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Algebraic Riccati equation based Q and R matrices selection algorithm for optimal LQR applied to tracking control of 3rd order magnetic levitation system

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Abstract: This paper presents an analytical approach for solving the weighting matrices selection problem of a linear quadratic regulator (LQR) for the trajectory tracking application of a magnetic levitation system. One of the challenging problems in the design of LQR for tracking applications is the choice of Q and R matrices. Conventionally, the weights of a LQR controller are chosen based on a trial and error approach to determine the optimum state feedback controller gains. However, it is often time consuming and tedious to tune the controller gains via a trial and error method. To address this problem, by utilizing the relation between the algebraic Riccati equation (ARE) and the Lagrangian optimization principle, an analytical methodology for selecting the elements of Q and R matrices has been formulated. The novelty of the methodology is the emphasis on the synthesis of time domain design specifications for the formulation of the cost function of LQR, which directly translates the system requirement into a cost function so that the optimal performance can be obtained via a systematic approach. The efficacy of the proposed methodology is tested on the benchmark Quanser magnetic levitation system and a detailed simulation and experimental results are presented. Experimental results prove that the proposed methodology not only provides a systematic way of selecting the weighting matrices but also significantly improves the tracking performance of the system.

Key words: algebraic Riccatti equation, linear quadratic regulator, magnetic levitation system, weighting matrices, command following, cost function

1. Introduction

In the last few decades, classical optimal control theory has evolved to formulate the well-known optimal state feedback controller called LQR, which minimizes the deviation in state trajectories of a system while maintaining a minimum control effort. The LQR design is considered the foundation of the Linear Quadratic Gaussian/Loop Transfer Recovery (LQG/LTR). Due to their inherent robustness and stability properties, such as a gain margin of $(-6, \infty)$ dB and a phase margin of $(-60°, 60°)$, LQR finds its application in many engineering and scientific domains [1-3]. In the last two decades several investigations have been reported on LQR, namely, self-adjusting LQR [4], switched LQR [5], hybrid LQR [6] and fuzzy LQR [7]. In addition, LQR techniques have been successfully implemented for a large number of complex systems such as the double inverted pendulum [8], fuel cell systems [9], vibration control system [10], electric vehicles [11], and aircraft [12].

However, two main issues of the LQR problem have been the subject of investigation since the 1960s until the present day: the solution of ARE, and the choice of Q and R weighting matrices [13]. The two tasks are known to be strongly time dependent and based on certain operational conditions. Even if all of the control strategies are optimal in nature, different values of Q and R will ultimately end up with a different system response, which indicates that the response is non-optimal in a true sense. Conventional optimization methods, such as the gradient search method, used for designing the state feedback controller are restricted to the eigen values of the linear system matrix that not only increases the difficulty but also consumes long time to find the global optimum solution [14]. Hence, researchers have considered employing evolutionary computation (EC) techniques for solving the LQR optimization problem. For instance, in [15] genetic algorithm (GA) has been used for optimally selecting the weighting matrices of LQR for state feedback control design of multi machine power system. GA based LQR design for ball and beam position control application has been reported in [16], and the results are reported to be encouraging. Similarly, the stabilizing controller design for inverted pendulum is solved using particle swarm optimization (PSO) in [17]. In [18] the performances of GA, PSO and APSO for tracking control application of single inverted pendulum are compared and reported that APSO yields better response than those of GA and PSO. However, the fundamental limitations of EC techniques such as high dependency on the parameters of optimization algorithm and time of computation limit the use of EC algorithms for solving real world optimization problems.

Therefore, in this paper, as an alternative to solve weight selection problem of LQR, we propose an analytical approach by making use of the relationship between the ARE and Lagrangian optimization principle. The advantages of the proposed methodology are twofold. First, it significantly reduces the time needed for optimal selection of Q and R matrices with the aid of simple mathematical expressions. Second, the analytical approach has the ability to translate the system's performance objectives in time domain into cost function which makes the design of LQR not only simple but also modular.

