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Optimisation of the preventive maintenance plan of a series components system with Weibull hazard function

Keywords

availability, hazard function, Weibull distribution, preventive maintenance, series components

Abstract

In this paper we propose an algorithm to calculate the optimum frequency to perform preventive maintenance in equipment that exhibits Weibull hazard function and constant repair rate in order to ensure its availability. Based on this algorithm we have developed another one to solve the problem of maintenance management of a series system based on preventive maintenance over the different system components. We assume that all components of the system still exhibit Weibull hazard function and constant repair rate and that preventive maintenance would bring the system to the as good as new condition. The algorithm calculates the interval of time between preventive maintenance actions for each component, minimizing the costs, and in such a way that the total downtime, in a certain period of time, does not exceed a predetermined value

1. Introduction

Throughout the years, there has been tremendous pressure on manufacturing and service organizations to be competitive and provide timely delivery of quality products. In many industries, heavily automated and capital intensive, any loss of production due to equipment unavailability strongly impairs the company profit. This new environment has forced managers and engineers to optimise all sectors involved in their organizations.

Maintenance, as a system, plays a key role in achieving organizational goals and objectives. It contributes to reducing costs, minimizing equipment downtime, improving quality, increasing productivity, and providing reliable equipment that are safe and well configured to achieve timely delivery of orders to costumers. In addition, a maintenance system plays an important role in minimizing equipment life cycle cost.

To achieve the target rate of return on investment, plant availability and equipment effectiveness have to be maximized.

Grag and Deshmukh [72] had recently review the literature on maintenance management and points out that, next to the energy costs, maintenance costs can be the largest part of any operational budget.

A brief bibliographic review, (Andrews & Moss [1], Elsayed [5], McCormick [9] and Modarres *et al* [10]), is enough to conclude that the discipline known as reliability was developed to provide methods that can guarantee that any product or service will function efficiently when its user needs it. From this point of view, reliability theory incorporates techniques to determine what can go wrong, what should be done in order to prevent that something goes wrong, and, if something goes wrong, what should be done so that there is a quick recovery and consequences are minimal.

So, reliability has several meanings. However it is usually associated to the ability of a system to perform successfully a certain function. To measure quantitatively the reliability of a system it is used a probabilistic metric, which we state next.

Reliability of a system is the probability that a system will operate without failure for a stated period of time under specified conditions.
Another measure of the performance of a system is its availability that reflects the proportion of time that we expect it to be operational. Availability of a system is the probability to guarantee the intended function, that is, the probability that the system is normal at time *t*. The availability of a system is a decreasing function of the failure rate and it is an increasing function of the repair rate.
According to Elsayed [5], reliability of a system depends mainly in the quality and reliability of its components and in the implementation and accomplishment of a suitable preventive maintenance and inspection program. If failures, degradation and aging are characteristics of any system, however, it is possible to prolong its useful lifetime and, consequently, to delay the wear-out period carrying out maintenance and monitoring programs.
This type of programs leads necessarily to expenses and so we are taken to a maintenance optimisation problem.

Basic maintenance approaches can be classified as:

- Unplanned (corrective): this amounts to the replacement or repair of failed units;
- Planned (preventive):
  - Scheduled: this amounts to performing inspections, and possibly repair, following a predefined schedule;
  - Conditioned: this amounts to monitor the health of the system and to decide on repair actions based on the degradation level assessed.

In the unplanned, corrective strategy, no maintenance action is carried out until the component or structure breaks down. Upon failure, the associated repair time is typically relatively large, thus leading to large downtimes and high costs. In this approach, efforts are undertaken to achieve small mean times to repair (MTTRs).
To avoid failures at occasions that have high cost consequences preventive maintenance is normally chosen. This allows that inspections and upgrading can be planned for periods, which have the lowest impact on production or availability of the systems.
The main function of planned maintenance is to restore equipment to the "as good as new" condition; periodical inspections must control equipment condition and both actions will ensure equipment availability. In order to do so it is necessary to determine:

