Linear Vibrations of Periodic Timoshenko and Rayleigh Beams

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Abstract

Elastic periodic structures with variable material and geometrical properties exhibit dynamic characteristics that are investigated in this contribution. The paper is devoted to analysis of geometrically linear vibrations of Rayleigh and Timoshenko beams with cross-sections and material properties periodically varying along the longitudinal axis. The period of inhomogeneity is assumed to be sufficiently small when compared to the beam length. Equations of motion in both beam theories under consideration have highly-oscillating coefficients. In order to derive the averaged model equations with constant coefficients for vibrations, the tolerance averaging approach is applied. The method of averaging differential operators with rapidly varying coefficients is applied to obtain averaged governing equations with constant coefficients. An assumed tolerance and indiscernibility relations and the definition of slowly varying function found the applied technique. Numerical results from the tolerance Rayleigh and Timoshenko beam model equations are compared.

Keywords: periodic beams, Timoshenko beams, Rayleigh beams

1. Introduction

The subject of this contribution is an investigation of linear vibrations of Timoshenko and Rayleigh beams with cross-sections periodically varying along the longitudinal axis. Equations of motion in both beam theories under consideration are described by differential equations with highly oscillating periodic coefficients. Governing equations of most of proposed averaged models neglect the effect of the microstructure size. Hence, in order to take into account this effect in model equations, the tolerance averaging technique is applied.

The analysis of the periodic Timoshenko or Rayleigh beams is restricted to geometrically linear problems. Nevertheless, dynamic response analysis of simply supported beams under moving load is presented in [8]. Free and forced vibrations of the

Timoshenko beam are described in [5] by a single equation as a function of vibration amplitudes.

The paper concerns the tolerance modelling method which is applied to obtain a mathematical model describing periodic beams behaviour by differential equations with constant coefficients. This approach enables analysis of the effect of the microstructure. Some applications of this method to various cases can be found in a series of monographs cf. [11, 13, 12]. Featured method deals well with geometrically nonlinear vibrations of beams with periodic structure, cf. [1, 2], where the model is based on the Rayleigh beam theory. Another non-asymptotic macro-model of micro-periodic elastic beams is proposed in [7] and used to solve a dynamic eigenvalue problem and steady state harmonic vibration. The tolerance modelling technique is used in [3, 4] to describe functionally graded plates with a special microstructure of an order of the plate thickness and functionally graded macrostructure along only one axis, respectively. Similarly, cf. [6], authors use above-mentioned technique in analysis of three-layered sandwich plates Hence, in the contribution, cf. [10], authors study linear-elastic thin plates behaviour with a tolerance-periodic structure in planes parallel to the plate midplane. The tolerance averaging procedure is also applied in [9] to derive governing equations with constant or slowly varying coefficients of dynamic problems for thin linear-elastic cylindrical shells having a periodic or almost-periodic structure.

The main aim of this contribution is to derive equations of the linear tolerance model of vibrations of a Rayleigh and Timoshenko beams with a periodic structure.

2. Governing equations of the model

Objects under consideration are linearly elastic Rayleigh and Timoshenko beams having a periodically inhomogeneous structure. Cross-sections and material properties periodically varying along the longitudinal axis. The period of inhomogeneity is assumed to be sufficiently small when compared to the beam length.

Let $Oxyz$ be an orthogonal Cartesian coordinate system, the Ox axis coincides with the axis of the beam. The cross section of the beam is specified as a height and width along z and y, respectively. The problem is treated as one dimensional, so that in order to describe the beam geometry define $\mathbf{\Omega} \equiv [0,L]$, where L stands for the beam length along x axis. The beam is assumed to be made of many repetitive small elements, called periodicity cells, each of which is defined as $\Delta(x) \equiv [-l/2,l/2]$, where $l<<L$ is the length of the cell Δ and l is named the microstructure parameter. The transverse vibrations denoted by $w = w(x,t)$ – for both Rayleigh and Timoshenko model cases and the slope of the deflection curve $\theta = \theta(x,t) = \partial w\text{-}\gamma$ – for the Timoshenko model case, are under consideration. When the shearing force is neglected, the angle of shear at the neutral axis $\gamma = \gamma(x,t)$. Hence, the flexural stiffness $EJ = E(x)J(x)$, where E stands for Young modulus, J for moment of inertia module, the Timoshenko shear stiffness $kGA = kG(x)A(x)$, where k stands for the shear factor, G for the shear modulus, A for cross-section area, and the mass density $\rho = \rho(x)$.