The rest of the paper is organized as follows. The problem formulation is presented in Section 2. Section 3 details the analytical procedure for selecting the Q and R matrices based on time domain specifications. System description and the mathematical modelling of magnetic levitation plant is given in Section 4. Simulation results of trajectory tracking response are given in Section 5, and the experimental results are detailed in Section 6. The paper ends with the concluding remarks in Section 7.

2. Problem formulation

Consider an LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad t \geq 0, \quad x(0) = 0, \\ y(t) &= Cx(t) + Du(t), \quad t \geq 0, \end{aligned} \tag{1}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$, are system matrix, input matrix, output matrix and feed forward matrix, respectively. X is the state vector, u is the control input vector, and y is the output vector. The conventional LQR problem is to obtain the control input u^* which minimizes the following cost function.

$$J(u) = \int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)]dt, \tag{2}$$

where $Q = Q^T$ is a positive semidefinite matrix that penalizes the departure of system states from the equilibrium, and $R = R^T$ is a positive definite matrix that penalizes the control input [19]. The solution of the LQR problem, the optimal control gain K, can be determined via the following Lagrange multiplier based optimization technique.

$$K = R^{-1}B^T P. \tag{3}$$

The optimal state feedback control gain matrix K of LQR can be found by solving the following ARE.

$$A^T P + PA + Q - PBR^{-1}B^T P = 0, \tag{4}$$

where $P \in R^{n\times n}$ is the solution of *ARE*. The elements of weighting matrices Q and R are important components of an LQR optimization process. The compositions of Q and R elements have great influence not only on system performance but also on system input [20-21]. The number of elements of Q and R matrices depends on the number of state variable (n) and the number of input variable (m), respectively. If the weighting matrices are chosen as diagonal matrices, the quadratic performance index is simply a weighted integral of the square of the states and inputs [22]. Commonly, a trial and error method has been used to select the elements of Q and R matrices. This method is cumbersome, time consuming and does not result in optimum performance. In order to address this issue, in the following section we propose an analytical procedure for selecting the weight matrices of LQR based on the time domain specifications of the system to be controlled.

3. Analytical approach for weighting matrices selection of LQR

Consider the third order LTI system represented in controllable canonical form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ B_{31} \end{bmatrix} u. \tag{5}$$

Designing a full state feedback control via LQR requires the minimization of the cost function given in (2), which places the weight not only on control input but also on states of the system. Hence, the state feedback control law is given by

$$u = -Kx. \tag{6}$$

By solving the ARE, the transformation matrix (P) between states and co-states can be obtained. One of the essential features of LQR is that Q should be a symmetric positive semi-definite matrix and R should be a positive definite matrix. So, the weighting matrices Q and R, and the solution of ARE are chosen as

$$Q = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{bmatrix}, \quad R = r, \quad P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{13} \\ p_{13} & p_{23} & p_{33} \end{bmatrix}. \tag{7}$$

Using the Lagrange optimization technique, the state feedback gain matrix K can be calculated as

$$K = R^{-1}B^T P = \frac{1}{r}\begin{bmatrix} 0 & 0 & B_{31} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix}, \tag{8}$$

$$K = \frac{B_{31}}{r}\begin{bmatrix} p_{13} & p_{23} & p_{33} \end{bmatrix}. \tag{9}$$

The elements of P matrix such as p_{13}, p_{23} and p_{33} are obtained using the ARE given in (4).

$$\begin{bmatrix} 2p_{13}A_{31} + -\dfrac{B_{31}^2}{r}p_{13}^2 + q_1 & p_{11}+p_{13}A_{32} + p_{23}A_{31} + -\dfrac{B_{31}^2}{r}p_{13}p_{23} & p_{12}+p_{13}A_{33} + p_{33}A_{31} + -\dfrac{B_{31}^2}{r}p_{13}p_{33} \\ p_{11}+p_{13}A_{32} + p_{23}A_{31} + -\dfrac{B_{31}^2}{r}p_{13}p_{23} & 2(p_{12}+A_{32}p_{23}) + -\dfrac{B_{31}^2}{r}p_{23}^2 + q_2 & p_{13}+p_{22} + p_{23}A_{33} + p_{33}A_{32} + -\dfrac{B_{31}^2}{r}p_{23}p_{33} \\ p_{12}+p_{13}A_{33} + p_{33}A_{31} + -\dfrac{B_{31}^2}{r}p_{13}p_{33} & p_{13}+p_{22} + p_{23}A_{33} + p_{33}A_{32} + -\dfrac{B_{31}^2}{r}p_{23}p_{33} & 2(p_{23}+A_{33}p_{33}) + -\dfrac{B_{31}^2}{r}p_{33}^2 + q_3 \end{bmatrix} = 0. \tag{10}$$

