

On positive reachability of time-variant linear systems on time scales

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Abstract. Positive reachability of time-variant linear positive systems on arbitrary time scales is studied. It is shown that the system is positively reachable if and only if a modified Gram matrix corresponding to the system is monomial. The general criterion is then specified for particular cases of continuous-time systems and various classes of discrete-time systems. It is shown that in the case of continuous-time systems with analytic coefficients the conditions for positive reachability are very restrictive, similarly as for time-invariant systems.

Key words: positive system, positive reachability, time scale, time-variant linear system, Gram matrix.

1. Introduction

In many applications the variables representing various phenomena take only nonnegative values. Examples from biology, economics, chemistry can be found e.g. in [1, 2]. Theory of such systems is significantly different from theory describing general systems and often requires a different language.

Positive reachability of linear positive systems has been studied since 1980's both for discrete- and continuous-time systems [1–11]. The results show that there is a significant difference between discrete-time and continuous-time systems with constant coefficients. In the continuous-time case positive reachability of the system $\dot{x} = Ax + Bu$ requires that A be diagonal and B contain a monomial matrix [10, 11], while in the discrete-time case the conditions are much milder. Positive fractional systems and, in particular, their reachability, have been studied in [12–14]

In [15] we have shown that positive reachability of discrete-time and continuous-time systems can be studied in a common framework using the theory of systems on time scales. Modified Gram matrices have been used to characterize positive reachability of systems with constant coefficients on arbitrary time scales. Besides standard discrete-time and continuous-time systems also discrete-time systems corresponding to nonuniform sampling or systems arising in quantum calculus are considered. Similar characterization of positive observability has been established in [16]. Standard controllability of linear systems on time scales has been studied in [17, 18], where standard Gram matrices on time scales have been used. Realizations of linear positive systems on time scales have been considered in [19].

In this paper we extend the results of [15] to time-variant systems. Positive reachability is characterized by the condition that a modified Gram matrix corresponding to the system is monomial. The modification is performed by selecting columns of the matrix B and assigning different intervals of integration to chosen columns. The essential difference between discrete-time and continuous-time systems is now represented by different properties of the standard integral that

is used in the continuous-time case and the discrete integral used in the discrete-time case.

From the general criterion, valid on all time scales, we infer different characterizations for particular time scales. We extend the result of [10] to time-variant continuous-time systems with analytic coefficients, showing that positive reachability requires that A be diagonal and B contain a monomial submatrix. We also deduce the criteria of positive reachability for systems on a discrete homogeneous and quantum time scales.

The paper is organized as follows. In Section 2 we present necessary information on positive math, calculus on time scales and systems on time scales. In Section 3 we introduce positive systems on time scales. Section 4 contains the main result of the paper: characterization of positive reachability.

2. Preliminaries

We introduce here the main concepts, recall definitions and facts, and set notation. For more information on positive continuous-time and discrete-time systems, the reader is referred to e.g. [1, 2], and for information on time scales calculus, to e.g. [20].

2.1. Positive math. By \mathbb{R} we shall denote the set of all real numbers, by \mathbb{Z} the set of integers, and by \mathbb{N} the set of natural numbers (without 0). We shall also need the set of nonnegative real numbers, denoted by \mathbb{R}_+ and the set of nonnegative integers \mathbb{Z}_+ , i.e. $\mathbb{N} \cup \{0\}$. Similarly, \mathbb{R}_+^k means the set of all column vectors in \mathbb{R}^k with nonnegative components and $\mathbb{R}_+^{k \times p}$ consists of $k \times p$ real matrices with nonnegative elements. If $A \in \mathbb{R}_+^{k \times p}$ we write $A \geq 0$ and say that A is *nonnegative*. A nonnegative matrix A is called *positive* if at least one of its elements is greater than 0. Then we shall write $A > 0$.

A positive column or row vector is called *monomial* if one of its components is positive and all the other are zero. A monomial column in \mathbb{R}_+^n has the form αe_k for some $\alpha > 0$ and $1 \leq k \leq n$, where e_k denotes the column with 1 at the k th position and other elements equal to 0. Then we say that

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the column is *k*-monomial. An $n \times n$ matrix A is called *monomial* if all columns and rows of A are monomial. Then A is invertible and its inverse is also positive. Moreover, we have the following important fact.