- Frequency of the maintenance, substitutions and inspections
- Rules of the components replacements
- Effect of the technological changes on the replacement decisions
- The size of the maintenance staff
- The optimum inventory levels of spare parts

There are several strategies for maintenance; the one we have just described and that naturally frames in what has been stated is known as Reliability Centered Maintenance - RCM. Gertsbakh [7] reviews some of the most popular models of preventive maintenance.
In theory, maintenance management, facing the problems stated above, could have benefited from the advent of a large area in operations research, called maintenance optimisation. This area was founded in the early sixties by researchers like Barlow and Proschan. Basically, a maintenance optimisation model is a mathematical model in which both costs and benefits of maintenance are quantified and in which an optimal balance between both is obtained. Well-known models originating from this period are the so-called age and the block replacement models.
Valdez-Flores & Feldman [12] presents a comprehensive review of these approaches. Dekker [2] gives an overview of applications of maintenance optimisation models published so far and Duffuaa [4] describes various advanced mathematical models in this area that have "high potential of being applied to improve maintenance operations".
More recently, Nakagawa [11] summarizes the results of maintenance policies and Garg and Deshmukh [6] describe the existence of 24 papers on the quantitative approach to maintenance optimisation
As we have already mentioned, one of the most critical problems in preventive maintenance is the determination of the optimum frequency to perform preventive maintenance in equipment, in order to ensure its availability.
The Preventive Maintenance policies are adapted to slow the degradation process of the system while the system is operating and to extend the system life. A number of Preventive Maintenance policies have been proposed in the literature. These policies are typically to determine the optimum interval between preventive maintenance tasks to minimize the average cost over a finite time span. But in many areas one complains about the gap between theory and practice. Practitioners say maintenance optimisation models are difficult to understand and to interpret [2]. Vatn *et al* [13] claim there exists a vast number of academic papers describing narrow maintenance models, which are rarely, if ever, used in practice. Most of these papers are too abstract, and the models that could be useful are difficult to identify among this large number of suggested models.

In this paper we propose an algorithm to solve the previous problem for equipment that exhibits Weibull hazard function and constant repair rate. Based on this algorithm we have developed another one to optimise maintenance management of a series system based on preventive maintenance over the different system components.
This is a problem with many applications in real systems and there are not many practical solutions for it. The main objective of this paper is to present an optimisation model understandable by practitioners, simple and useful for practical applications.
Duarte and Craveiro [3] have already outlined a solution for this problem for equipment that exhibit linearly increasing hazard rate and constant repair rate.
We assume that all components of the system still exhibit Weibull hazard function and constant repair rate and that preventive maintenance would bring the system to the as good as new condition. We define a cost function for maintenance tasks (preventive and corrective) for the system. The algorithm calculates the interval of time between preventive maintenance actions for each component, minimizing the costs, and in such a way that the total downtime, in a certain period of time, does not exceed a predetermined value.

2. Previous concepts and results

In this section we present the classical concept of availability, while describing how to calculate it.
Point-wise availability of a system at time *t*, *A(t)*, *is* the probability of the system being in a working state (operating properly) at time *t*. The unavailability of the system, *Q(t)*, is *Q(t) = 1 – A(t)*.
The pointwise availability of a system that has constant failure rate λ and constant repair rate μ is

$$A(t) = \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}\exp\left[-(\mu+\lambda)t\right]. \quad (1)$$

and the limiting availability is

$$A = \lim_{t\to+\infty} A(t) = \frac{\mu}{\mu+\lambda}. \quad (2)$$

The second parcel in formula (1) decreases rapidly to zero as time *t* increases; so, we can state

$$A(t) \approx \frac{\mu}{\mu+\lambda}$$

and this means that the availability of such a system is almost constant.

Example. A system is found to exhibit a constant failure rate of 0,000816 failures per hour and a constant repair rate of 0,02 repairs per hour.