3.1. Closed loop response of the system

The closed loop state equation of the system can be written as

$$\dot{x}(t) = [A - BK]\,x(t) = \left[A - BR^{-1}B^{T}P\right]x(t). \tag{11}$$

According to direct substitution method for the pole placement controller design, the actual characteristic equation of the system is compared with the desired characteristic equation to obtain the expressions for p_{13}, p_{23} and p_{33} in terms of state model and design specifications in time domain. Therefore, the actual characteristic equation of the system can be represented as

$$|\lambda I - A + BK| = 0. \tag{12}$$

Substitution of the corresponding system matrix A, input matrix B and the state feedback controller gain matrix K in the above characteristic equation results in

$$s^3 + s^2\left(\frac{p_{33}B_{31}^2}{r} - A_{33}\right) + s\left(\frac{p_{23}B_{31}^2}{r} - A_{32}\right) + \left(\frac{p_{13}B_{31}^2}{r} - A_{31}\right) = 0. \tag{13}$$

The general form of desired characteristic equation of the third order system is

$$\begin{aligned}&(s + \delta\omega_n)\left(s^2 + 2\delta\omega_n s + \omega_n^2\right) = 0.\\ &s^3 + 3\delta\omega_n s^2 + \left(2\delta^2 + 1\right)\omega_n^2 s + \delta\omega_n^3 = 0.\end{aligned} \tag{14}$$

Comparing Equations (13) and (14), the expressions for p_{13}, p_{23} and p_{33} can be obtained as given below

$$p_{13} = \frac{r}{B_{31}^2}\left(A_{31} + \delta\omega_n^3\right), \tag{15}$$

$$p_{23} = \frac{r}{B_{31}^2}\left(A_{32} + (2\delta^2 + 1)\,\omega_n^2\right), \tag{16}$$

$$p_{33} = \frac{r}{B_{31}^2}\left(A_{33} + 3\delta\omega_n\right), \tag{17}$$

From the element available at first row and first column of Equation (10),

$$2p_{13}A_{31} - \frac{B_{31}^2}{r}p_{13}^2 + q_1 = 0. \tag{18}$$

Substituting (15) into (18) results in

$$2\frac{r}{B_{31}^2}A_{31}\left(A_{31} + \delta\omega_n^3\right) - \frac{B_{31}^2}{r}\frac{r^2}{B_{31}^4}\left(A_{31} + \delta\omega_n^3\right)^2 + q_1 = 0. \tag{19}$$

$$\frac{1}{B_{31}^2}\left(A_{31} + \delta\omega_n^3\right)\left(A_{31} - \delta\omega_n^3\right) + \frac{q_1}{r} = 0. \tag{20}$$

By rearranging, the expression for q_1/r can be obtained as

$$\frac{q_1}{r} = \frac{1}{B_{31}^2}\left((\delta\omega_n^3)^2 - A_{31}^2\right). \tag{21}$$

Similarly, from (10), the element available at third row and first column can be rearranged as

$$p_{12} + p_{13}A_{33} + p_{33}A_{31} - \frac{B_{31}^2}{r} p_{13}p_{33} = 0, \tag{22}$$

$$p_{12} = -p_{13}A_{33} - p_{33}A_{31} + \frac{B_{31}^2}{r} p_{13}p_{33}. \tag{23}$$

The element in second row second column of ARE (10) can be rearranged in terms of known quantities such as p_{13}, p_{23} and p_{33} by substituting (23) into the following equation

$$2(p_{12} + A_{32}p_{23}) - \frac{B_{31}^2}{r} p_{23}^2 + q_2 = 0, \tag{24}$$

$$2\left(-p_{13}A_{33} - p_{33}A_{31} + \frac{B_{31}^2}{r} p_{13}p_{33} + p_{23}A_{32}\right) - \frac{B_{31}^2}{r} p_{23}^2 + q_2 = 0. \tag{25}$$