Here and thereafter coefficients related to the Timoshenko beam model are denoted by underline, however for the Rayleigh beam model will be assumed as neglected. Thus, the strain – displacement relations: the radius of curvature κ remains for both Rayleigh and Timoshenko beams:

$$\kappa = \partial\theta = \partial\left(\partial w - \underline{\gamma}\right). \tag{1}$$

Therefore, internal forces: the bending moment and shear force, are given respectively:

$$M = EJ\kappa, \quad \underline{Q = kGA\gamma}. \tag{2}$$

The external damping force is assumed in the form:

$$p = p(x,t,) = c(x)\dot{w}(x,t). \tag{3}$$

In the framework of Rayleigh and Timoshenko beam theory, strain energy function U, kinetic energy K and potential of external loading F defines the Lagrangian functional. which is given by

$$L = U - K - F, \tag{4}$$

where:

$$U = \frac{1}{2}\int_0^L \left(M\kappa + \underline{Q\gamma}\right)\mathrm{d}x, \quad K = \frac{1}{2}\int_0^L \left(\rho A\dot{w}^2 + \rho J\dot{\theta}^2\right)\mathrm{d}x, \quad F = -\int_0^L qw\,\mathrm{d}x. \tag{5}$$

The equations of motion are obtained from the extended (Woźniak et al., 2010) principle of stationary action $\mathcal{A} = \mathcal{A}(w)$ formulated as:

$$\delta\mathcal{A} = \delta\int_0^1\int_0^L \mathcal{L}dxdt = \int_0^1\int_0^L \delta\mathcal{L}dxdt. \tag{6}$$

3. Basic assumptions and introductory concepts of the tolerance averaging technique

The averaged equations of periodic beams are derived using the concepts and assumptions of the tolerance modelling technique, see [12]. The fundamental concepts are: the tolerance system, averaging operation and certain classes of functions such as tolerance-periodic (*TP*), slowly-varying (*SV*), highly oscillating (*HO*) and fluctuation shape (*FS*) functions. The highest order of function derivative that can be included into a certain function class is denoted by α. Let $\Delta(x) = x + \Delta$, $\Omega_\Delta = \{x \in \Omega : \Delta(x) \subset \Omega\}$ be a cell with its center at $x \in \Omega_\Delta$. The averaging operator for an arbitrary integrable function f is defined by

$$\langle f \rangle(x) = \frac{1}{l}\int_{\Delta(x)} f(y)\mathrm{d}y, \quad x \in \Omega_\Delta, \quad y \in \Delta(x). \tag{7}$$

The first of the basic assumptions is the micro-macro decomposition of the unknown transverse deflection and longitudinal displacement

$$\begin{aligned} w(x,t) &= W(x,t) + h^A(x)V^A(x,t), \quad A = 1,\ldots,N, \\ \theta(x,t) &= \Theta(x,t) + p^R(x)Z^R(x,t), \quad R = 1,\ldots,N. \end{aligned} \tag{8}$$

New unknown macro-displacements and fluctuation amplitudes are given respectively:

$$\begin{aligned} & W(\cdot), V^A(\cdot), \Theta(\cdot), Z^R(\cdot) \in WSV_d^2(\Omega,\Delta), \\ & h^A(\cdot), p^R(\cdot) \in FS_d^2(\Omega,\Delta), \end{aligned} \tag{9}$$

The highly oscillating fluctuation shape functions (*FSFs*) h^A and p^R are proposed *a priori* for each problem under consideration.

4. Equations of the tolerance model

Substituting micro-macro decompositions (8) into Lagrangian (4), and averaging over an arbitrary periodicity cell in (7) the averaged action functional has the following form:

$$\delta \mathcal{A}_h = \delta \int_0^1 \int_0^L \langle \mathcal{L}_h \rangle \, dx dt = \int_0^1 \int_0^L \delta \langle \mathcal{L}_h \rangle \, dx dt = 0. \tag{10}$$