Using formula 1, the availability of such a system (see *Figure 1*) is obtained as

$$A(t) = 0{,}9608 + 3.9201\times10^{-2}\exp(-2{,}0816\times10^{-2}t).$$

and the limiting availability is

$$\lim_{t\to\infty} A(t) = 0{,}9608.$$

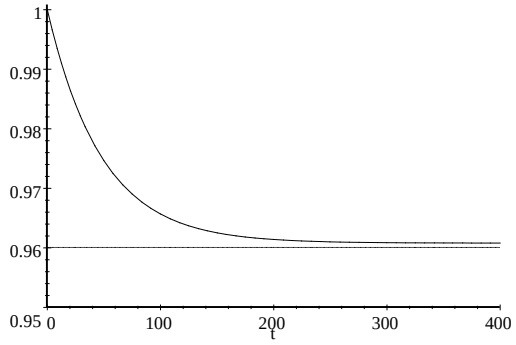

Figure 1. The availability function

It should be notice that, in this in case, we do not have almost any variation in the value of component's availability for *t > 200*.
We can therefore conclude that, to guarantee a value of availability *A*, known the constant repair rate, μ, the value of the constant failure rate of the system it will have to satisfy the relationship

$$A \approx \frac{\mu}{\mu+\lambda} \Leftrightarrow \lambda \approx \frac{\mu(1-A)}{A}. \quad (3)$$

3. Model and assumptions

Suppose a system is found to exhibit an increasing hazard rate,

$$h(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}, \theta > 0, \beta > 0, t \geq 0,$$

and a constant repair rate μ.
Our goal is to determine the interval time between preventive maintenance tasks (we assume that the system is restored to the "as good as new" condition after each maintenance operation) in such a way that the availability of the system is no lesser than *A*.
The main idea for the solution of this problem consists of determining the time interval during which the increasing hazard rate can be substituted by a constant

failure rate in order to guarantee the pre-determinate availability level.
Applying the mean value theorem of integral calculus to the function

$$h(t)=\frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1},\theta>0,\beta>0,t\geq 0,$$

we obtain

$$\int_0^x \frac{\beta}{\theta^\beta}t^{\beta-1}dt=\lambda x \Rightarrow \left[\frac{t^\beta}{\theta^\beta}\right]_0^x=\lambda x$$

$$\Rightarrow \frac{x^\beta}{\theta^\beta}=\lambda x \qquad (4)$$

$$\Rightarrow x=0 \vee x=\sqrt[\beta-1]{\lambda\theta^\beta}.$$

Substituting λ in (4) by its approximate value given in formula 3, we have

$$x=\sqrt[\beta-1]{\frac{\mu(1-A)}{A}\theta^\beta}.$$

We can therefore conclude that in the time interval

$$\left[0,\sqrt[\beta-1]{\frac{\mu(1-A)}{A}\theta^\beta}\right],$$

the hazard functions,

$$h(t)=\frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1},\theta>0,\beta>0,t\geq 0\ ,$$

and

$$h(t)=\frac{\mu(1-A)}{A}$$

guarantee approximately the same value of availability.
What we have just demonstrated can formally be stated on the following form:
Proposition: Let *S* be a system exhibiting an increasing hazard rate,

$$h(t)=\frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1},\theta>0,\beta>0,t\geq 0$$

and a constant repair rate μ. To guarantee an availability for the system equal or greater than *A* the interval of time between two consecutive preventive maintenance tasks must be equal or lesser than

$$\sqrt[\beta-1]{\frac{\mu(1-A)}{A}\theta^\beta}.$$

Example. A system is found to exhibit an increasing hazard rate, $h(t)=5\times10^{-8}\times t^{1,25}$, and a constant repair rate $m(t)=4\times10^{-2}$.
What should be the greatest time interval between preventive maintenance tasks (we assume that the system is restored to the "as good as new" condition after each maintenance operation) in such a way that the availability of the system is at least *98%*?
If the system had a constant failure rate, to guarantee such availability it should be

$$\lambda=\frac{4\times10^{-2}(1-0{,}98)}{0{,}98}=0{,}0008163$$

We want to calculate the instant *x* in order to satisfy the following condition

$$\int_0^x 5\times10^{-8}\times t^{1,25}dt=0{,}0008163x$$

$$\Rightarrow \frac{5\times10^{-8}\times x^{2,25}}{2{,}25}=0{,}0008163x$$

$$\Rightarrow x=0 \vee x=4488.$$

We can therefore conclude that the system must be restored to the "as good as new" condition after each maintenance task every *4488* hours in order to achieve the availability target of *98%*.
Figure 2 illustrates this example.