Proposition 2.1. A positive matrix A has a positive inverse if and only if A is monomial.

2.2. Calculus on time scales. Calculus on time scales is a generalization of the standard differential calculus and the calculus of finite differences.

A *time scale* \mathbb{T} is an arbitrary nonempty closed subset of the set \mathbb{R} of real numbers. In particular $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = h\mathbb{Z}$ for $h > 0$ and $\mathbb{T} = q^{\mathbb{N}} := \{q^k, k \in \mathbb{N}\}$ (*quantum scale*) for $q > 1$ are time scales. We assume that \mathbb{T} is a topological space with the relative topology induced from \mathbb{R} . If $t_0, t_1 \in \mathbb{T}$, then $[t_0, t_1]_{\mathbb{T}}$ denotes the intersection of the ordinary closed interval with \mathbb{T} . Similar notation is used for open, half-open or infinite intervals.

For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ if $t \neq \sup \mathbb{T}$ and $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$ when $\sup \mathbb{T}$ is finite; the *backward jump operator* $\rho : \mathbb{T} \rightarrow \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ if $t \neq \inf \mathbb{T}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$ when $\inf \mathbb{T}$ is finite; the *forward graininess function* $\mu : \mathbb{T} \rightarrow [0, \infty)$ by $\mu(t) := \sigma(t) - t$; the *backward graininess function* $\nu : \mathbb{T} \rightarrow [0, \infty)$ by $\nu(t) := t - \rho(t)$.

If $\sigma(t) > t$, then t is called *right-scattered*, while if $\rho(t) < t$, it is called *left-scattered*. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$ then t is called *right-dense*. If $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is *left-dense*.

The time scale \mathbb{T} is *homogeneous*, if μ and ν are constant functions. When $\mu \equiv 0$ and $\nu \equiv 0$, then $\mathbb{T} = \mathbb{R}$ or \mathbb{T} is a closed interval (in particular a half-line). When μ is constant and greater than 0, then $\mathbb{T} = \mu\mathbb{Z}$.

Let $\mathbb{T}^k := \{t \in \mathbb{T} : t \text{ is nonmaximal or left-dense}\}$. Thus \mathbb{T}^k is got from \mathbb{T} by removing its maximal point if this point exists and is left-scattered.

Let $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^k$. The *delta derivative of f at t* , denoted by $f^{\Delta}(t)$, is the real number with the property that given any ε there is a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ such that

$$|(f(\sigma(t)) - f(s)) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$. If $f^{\Delta}(t)$ exists, then we say that f is *delta differentiable at t* . Moreover, we say that f is *delta differentiable on \mathbb{T}^k* provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$.

Example 2.2. If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = f'(t)$. If $\mathbb{T} = h\mathbb{Z}$, then $f^{\Delta}(t) = \frac{f(t+h) - f(t)}{h}$. If $\mathbb{T} = q^{\mathbb{N}}$, then $f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t}$.

A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \rightarrow \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Let $a, b \in \mathbb{T}$. Then the *delta integral* of f on the interval $[a, b]_{\mathbb{T}}$ is defined by

$$\int_a^b f(\tau)\Delta\tau := \int_{[a,b]_{\mathbb{T}}} f(\tau)\Delta\tau := F(b) - F(a).$$

It is more convenient to consider the half-open interval $[a, b)_{\mathbb{T}}$ than the closed interval $[a, b]_{\mathbb{T}}$ in the definition of the integral. If b is a left-dense point, then the value of f at b would not affect the integral. On the other hand, if b is left-scattered, the value of f at b is not essential for the integral (see Example 2.3). This is caused by the fact that we use delta integral, corresponding to the forward jump function.

Riemann and Lebesgue delta integrals on time scales have been also defined (see e.g. [21]). It can be shown that every continuous function has an antiderivative and its Riemann and Lebesgue integrals agree with the delta integral defined above.