Then, we derive the system of equations for the Timoshenko beam model in (11)

$$\begin{aligned} & -\partial\langle Q\rangle - \langle N\rangle \partial^2 W + \langle \rho A\rangle \ddot{W} + \langle \rho A h^A\rangle \ddot{V}^A - \langle q\rangle = 0, \\ & -\partial\langle M\rangle - \langle Q\rangle + \langle \rho J\rangle \ddot{\Theta} + \langle \rho J p^S\rangle \ddot{Z}^S = 0, \\ & \langle Q\partial h^A\rangle - \partial\langle Q h^A\rangle + \langle N\partial h^A\rangle \partial W + \langle N\partial h^A \partial h^B\rangle V^B - \langle q h^A\rangle + \\ & + \langle \rho A h^A\rangle \ddot{W} + \langle \rho A h^A h^B\rangle \ddot{V}^B = 0, \\ & \langle M\partial p^R\rangle - \partial\langle M p^R\rangle - \langle Q p^R\rangle + \langle \rho J p^R\rangle \ddot{\Theta} + \langle \rho J p^R p^Z\rangle \ddot{Z}^S = 0, \end{aligned} \tag{11}$$

and Rayleigh beam model in (12)

$$\begin{aligned} & -\partial^2\langle M\rangle - \langle \vartheta\rangle \partial^2 \ddot{W} + \langle \vartheta \partial h^A\rangle \partial \ddot{V}^A + \langle \mu\rangle \ddot{W} + \langle \mu h^A\rangle \ddot{V}^A - \langle q\rangle = 0, \\ & \langle M\partial^2 h^A\rangle - 2\partial\langle M\partial h^A\rangle + \langle \mu h^A\rangle \ddot{W} + \langle \mu h^A h^B\rangle \ddot{V}^B + \langle \vartheta \partial h^A\rangle \partial \ddot{W} + \\ & + \langle \vartheta \partial h^A \partial h^B\rangle \ddot{V}^B + \partial^2\langle M h^A\rangle - \langle q h^A\rangle = 0. \end{aligned} \tag{12}$$

5. Solution methods

As an example there is considered a simply supported beam, which fragment and periodicity cell is shown in Fig. 1. The beam's cross section, moment of inertia, Young's modulus and mass per unit length are variable in this analysis. It is assumed that cross section of the beam is rectangular.

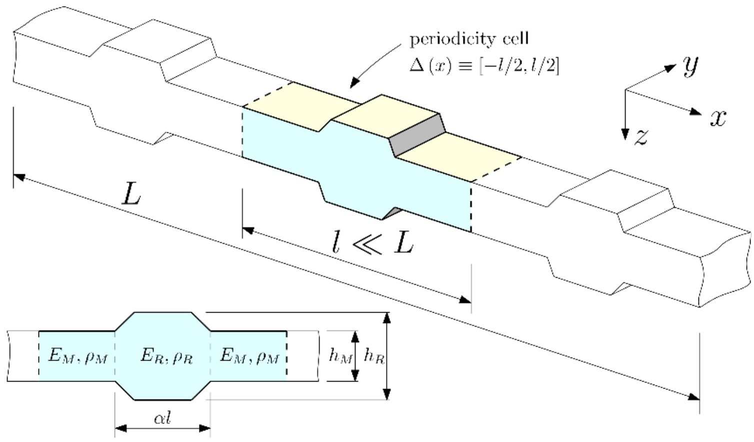

Figure 1. Fragment of considered beam

The fluctuation shape functions play important role in the analysis. These functions represent the oscillations of displacements in the periodicity cell. The common practice is to use approximate functions, defined by trigonometric sine and cosine functions. Transverse and longitudinal approximate *l*-periodic trigonometric functions are introduced for the symmetric periodicity cell:

$$\begin{aligned} h^A(y) &= l^2\left(\cos\left(\frac{2A\pi y}{l}\right) + c_h^A\right), \quad A = 1,\dots,N \\ p^R(y) &= l^2\left(\cos\left(\frac{2R\pi y}{l}\right) + c_p^R\right), \quad R = 1,\dots,N \end{aligned} \tag{13}$$

6. Application

The beam length is L = 1 m, shear factor k = 5/6, the mass density of the material ρ = 7850 kg/m^3, Young's modulus E = 210 GPa. The cross-section is rectangular. The saturation parameter α changes in range 0.1-0.9, section height is h_R = 8 mm, h_M = {4 mm, 5 mm, 6 mm}. The number of the cells is 10.

7. Linear vibrations

The first natural frequency is computed for different values of cross section height, Fig. 2. The frequency gap between 9th and 10th natural frequency is the largest for the α parameter being around 0.7. The thicker the considered beam is, the frequency gap is smaller. Both Rayleigh and Timoshenko results give satisfying accuracy.