Substituting p_{13}, p_{23} and p_{33} into the above equation,

$$2\left(-\frac{r}{B_{31}^2}\left(A_{31} + \delta\omega_n^3\right)A_{33} - \frac{r}{B_{31}^2}\left(A_{33} + 3\delta\omega_n\right)A_{31} + \frac{r}{B_{31}^2}\left(A_{31} + \delta\omega_n^3\right)\left(A_{33} + 3\delta\omega_n\right) + \frac{r}{B_{31}^2}\left(A_{32} + (2\delta^2+1)\omega_n^2\right)A_{32}\right)$$
$$-\frac{B_{31}^2}{r}\left(\frac{r}{B_{31}^2}\left(A_{32} + (2\delta^2+1)\omega_n^2\right)\right)^2 q_2 = 0. \tag{26}$$

Rearranging (26), the expression for q_2/r can be obtained as given below.

$$\frac{q_2}{r} = \frac{2A_{31}(A_{33} - 2\delta\omega_n) + \delta\omega_n^3 A_{33} - 3\delta^2\omega_n^4 + (2\delta^2+1)^2\omega_n^4 + A_{32}((2\delta^2+1)\omega_n^2 + 1)}{B_{31}^2}. \tag{27}$$

Likewise, the element available in third row and third column of ARE (10) can be written as,

$$2(p_{23} + A_{33}p_{33}) - \frac{B_{31}^2}{r} p_{33}^2 + q_3 = 0. \tag{28}$$

Substituting (16) and (17) into the above equation,

$$2\left(\frac{r}{B_{31}^2}\left(A_{32} + (2\delta^2+1)\omega_n^2\right) + \frac{r}{B_{31}^2}\left(A_{33} + 3\delta\omega_n\right)A_{33}\right) - \frac{B_{31}^2}{r}\left(\frac{r}{B_{31}^2}\left(A_{33} + 3\delta\omega_n\right)\right)^2 + q_3 = 0. \tag{29}$$

By reordering the above equation,

$$\frac{q_3}{r} = \frac{-2(A_{32} + (2\delta^2 + 1)\omega_n^2) - A_{33}^2 + 9\delta^2\omega_n^2}{B_{31}^2}. \tag{30}$$

The elements of Q matrix can be obtained from the desired specifications by fixing the value of R matrix which is taken as scalar in the present example. The complete design procedure of LQR weight selection based on time domain specification is summarized below.

3.2. Design procedure

1) Obtain the mathematical model of the system in controllable canonical form.
2) Specify the required damping ratio (δ) and natural frequency (ω_n) of the system.
3) Represent the state feedback gain matrix K in terms of solution of ARE.

$$K = \frac{B_{31}}{r}[p_{13} \ \ p_{23} \ \ p_{33}].$$

4) Determine the actual characteristic equation of the system.

$$|\lambda I - A + BK| = 0.$$

5) Evaluate the desired characteristic equation of the system from the given time domain specifications.

$$(s + \delta\omega_n)(s^2 + 2\delta\omega_n s + \omega_n^2) = 0,$$

$$s^3 + 3\delta\omega_n s^2 + (2\delta^2 + 1)\omega_n^2 s + \delta\omega_n^3 = 0.$$

6) Compare the desired characteristic equation with the actual characteristic equation and obtain the elements of P matrix

$$p_{13} \frac{r}{B_{31}^2}\left(A_{31} + \delta\omega_n^3\right),$$

$$p_{23} \frac{r}{B_{31}^2}\left(A_{32} + (2\delta^2 + 1)\omega_n^2\right),$$

$$p_{33} \frac{r}{B_{31}^2}\left(A_{33} + 3\delta\omega_n\right).$$

7) Substitute the values of p_{13}, p_{23} and p_{33} in ARE and obtain the expression for q_1/r, q_2/r and q_3/r .