We have a natural property:

$$\int_a^b f(\tau)\Delta\tau = \int_a^c f(\tau)\Delta\tau + \int_c^b f(\tau)\Delta\tau$$

for any $c \in (a, b)_{\mathbb{T}}$. Moreover, if f is continuous, $f(t) \geq 0$ for all $a \leq t < b$ and $\int_a^b f(\tau)\Delta\tau = 0$, then $f \equiv 0$.

Example 2.3. If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(\tau)\Delta\tau = \int_a^b f(\tau)d\tau$, where the integral on the right is the usual Riemann integral. If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\int_a^b f(\tau)\Delta\tau = \sum_{t=\frac{a}{h}}^{\frac{b}{h}-1} f(th)h$ for $a < b$.

2.3. Linear systems on time scale. Let us consider the system of delta differential equations on a time scale \mathbb{T} :

$$x^{\Delta}(t) = A(t)x(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $A(t)$ is a $n \times n$ matrix. We assume that A is continuous on \mathbb{T} .

If $\mathbb{T} = \mathbb{R}$ then (1) is the standard differential equation

$$\dot{x}(t) = A(t)x(t).$$

On the other hand, for $\mathbb{T} = \mathbb{Z}$ we get the discrete-time system

$$x(k+1) - x(k) = A(k)x(k),$$

which can be rewritten in a more standard shift form

$$x(k+1) = (I + A(k))x(k).$$

Proposition 2.4. Equation (1) with initial condition $x(t_0) = x_0$ has a unique forward solution defined for all $t \in [t_0, +\infty)_{\mathbb{T}}$.

The *matrix exponential function* (at t_0) for A is defined as the unique forward solution of the matrix differential equation $X^{\Delta}(t) = A(t)X(t)$, with the initial condition $X(t_0) = I$. Its value at t is denoted by $e_A(t, t_0)$.

Proposition 2.5. The following properties hold for every $t, s, r \in \mathbb{T}$ such that $r \leq s \leq t$:

- i) $e_A(t, t) = I$;
- ii) $e_A(t, s)e_A(s, r) = e_A(t, r)$;

Remark 2.6. If A is a constant matrix, then for $\mathbb{T} = \mathbb{R}$, $e_A(t, t_0) = e^{A(t-t_0)}$, while for $\mathbb{T} = \mathbb{Z}$, $e_A(k, k_0) = A^{k-k_0}$. Observe that in the second case the exponential matrix may be not defined for $k < k_0$.

Let us consider now a nonhomogeneous system

$$x^\Delta(t) = A(t)x(t) + f(t), \quad (2)$$

where A and f are continuous.

Theorem 2.7. Let $t_0 \in \mathbb{T}$. System (2) for the initial condition $x(t_0) = x_0$ has a unique forward solution of the form

$$x(t) = e_A(t, t_0)x_0 + \int_{t_0}^t e_A(t, \sigma(\tau))f(\tau)\Delta\tau. \quad (3)$$

3. Positive control systems

Let $n \in \mathbb{N}$ be fixed. From now on we shall assume that the time scale \mathbb{T} consists of at least $n + 1$ elements.

Let us consider a linear control system, denoted by Σ , and defined on the time scale \mathbb{T} :

$$x^\Delta(t) = A(t)x(t) + B(t)u(t), \quad (4)$$

where $t \in \mathbb{T}$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, A and B are continuous matrix-valued functions.

We assume that the control u is a piecewise continuous function defined on some interval $[t_0, t_1]_{\mathbb{T}}$, depending on u , where $t_0 \in \mathbb{T}$ and $t_1 \in \mathbb{T}$ or $t_1 = \infty$. We shall assume that at each point $t \in [t_0, t_1]_{\mathbb{T}}$, at which u is not continuous, u is right-continuous and has a finite left-sided limit if t is left-dense. This allows to solve (4) step by step. Moreover, for a finite t_1 we can always evaluate $x(t_1)$. For t_1 being left-scattered we do not need the value of u at t_1 , and for a left-dense t_1 we just take a limit of $x(t)$ at t_1 .

