuracy of C model.

The lower accuracy of CF model in contrast to C model can also be explained by the fact that the CF operator gives only an "approximation" of the fractional order behaviour. Simultaneously, if the FO model of a particular dynamic system is more accurate than analogical integer order model, then it can be expected that the use of an approximation of FO operator will give worse results.

5. Experimental results

Experiments were done with the use of the system presented above. Both compared models were examined using the typical MSE cost function (see for example [25]):

$$\mathrm{MSE} = \frac{1}{3K_s}\sum_{j=1}^{3}\sum_{k=1}^{K_s} e_j^{+}(k). \tag{38}$$

In (38) $e^{+}(k)$ is the difference between experimental step response $y_e^{+}(k)$ measured in time moments $k = 1, ..., K_s$ and step response of model $y^{+}(k)$, calculated with the use of (26) and MATLAB function *step* along the same time grid.

The goal of the experiments was to verify results discussed in the previous section. Additionally, the order of each model N assuring the sensible accuracy can be estimated. It has been done via optimization of cost function (38) with respect to models parameters for different values of N with the use of simplex method available at MATLAB as *fminsearch* function. Results are given in Tables 1, 2 and illustrated by

Table 1
Cost function MSE (38) for different N and all tested models using Caputo operator

Order N	$\alpha = 1$, $\beta = 2$	$\alpha = 1$, $\beta \in \Re$	$\alpha \in \Re$, $\beta = 2$	$\alpha \in \Re$, $\beta \in \Re$
6	0.296082	0.210358	0.296048	0.210324
8	0.175856	0.112488	0.143381	0.061347
10	0.140216	0.120017	0.080125	0.078878
12	0.153525	0.138439	0.131474	0.125623
14	0.138942	0.115968	0.074268	0.060765
16	0.134770	0.119325	0.064554	0.063440
18	0.144883	0.116270	0.074788	0.044133
20	0.140243	0.116270	0.050416	0.041390
22	0.139945	0.117205	0.024154	0.007108
24	0.140586	0.117074	0.034738	0.010223
26	0.138594	0.117506	0.038179	0.012364
28	0.139619	0.117264	0.026374	0.007762
30	0.139486	0.117294	0.028256	0.008315

Table 2
Cost function MSE (38) for different N and all tested models using Caputo-Fabrizio operator

Order N	$\alpha = 1$, $\beta = 2$	$\alpha = 1$, $\beta \in \Re$	$\alpha \in \Re$, $\beta = 2$	$\alpha \in \Re$, $\beta \in \Re$
6	0.296082	0.210358	0.265889	0.191813
8	0.175855	0.112488	0.160844	0.110222
10	0.140216	0.1200177	0.136683	0.116831
12	0.153525	0.138439	0.135442	0.122844
14	0.138942	0.115968	0.128992	0.109914
16	0.134770	0.119325	0.127976	0.113607
18	0.144883	0.116270	0.137764	0.112482
20	0.140243	0.116270	0.135584	0.113344
22	0.139945	0.117205	0.136006	0.114582
24	0.140586	0.1170742	0.137140	0.114870
26	0.138594	0.117506	0.135797	0.115564
28	0.139619	0.117264	0.137108	0.115616
30	0.139486	0.117294	0.137309	0.115875

Figs. 3, 4. The same results given in the first two columns of both tables represent the integer order $\alpha = 1$, when both tested CF and C operators are equivalent. The non pure monotonic form of diagrams presented in Figs. 3 and 4 can be explained by the existence of non observable modes in state equation, which decrease the accuracy. The loss of observability can be detected by the fact that some elements of C matrix are approaching to zero or close to zero. This implies that increasing the order N not always increases the accuracy of the presented model. This problem has been also observed for integer order model of the considered heat plant.

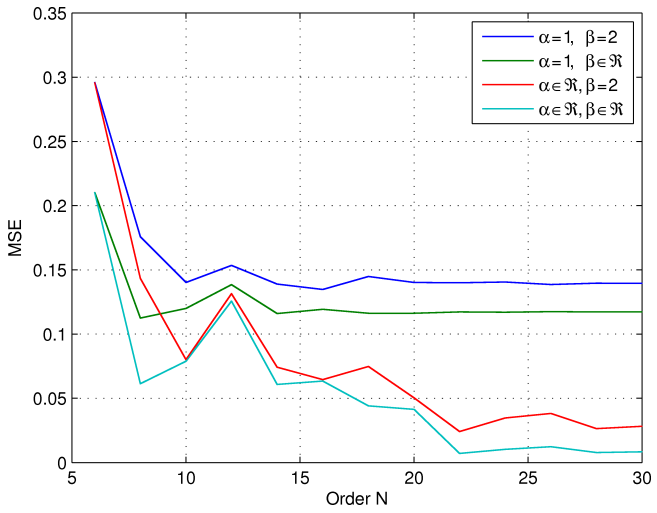

Fig. 3. The dependence of the MSE of the order N for model using Caputo operator

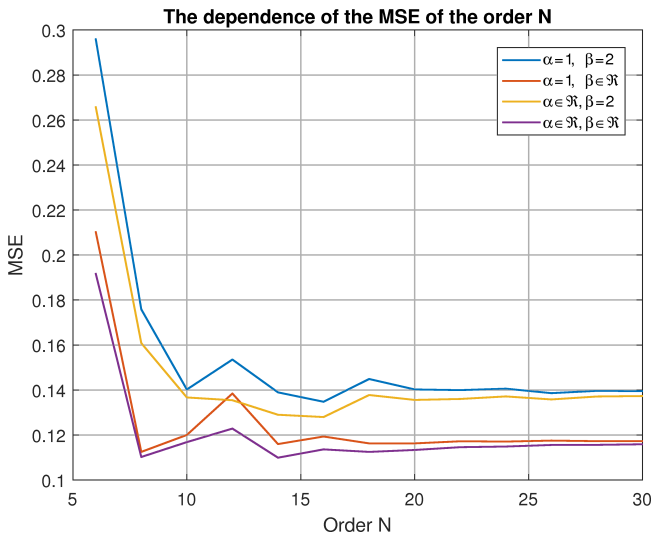

Fig. 4. The dependence of the MSE of the order N for model using Caputo-Fabrizio operator

From Tables 1, 2 and Figs. 3, 4 it can be concluded that the model using CF operator is faster convergent: its size assuring satisfying accuracy, equals $N_{CF} = 15$, but its accuracy is worse than the accuracy of the model employing C operator. This confirms analytical results given in the previous section.

6. Final conclusions

Final conclusions of the paper are the following:

- The convergence of the model using CF operator is faster, but its accuracy in the sense of MSE cost function is worse than model applying C operator.
- The model employing CF operator is much easier to implement in MATLAB with the use of typical functions, because it does not employ Mittag-Leffler function or any approximation necessary to use during the application of Caputo operator.
- The use of CF model facilitates the analysis of the exponential stability due to the fact that state trajectories of CF system are limited by exponential function. This problem is being further analysed by the authors.

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