$$\frac{q_1}{r} = \frac{1}{B_{31}^2}\left((\delta\omega_n^3)^2 - A_{31}^2\right).$$

$$\frac{q_2}{r} = \frac{2A_{31}(A_{33} - 2\delta\omega_n) + \delta\omega_n^3 A_{33} - 3\delta^2\omega_n^4 + (2\delta^2 + 1)^2\omega_n^4 + A_{32}((2\delta^2 + 1)\omega_n^2 + 1)}{B_{31}^2}.$$

$$\frac{q_3}{r} = \frac{-2(A_{32} + (2\delta^2 + 1)\omega_n^2) - A_{33}^2 + 9\delta^2\omega_n^2}{B_{31}^2}.$$

8) Fix the value of r and find the elements of Q matrix using the above expressions.
9) Calculate the state feedback gain matrix (K) using the Lagrange multiplier based optimization technique.

One of the distinguishing features of the proposed analytical methodology, compared to the conventional pole placement techniques, is that it provides simple mathematical expressions by correlating the Q and R matrices elements with the design specifications and system matrices such that the elements of weighting matrices can be systematically computed. Moreover, by synthesizing the cost function with the time domain specifications it results in optimal response between control input and speed of response of the system. Hence, it largely reduces the time required to tune the values of weighting matrices in obtaining the optimal performance of the controller. To assess the performance of the proposed approach, a benchmark magnetic levitation system is chosen for experimentation and the mathematical modelling of the system and LQR controller design based on the analytical approach are explained in the following section.

4. Magnetic levitation system

Magnetic levitation (maglev) systems have received wide attention recently because of their practical importance in many engineering systems such as high-speed maglev passenger trains, frictionless bearings, levitation of wind tunnel models, vibration isolation of sensitive machinery, levitation of molten metal in induction furnaces, and levitation of metal slabs during manufacturing [23]. Magnetic levitation technology reduces the physical contact between moving and stationary parts and in turn eliminates the friction problem. Maglev systems are inherently nonlinear, unstable and are described by highly nonlinear differential equations which present additional difficulties in controlling these systems. So the design of feedback controller for regulating the position of the levitated object is always a challenging task.

4.1. System model

Maglev system consists of an electromagnet, a steel ball, a ball post, and a ball position sensor. The schematic diagram of the magnetic levitation system is shown in Figure 1, and the parameters of the maglev system are given in Table 1. The entire system is encased in a rectangular enclosure which contains three distinct sections. The upper section contains an electromagnet, made of a solenoid coil with a steel core. The middle section consists of a chamber where the ball suspension takes place. One of the electro magnet poles faces the top of a black post upon which a one inch steel ball rests. A photo sensitive sensor embedded in the post measures the ball elevation from the post. The last section of maglev system houses the signal conditioning circuitry needed for light intensity position sensor. The entire system is decomposed into two subsystems, namely, mechanical subsystem and electrical subsystem. The coil current is adjusted to control the ball position in the mechanical system, whereas the coil voltage is varied to control the coil current in an electrical system [24]. Thus, the voltage applied to the electromagnet indirectly controls the ball position. In the following section, the nonlinear mathematical model of the maglev system is obtained and linearized around the operating region to design a stabilizing controller.

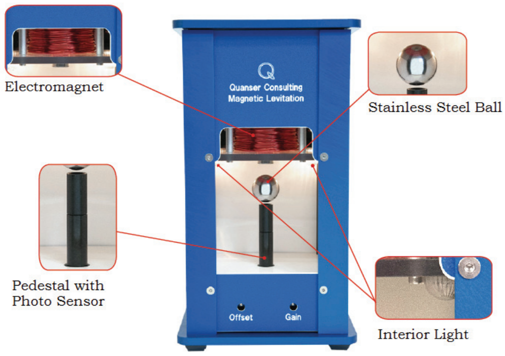

Fig. 1. Magnetic levitation system diagram

Table 1. System parameters

Symbol	Description	Value
L_c	coil inductance	412.5 mH
R_c	coil resistance	10 Ω
N_c	number of turns in the coil wire	2450
l_c	coil length	0.0825 m
r_c	coil steel core radius	0.008 m
R_s	current sense resistance	1 Ω
K_m	electromagnet force constant	6.53 E-005 N.m^2/A^2
r_b	steel ball radius	1.27 E-002 m
M_b	steel ball mass	0.068 kg
K_b	ball position sensor sensitivity	2.83 E-003 m/V
g	gravitational constant	9.81 m/s^2