Definition 3.1. We say that system Σ is *positive* if for any $t_0 \in \mathbb{T}$, any initial condition $x_0 \in \mathbb{R}_+^n$, any control $u : [t_0, t_1]_{\mathbb{T}} \rightarrow \mathbb{R}_+^m$ and any $t \in [t_0, t_1]_{\mathbb{T}}$, the solution x of (4) satisfies $x(t) \in \mathbb{R}_+^n$.

By the separation principle we have the following characterization.

Proposition 3.2. The system Σ is positive if and only if $e_A(t, t_0) \in \mathbb{R}_+^{n \times n}$ for every $t, t_0 \in \mathbb{T}$ such that $t \geq t_0$, and $B(t) \in \mathbb{R}_+^{n \times m}$ for $t \in \mathbb{T}$.

Remark 3.3. If A is constant, then the conditions for nonnegativity of the exponential matrix are known (see [2, 15]). For $\mathbb{T} = \mathbb{R}$, A has to be a Metzler matrix (all elements outside the diagonal are nonnegative), for $\mathbb{T} = \mu\mathbb{Z}$, $I + \mu A$ has to be nonnegative, and for $\mathbb{T} = q^{\mathbb{N}}$, A has to be nonnegative.

4. Reachability

If Σ is a positive system, then for a nonnegative initial condition x_0 and a nonnegative control u , the trajectory x stays in \mathbb{R}_+^n . One may be interested in properties of the reachable sets of the system. For simplicity we assume that the initial condition is $x_0 = 0$. Let $x(t_1, t_0, 0, u)$ mean the trajectory of

the system corresponding to the initial condition $x(t_0) = 0$ and the control u , and evaluated at time t_1 .

Definition 4.1. Let $t_0, t_1 \in \mathbb{T}$, $t_0 < t_1$. The *positive reachable set* (from 0) of the system Σ on the interval $[t_0, t_1]_{\mathbb{T}}$ is the set $\mathcal{R}_+^{[t_0, t_1]}$ consisting of all $x(t_1, t_0, 0, u)$, where u is a nonnegative control on $[t_0, t_1]_{\mathbb{T}}$.

The system Σ is *positively reachable on* $[t_0, t_1]_{\mathbb{T}}$ if $\mathcal{R}_+^{[t_0, t_1]} = \mathbb{R}_+^n$.

To study positive reachability let us introduce a modified Gram matrix related to the control system.

Definition 4.1. Let $M \subseteq \{1, \dots, m\}$ and $t_0, t_1 \in \mathbb{T}$, $t_0 < t_1$. For each $k \in M$ let S_k be a subset of $[t_0, t_1]_{\mathbb{T}}$ that is a union of finitely many disjoint intervals of \mathbb{T} of the form $[\tau_0, \tau_1]_{\mathbb{T}}$, and let $\mathcal{S}_M = \{S_k : k \in M\}$. By the *Gram matrix of system* (4) corresponding to t_0, t_1, M and \mathcal{S}_M we mean the matrix

$$W := W_{t_0}^{t_1}(M, \mathcal{S}_M) := \sum_{k \in M} \int_{S_k} e_A(t_1, \sigma(\tau))b_k(\tau)b_k^T(\tau)e_A(t_1, \sigma(\tau))^T \Delta\tau, \quad (5)$$

where $b_k(\tau)$ is the k th column of $B(\tau)$.

Then we have the following characterization:

Theorem 4.3. Let $t_0, t_1 \in \mathbb{T}$, $t_0 < t_1$. System (4) is positively reachable on $[t_0, t_1]_{\mathbb{T}}$ iff there are $M \subseteq \{1, \dots, m\}$ and the family $\mathcal{S}_M = \{S_k : k \in M\}$ of subsets of $[t_0, t_1]_{\mathbb{T}}$ such that the matrix $W = W_{t_0}^{t_1}(M, \mathcal{S}_M)$ is monomial.