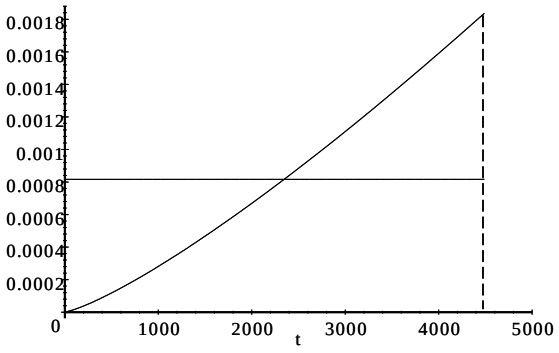

Figure 2. Hazard functions $h(t)=5\times10^{-8}t^{1,25}$ and $h(t)=0{,}0008163$ over the interval [0, 4488]

4. Optimisation of the preventive maintenance plan of a series components system

In this section we will present a model for the preventive maintenance management of a series system.
The system is composed by a set of n components in series as *Figure 3* shows.

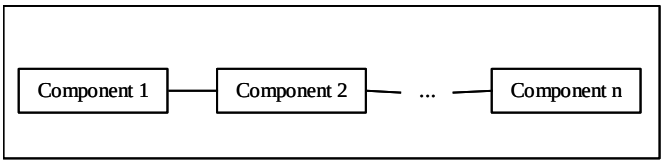

Figure 3. A series system of n components

Let τ_1, τ_2, ..., τ_n be the time units between preventive maintenance tasks on components *1, 2, ..., n*, respectively (*Figure 4*); assuming that these actions will restore periodically the components to the "as good as new" condition, they will have, therefore, consequences at the reliability and availability levels of the system.

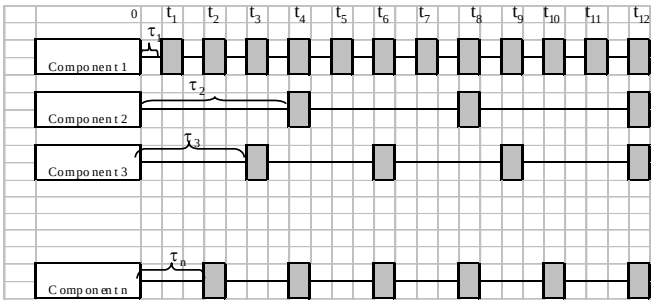

Figure 4. A preventive maintenance plan.

Our goal is to calculate the vector

$$[\tau_1 \quad \tau_2 \quad \tau_3 \quad \cdots \quad \tau_n]^T$$

in such a way that the total down time in a certain period of time does not exceed a predetermined value, that is to say, that it guarantees the specified service level and simultaneously minimizes the maintenance costs.
We assume that each component has a Weibull hazard function,

$$h_i(t) = \frac{\beta_i}{\theta_i}\left(\frac{t}{\theta_i}\right)^{\beta_i - 1}, \theta_i > 0, \beta_i > 0, t \geq 0$$

and a constant repair rate

$$m_i(t) = \mu_i .$$

The cost of each preventive maintenance task is cmp_i and the cost of each corrective maintenance task is cmc_i.
Since the availability of the system consisting of n components in series requires that all units must be available (assuming that components' failures are independent), system availability A is

$$A = \prod_{i=1}^{n} A_i$$

where A_i is the availability of component i.
Applying proposition presented in section 3 we can write that the availability of each component i is A_i over the interval

$$\left[0, \sqrt[\beta_i - 1]{\frac{\mu_i(1-A_i)}{A_i}\theta_i^{\beta_i}}\right],$$

and its hazard function can be approximated by the constant function

$$h_i(t) = \frac{\mu_i(1-A_i)}{A_i}$$

Then, the expected number of failures in that time interval is

$$\sqrt[\beta_i - 1]{\frac{\mu_i(1-A_i)}{A_i}\theta_i^{\beta_i}} \times \frac{\mu_i(1-A_i)}{A_i}$$