4.2. Equation of motion of a ball

i) Electrical system modelling

From Figure 2, using Kirchoff's voltage law, the following first order differential equation can be obtained.

$$V_c(t) = (R_c + R_s)I_c(t) + L_c \frac{d}{\mathrm{d}t} I_c(t), \tag{31}$$

$$\frac{d}{\mathrm{d}t} I_c(t) = -\frac{(R_c + R_s)}{L_c} I_c(t) + \frac{V_c(t)}{L_c}. \tag{32}$$

ii)Electro-mechanical system modelling

Force applied on the ball due to gravity can be expressed as

$$F_g = M_b g. \tag{33}$$

The force created by the coil is given as

$$F_c = -\frac{K_m I_c^2}{2x_b^2} \tag{34}$$

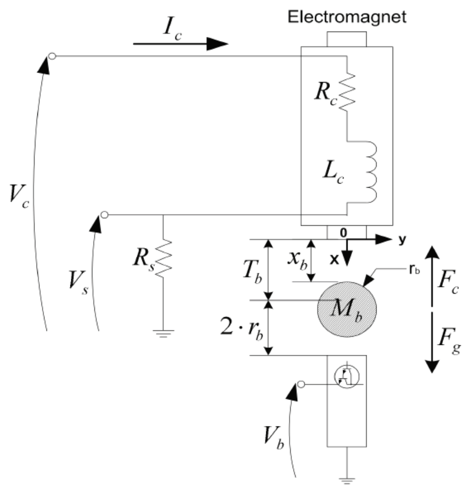

Fig. 2. Schematic diagram of the Maglev plant

Therefore, the total external force experienced by the ball is the sum of the gravity and coil forces.

$$F_g + F_c = -M_b g - \frac{K_m I_c^2}{2x_b^2}. \tag{35}$$

Applying Newton's second law, we obtain the following equation of motion of the ball.

$$\frac{d^2 x_b}{\mathrm{d}t^2} = \frac{2g x_b}{x_{b0}} - \frac{2g I_c}{I_{c0}}, \tag{36}$$

where x_{b0} and I_{c0} are the ball position and coil current at equilibrium point. Considering the ball position (x_b), velocity of the ball ($dx_b/\mathrm{d}t$) and coil current (I_c) as the state variables the following state model of the system is obtained.

$$\begin{bmatrix} \dot{x}_b \\ \ddot{x}_b \\ \dot{I}_c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \dfrac{2g(R_c + R_s)}{x_{b0}L_c} & \dfrac{2g}{x_{b0}} & \dfrac{-(R_c + R_s)}{L_c} \end{bmatrix} \begin{bmatrix} x_b \\ \dot{x}_b \\ I_c \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{-2g}{I_{c0}L_c} \end{bmatrix} V_c. \tag{37}$$

The objective of the LQR control strategy is to levitate the ball in mid-air and make it track the reference trajectory, while maintaining minimum control input. Figure 3 illustrates the closed loop structure of the analytically tuned LQR control scheme. As the implementation of full state feedback controller using LQR requires full state vector, a second order derivative filter is designed to estimate the velocity of the ball. The transfer function of the derivative filter is given by

$$H(s) = \frac{\omega_{cf}^2 s}{s^2 + 2\zeta\omega_{cf} s + \omega_{cf}^2}. \tag{38}$$

Empirically, the damping ratio (ζ) and cut off frequency (ω_{cf}) of the filter are chosen as 0.8 and 314.57 rad/sec respectively.