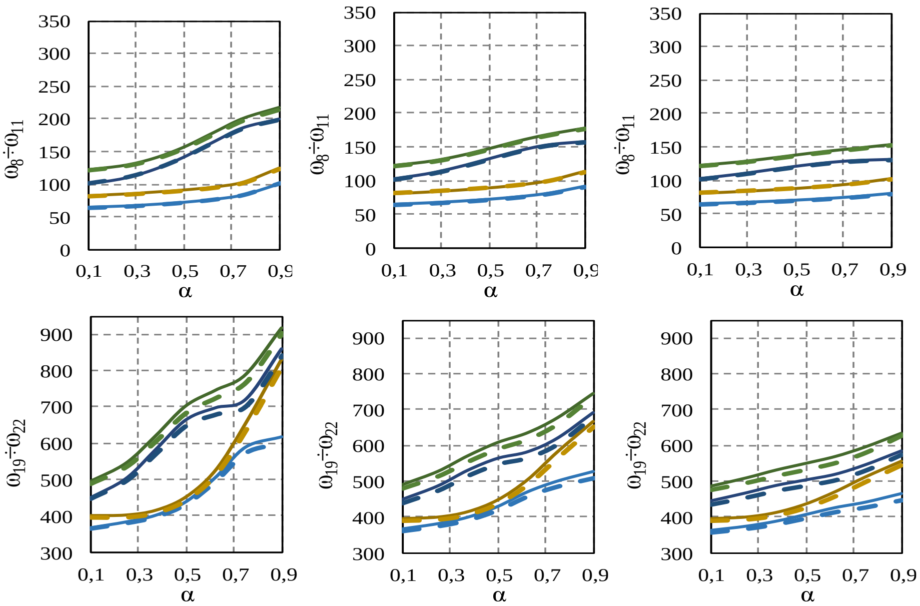

Figure 2. Section height sensitivity: Rayleigh (solid line) and Timoshenko (dashed line)

8. Conlusions

In this paper linear vibrations of a Timoshenko and Rayleigh beam have been presented. Equations of motion consists of differential equations with constant coefficients, which explicitly depend on the microstructure parameter. The proposed method may be applicable in parametric analysis of natural vibrations. The differences between Rayleigh and Timoshenko solutions increase with increasing cross-section height and with number of half-waves of considered eigenmodes.

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References

1. Ł. Domagalski, J. Jędrysiak, *Geometrically nonlinear vibrations of slender meso-periodic beams. The tolerance modeling approach*, Composite Structures, **136** (2016) 270 – 277.

2. Ł. Domagalski, J. Jędrysiak, *Nonlinear vibrations of periodic beams*, Journal of Theoretical and Applied Mechanics, **54** (2016) 1095 – 1108.
3. J. Jędrysiak, *Tolerance modelling of free vibration frequencies of thin functionally graded plates with one-directional microstructure,* Composite Structures, **161** (2017) 453 – 468.
4. J. Jędrysiak, *Modelling of dynamic behaviour of microstructured thin functionally graded plates*, Thin-Walled Structures, **71** (2013) 102 – 107.
5. L. Majkut, *Free and forced vibrations of Timoshenko beams described by single difference equation*, Journal of Theoretical and Applied Mechanics, **47** (2009) 193 – 210.
6. J. Marczak, J. Jędrysiak, *Tolerance modelling of vibrations of periodic three-layered plates with inert core*, Composite Structures, **134** (2015) 854 – 861.
7. K. Mazur-Śniady, *Macro-dynamic of micro-periodic elastic beams*, Journal of Theoretical and Applied Mechanics, **31** (1993) 781 – 793.
8. G. G. Sheng, X. Wang, *The geometrically nonlinear dynamic responses of simply supported beams under moving loads*, Applied Mathematical Modelling, **48** (2017) 183 – 195.
9. B. Tomczyk, *On the modelling of thin uniperiodic cylindrical shells*, Journal of Theoretical and Applied Mechanics, **41** (2003) 755 – 774.
10. A. Wirowski, J. Jędrysiak, M. Kaźmierczak, *Vibrations of thin tolerance-periodic plates*. PAMM **10** (2010).
11. C. Woźniak, E. Wierzbicki, *Averaging techniques in thermomechanics of composite solids. Tolerance averaging versus homogenization,* Technical University of Czestochowa Press 2000.
12. C. Woźniak (eds.). *Mathematical Modelling and Analysis in Continuum Mechanics of Microstructured Media*, Silesian University Press, 2010.
13. C. Woźniak, J. Jędrysiak, B. Michalak, *Thermomechanics of heterogeneous solids and structures, tolerance averaging approach*, Lodz University of Technology, 2008.