Proof. “ \Leftarrow ” Let $\bar{x} \in \mathbb{R}_+^n$. By $\tilde{e}_1, \dots, \tilde{e}_m$ we denote the vectors of the standard basis in \mathbb{R}^m . Define control $u : [t_0, t_1]_{\mathbb{T}} \rightarrow \mathbb{R}_+^m$ by $u(\tau) = \sum_{k \in M} u_k(\tau)\tilde{e}_k$, where $u_k(\tau) = b_k(\tau)^T e_A(t_1, \sigma(\tau))^T W^{-1} \bar{x}$ for $t \in S_k$ and $u_k(\tau) = 0$ for $t \notin S_k$. The control u is nonnegative and

$$\begin{aligned} x(t_1) &= \int_{t_0}^{t_1} e_A(t_1, \sigma(\tau))B(\tau)u(\tau)\Delta\tau \\ &= \sum_{k \in M} \int_{t_0}^{t_1} e_A(t_1, \sigma(\tau))b_k(\tau)u_k(\tau)\Delta\tau \\ &= \sum_{k \in M} \int_{S_k} e_A(t_1, \sigma(\tau))b_k(\tau)b_k^T(\tau)e_A(t_1, \sigma(\tau))^T \\ &\quad \cdot W^{-1} \bar{x} \Delta\tau = \bar{x}. \end{aligned}$$

Thus (4) is positively reachable on $[t_0, t_1]_{\mathbb{T}}$.

“ \Rightarrow ” Positive reachability implies that all the vectors e_1, \dots, e_n can be reached using nonnegative controls. Let us fix some e_i . Then there is a piecewise continuous nonnegative control $u = (u_1, \dots, u_m)$ on $[t_0, t_1]_{\mathbb{T}}$ such that

$$e_i = \sum_{j=1}^m \int_{t_0}^{t_1} e_A(t_1, \sigma(\tau))b_j(\tau)u_j(\tau)\Delta\tau.$$

Since all the integrals in the sum are nonnegative, for some k_i the integral

$$\int_{t_0}^{t_1} e_A(t_1, \sigma(\tau)) b_{k_i}(\tau) u_{k_i}(\tau) \Delta\tau$$

is an i -monomial vector. Then for every $\tau \in [t_0, t_1]_{\mathbb{T}}$ the vector $e_A(t_1, \sigma(\tau)) b_{k_i}(\tau) u_{k_i}(\tau)$ is either i -monomial or 0. Let T_i be the set of all τ for which $e_A(t_1, \sigma(\tau)) b_{k_i}(\tau) u_{k_i}(\tau)$ is i -monomial. Then for $\tau \in T_i$ the matrix

$$e_A(t_1, \sigma(\tau)) b_{k_i}(\tau) b_{k_i}^T(\tau) e_A(t_1, \sigma(\tau))^T$$

is diagonal with the only nonzero element at the i th place. The same is true for the matrix $\int_{T_i} e_A(t_1, \sigma(\tau)) b_{k_i}(\tau) b_{k_i}^T(\tau) e_A(t_1, \sigma(\tau))^T \Delta\tau$. This implies that the matrix

$$C := \sum_{i=1}^n \int_{T_i} e_A(t_1, \sigma(\tau)) b_{k_i}(\tau) b_{k_i}^T(\tau) e_A(t_1, \sigma(\tau))^T \Delta\tau$$

is monomial (and diagonal). Let M consist of all k_i for $i = 1, \dots, n$. Observe that if $k_i = k_j$ for $i \neq j$, then $T_i \cap T_j = \emptyset$. Define $S_k = \bigcup_{k_i=k} T_i$ and let $\mathcal{S}_M = \{S_k : k \in M\}$. Then

$$C = \sum_{k \in M} \int_{S_k} e_A(t_1, \sigma(\tau)) b_k(\tau) b_k^T(\tau) e_A(t_1, \sigma(\tau))^T \Delta\tau = W_{t_0}^{t_1}(M, \mathcal{S}_M),$$

so $W_{t_0}^{t_1}(M, \mathcal{S}_M)$ is monomial.

Corollary 4.4. If the ordinary Gram matrix

$$W_{t_0}^{t_1} = \int_{t_0}^{t_1} e_A(t_1, \sigma(\tau)) B(\tau) B^T(\tau) e_A(t_1, \sigma(\tau))^T \Delta\tau$$

is monomial, then system (4) is positively reachable on $[t_0, t_1]_{\mathbb{T}}$.