The objective function (defined as a cost function per unit time) is

$$c(A_1, A_2, \ldots, A_n) = \sum_{i=1}^{n}\left[\frac{cmp_i}{\sqrt[\beta_i - 1]{\frac{\mu_i(1-A_i)}{A_i}\theta_i^{\beta_i}}} + \frac{cmc_i}{\frac{\mu(1-A_i)}{A_i}}\right]$$

subject to

$$\begin{cases} \prod_{i=1}^{n} A_i \geq A, \\ 0 < A_i < 1, i = 1,2,\ldots,n. \end{cases}$$

5. Numerical example

The model described on section 4 was implemented to a three components series system.
We assume that each component has a Weibull hazard function and a constant repair rate. Components are maintained preventively at periodic times.

Data is presented on *Table 1*.
First we present the nomenclature.
θ_i, β_i – parameters of hazard function.
TTR – Mean Time to Repair (corrective maintenance).
TTP – Time of one preventive maintenance action.
PMC – Preventive maintenance cost.
CMC – Corrective maintenance cost.
τ - time between two consecutive preventive maintenance tasks.

Table 1. Initial conditions

Components	θ_i	β_i	TTR	TTP	PM Cost	CM Cost	τ_i
1	4472,136	2	100	10	2000	4000	2000
2	1873,1716	2	50	40	2500	5000	1500
3	500,94	2	80	10	1000	2000	250

With this preventive maintenance plan the availability achieved is about *90,30%* and the life cycle cost is *122055,79*.
The target for availability is 90%.
The objective function was slightly modified in order to include the cost of down time.
MATLAB was used to optimise the objective function. Table 2 shows the results. With this new preventive maintenance policy we have a reduction of 5,5% in Life Cycle Cost (LCC) and simultaneously the availability *A* achieved (*92,70%*) is greater than the existing one (*90,30%*).

Table 2. Results of MatLab optimisation

MatLab Optimisation	τ_1	1600.2
	τ_2	1246.8
	τ_3	170.7535
	A - %	92.70
	LCC	115345.22
	Δ LCC - %	-5.5

With these results as initial conditions we have applied the tool "SOLVER" of Excel and we got a better solution (Table 3).

Table 3. Results of MatLab + Excel optimisation

MatLab + Excel Optimisation	τ_1	1606.498
	τ_2	1255.498
	τ_3	175.4996
	A - %	93.02
	LCC	113809.75
	Δ LCC - %	-6.8

5. Conclusion

This paper deals with a maintenance optimisation problem for a series system. First we have developed an algorithm to determine the optimum frequency to perform preventive maintenance in systems exhibiting Weibull hazard function and constant repair rate, in order to ensure its availability. Based on this algorithm we have developed another one to optimise maintenance management of a series system based on preventive maintenance over the different system components. We assume that all components of the system still exhibit Weibull hazard function and constant repair rate and that preventive maintenance would bring the system to the as good as new condition. We define a cost function for maintenance tasks (preventive and corrective) for the system. The algorithm calculates the interval of time between preventive maintenance actions for each component, minimizing the costs, and in such a way that the total downtime, in a certain period of time, does not exceed a predetermined value. The maintenance interval of each component depends on factors such as failure rate, repair and maintenance times of each component in the system. In conclusion, the proposed analytical method is a feasible technique to optimise preventive maintenance scheduling of each component in a series system.
Currently we are developing a software package for the implementation of the algorithms presented in this paper.

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