Proof. Observe that $W_{t_0}^{t_1} = W_{t_0}^{t_1}(M, \mathcal{S}_M)$ for $M = \{1, \dots, m\}$ and $S_k = [t_0, t_1]_{\mathbb{T}}$ for all $k \in M$. Thus positive reachability follows from Theorem 4.

Remark 4.5. The condition that $W_{t_0}^{t_1}$ is monomial is not necessary for positive reachability on $[t_0, t_1]_{\mathbb{T}}$. Consider the system (see [15])

$$x^\Delta = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} u \quad (6)$$

on $\mathbb{T} = \mathbb{Z}$. Choose $t_0 = 0$ and $t_1 = 2$. System (6) is positively reachable on $[t_0, t_1]_{\mathbb{T}}$. Indeed, let $M = \{1\}$ and $S_1 = [0, 2]_{\mathbb{T}}$. Then

$$W = b_1 b_1^T + (I + A) b_1 b_1^T (I + A)^T = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1, 0) + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1, 0) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is a monomial matrix. However

$$W_{t_0}^{t_1} = BB^T + (I + A)BB^T(I + A)^T = \begin{pmatrix} 3 & 3 \\ 3 & 6 \end{pmatrix}$$

is not monomial.

Corollary 4.6. If there exists $M \subseteq \{1, \dots, m\}$ such that the matrix

$$W_{t_0}^{t_1}(M) = \int_{t_0}^{t_1} e_A(t_1, \sigma(\tau)) \tilde{B}(\tau) \tilde{B}^T(\tau) e_A(t_1, \sigma(\tau))^T \Delta\tau$$

is monomial, where \tilde{B} is a submatrix of B consisting of columns $b_k, k \in M$, then system (4) is positively reachable on $[t_0, t_1]_{\mathbb{T}}$.

Proof. Observe that $W_{t_0}^{t_1}(M) = W_{t_0}^{t_1}(M, \mathcal{S}_M)$ where $S_k = [t_0, t_1]_{\mathbb{T}}$ for all $k \in M$. Thus positive reachability follows from Theorem 4.3.

Remark 4.7. The condition that $W_{t_0}^{t_1}(M)$ is monomial is not necessary for positive reachability on $[t_0, t_1]_{\mathbb{T}}$. Let the time scale $\mathbb{T} = \{0\} \cup [1, 2] \cup \{3\}$. Consider the system (see [15])

$$x^\Delta = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u. \quad (7)$$

The system is positively reachable on $[0, 3]_{\mathbb{T}}$. Indeed, let $M = \{1\}$ and let $S_1 = [0, 1]_{\mathbb{T}} \cup [2, 3]_{\mathbb{T}}$. Then

$$W = \int_{[0,1]_{\mathbb{T}}} e_A(3, \sigma(\tau)) B B^T e_A(3, \sigma(\tau))^T \Delta\tau + \int_{[2,3]_{\mathbb{T}}} e_A(3, \sigma(\tau)) B B^T e_A(3, \sigma(\tau))^T \Delta\tau = \begin{pmatrix} 0 & 0 \\ 0 & e^{-2} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2} \end{pmatrix}$$

is monomial. Observe that we remove here the points t with $\mu(t) = 0$. This is essential in order to get a monomial matrix. To calculate the full Gram matrix we have to add to W the following matrix

$$\int_{[1,2]} e_A(3, \sigma(\tau)) B B^T e_A(3, \sigma(\tau))^T d\tau.$$

Its off-diagonal elements are equal to $\int_1^2 (3 - \tau) e^{-2(3-\tau)} d\tau$.

Since they are positive, $W_{t_0}^{t_1}(M)$ is not monomial.

From the general characterization of positive reachability presented in Theorem 4.3 we can deduce more concrete results for particular time scales. For $\mathbb{T} = \mathbb{R}$ we get very restrictive conditions for positive reachability. The following result was first obtained in [10] for constant matrices A and B .

Proposition 4.8. Let $\mathbb{T} = \mathbb{R}$ and $t_0, t_1 \in \mathbb{R}, t_0 < t_1$. Let A and B be analytic. System (4) is positively reachable on $[t_0, t_1]$ iff A is diagonal and B contains an $n \times n$ submatrix that is monomial for almost every $t \in [t_0, t_1]$ (so $m \geq n$).

Proof. “ \Leftarrow ” Let $\tilde{B}(t)$ denote the monomial submatrix of $B(t)$ and let the indices of columns of $\tilde{B}(t)$ form the set M . Then $\tilde{B}(t) \tilde{B}^T(t)$ is a diagonal matrix with all the diagonal elements being positive and so is

$$W_{t_0}^{t_1}(M) = \int_{t_0}^{t_1} e_A(t_1, \sigma(\tau)) \tilde{B}(\tau) \tilde{B}^T(\tau) e_A(t_1, \sigma(\tau))^T \Delta\tau.$$

Thus $W_{t_0}^{t_1}(M)$ is monomial, so system (4) is positively reachable by Corollary 4.6. Observe that the proof of this implication works for all time scales.

“ \Rightarrow ” Assume that the system is positively reachable on $[t_0, t_1]$. From Theorem 4.3 it follows that for some set M and some family \mathcal{S}_M the Gram matrix $W = W_{t_0}^{t_1}(M, \mathcal{S}_M)$ is monomial. Let j th column of W be i -monomial. Then for some $k \in M$ and for τ from some subinterval of $[t_0, t_1]$ the j th column of the matrix $e_A(t_1, \tau)b_k(\tau)b_k^T(\tau)(e_A(t_1, \tau))^T$ is i -monomial. Let $c(\tau) = e_A(t_1, \tau)b_k(\tau)$. Since the j th column of the matrix $c(\tau)c(\tau)^T$ is i -monomial, then $c(\tau)$ must be j -monomial and eventually $i = j$. This means that at least one column of $e_A(t_1, \tau)$ must be i -monomial. As the exponential matrix is invertible such a column must be unique. This implies that $b_k(\tau)$ is monomial. Moreover the i -monomial column of $e_A(t_1, \tau)$ must be its i th column. Otherwise we would get 0 on the diagonal of the analytic exponential matrix for all τ from some interval, which is impossible. Thus $e_A(t_1, \tau)$ is diagonal on some interval, which means that $A(t)$ is also diagonal. Now to get all n monomial columns in W we need n different monomial columns $b_k(t)$. Thus $B(t)$ contains an $n \times n$ monomial submatrix.

For discrete homogeneous time scales the conditions for positive reachability are much less restrictive. The following result and its equivalent formulations are well known for $\mu = 1$ (see e.g. [7]). We derive the conditions from our general result on positive reachability.

Observe that for $\mu = 1$ the system $x^\Delta = Ax + Bu$ can be rewritten in the shift form as $x(k+1) = (I+A)x(k) + Bu(k)$, which is more common in the literature. Thus the matrix $I + A$ naturally appears in the condition of positive reachability.

Proposition 4.9. Let $\mathbb{T} = \mu\mathbb{Z}$ for a constant $\mu > 0$. Let A and B be constant. Let $t_0 \in \mathbb{T}$ and $t_1 = t_0 + k\mu$ for some $k \in \mathbb{N}$. System (4) is positively reachable on $[t_0, t_1]_{\mathbb{T}}$ iff the matrix $[B, (I + \mu A)B, \dots, (I + \mu A)^{k-1}B]$ contains a monomial submatrix.

Proof. “ \Leftarrow ” Observe that $x(t_1) = \sum_{i=0}^{k-1} \sum_{j=1}^m (I + \mu A)^i b_j u_j(k-1-i)$. If $(I + \mu A)^i b_j = \gamma e_s$ for some $\gamma > 0$, then setting $u_j(k-1-i) = 1/\gamma$ and all other components and values at different times putting to 0 we get $x(t_1) = e_s$. This means positive reachability on $[t_0, t_1]_{\mathbb{T}}$.

“ \Rightarrow ” By Theorem 4 positive reachability implies existence of a set M and subsets S_k of $[t_0, t_1]$ for $k \in M$ such that the matrix

$$W = \sum_{k \in M} \int_{S_k} e_A(t_1, \sigma(\tau)) b_k b_k^T e_A(t_1, \sigma(\tau))^T \Delta \tau$$

is monomial. Moreover

$$\begin{aligned} & \int_{S_k} e_A(t_1, \sigma(\tau)) b_k b_k^T e_A(t_1, \sigma(\tau))^T \Delta \tau \\ &= \sum_{t \in S_k} (I + \mu A)^{(t_1-t-\mu)/\mu} b_k b_k^T ((I + \mu A)^{(t_1-t-\mu)/\mu})^T \mu. \end{aligned}$$

This implies that for every $i = 1, \dots, n$ there are $k \in M$, $t \in S_k$ and $0 \leq j \leq n$ such that the j th column of $(I + \mu A)^{(t_1-t-\mu)/\mu} b_k b_k^T ((I + \mu A)^{(t_1-t-\mu)/\mu})^T$ is i -monomial. This means that the column $(I + \mu A)^{(t_1-t-\mu)/\mu} b_k$ is i -monomial. But this column is one of the columns of the matrix $[B, (I + \mu A)B, \dots, (I + \mu A)^{k-1}B]$.

Proposition 4.9 may be extended to nonhomogeneous discrete-time scales and nonconstant matrices A and B .

Proposition 4.10. Assume that $\mu(t) > 0$ for all $t \in \mathbb{T}$, $t_0 \in \mathbb{T}$ and $t_1 = \sigma^k(t_0)$. System (4) is positively reachable on $[t_0, t_1]_{\mathbb{T}}$ iff the matrix

$$\begin{aligned} & [B(\sigma^{k-1}(t_0)), (I + \mu(\sigma(t_0))A(\sigma(t_0)))B(\sigma^{k-2}(t_0)), \\ & \quad (I + \mu(\sigma^2(t_0))A(\sigma^2(t_0))) \\ & \quad (I + \mu(\sigma(t_0))A(\sigma(t_0)))B(\sigma^{k-3}(t_0)), \dots, \\ & \quad (I + \mu(\sigma^{k-1}(t_0))A(\sigma^{k-1}(t_0))) \dots \\ & \quad (I + \mu(\sigma(t_0))A(\sigma(t_0)))B(t_0)] \end{aligned}$$

contains a monomial submatrix.

The proof is similar to the proof of Proposition 4.9, but we have to take into account that the exponential matrix is no longer a power of $I + \mu A$ for a constant μ but rather a product of such terms with possibly different values of μ and A . This criterion may be used for systems on $\mathbb{T} = q^{\mathbb{N}}$. When the functions A and B are constant, this implies a very restrictive condition of positive reachability.

Proposition 4.11. Let $\mathbb{T} = q^{\mathbb{N}}$, $t_0 \in \mathbb{T}$ and $t_1 = q^k t_0$ for some $k \geq 1$. Assume that A and B are constant. System (4) is positively reachable on $[t_0, t_1]_{\mathbb{T}}$ iff the matrix B contains a monomial submatrix.

Proof. Observe that under the assumptions made in the proposition, the matrix in Proposition 4.10 is now

$$\begin{aligned} & [B, (I + (q-1)qt_0A)B, \\ & \quad (I + (q-1)q^2t_0A)(I + (q-1)qt_0A)B, \dots, \\ & \quad (I + (q-1)q^{k-1}t_0A) \dots (I + (q-1)qt_0A)B]. \end{aligned}$$

The sufficiency is obvious. To show that the condition is also necessary, let us recall that the necessary and sufficient condition for positivity of the exponential matrix e_A on $\mathbb{T} = q^{\mathbb{N}}$ is that A be nonnegative (see [15]). This implies that each matrix $I + (q-1)q^i t_0 A$ is nonnegative and has a positive diagonal. Thus for any nonnegative vector v , $(I + (q-1)q^i t_0 A)v = v + (q-1)q^i t_0 Av$. This implies that we are not able to acquire new monomial vectors besides those that sit in B .

Remark 4.12. Proposition 4.11 can be extended to an arbitrary time scale \mathbb{T} for which $\sup \mu(t) = +\infty$. For such a time scale we have the same requirement for positivity of the exponential matrix: nonnegativity of A (see [15]), which is the essential ingredient of the proof.

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