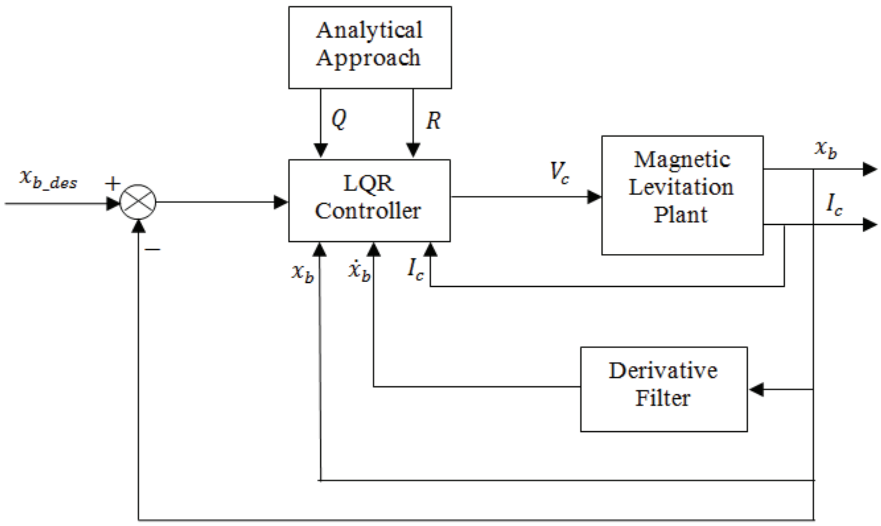

Fig. 3. LQR control scheme for maglev plant

5. Simulation results

To assess the efficacy of the proposed methodology, simulation is carried out in MATLAB. By substituting the parameter values of maglev system from Table 1, into (37), the following numerical state space model is obtained.

$$\begin{bmatrix} \dot{x}_b \\ \ddot{x}_b \\ \dot{I}_c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 87200 & 3270 & -2.667 \end{bmatrix} \begin{bmatrix} x_b \\ \dot{x}_b \\ I_c \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -5.549 \end{bmatrix} V_c .$$

$$Y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_b \\ \dot{x}_b \\ I_c \end{bmatrix} .$$

The open loop eigen values of the system are found to be –57.87, –26.67 and 57.18. The positive real part of eigen value suggests that the open loop system is unstable in nature and implies the necessity of a feedback controller. To assess the step response of the system, we have taken the damping ratio of the desired system to be 0.75 and a settling time of 0.5 s. Since the given system has only one input, the voltage applied to coil, the value of R matrix is chosen as 0.001. Then, the corresponding Q matrix obtained via the proposed methodology is

$$Q = \begin{bmatrix} -2483.91 & 0 & 0 \\ 0 & -4.92 & 0 \\ 0 & 0 & -2.33 \end{bmatrix}.$$

The optimal state feedback controller gain which satisfies the given time domain specification is

$$K = \begin{bmatrix} -1696.2 & -75.02 & -0.29 \end{bmatrix}.$$

Figure 4 depicts the step response of the system, and it can be observed that the settling time of the response is 0.4 s, which proves that the design is satisfactory. Moreover, the ball position response yields zero steady state error. Further to test the convergence of the analytical methodology, three different values of damping ratio and natural frequency are considered to evaluate the impulse response of the system. Values of time domain specification considered for the design and the corresponding weighting matrices calculated according to the design procedure are given in Table 2. One can note that the value of R can be set to any scalar value because the approach does not place any constraint on it.

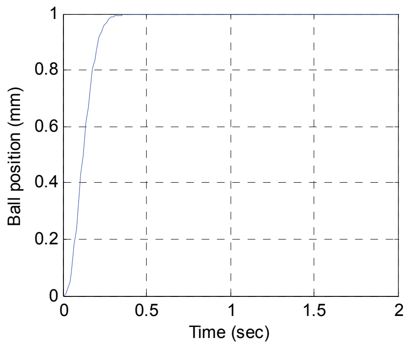

Fig. 4. Step response of LQR design

Table 2. Weighting matrices and controller gains for R = 0.001

t_s	δ	$\%M_p$	Q	K	**Closed loop eigen values**
0.28 s	0.65	6.55	$\begin{bmatrix} -2483.91 & 0 & 0 \\ 0 & -4.92 & 0 \\ 0 & 0 & -2.33 \end{bmatrix}$	$[-1696.2\ -75.02\ -0.29]$	$-14.30 + 16.72i$ $-14.30 + 16.72i$ -14.30
0.4 s	0.7	4.2	$\begin{bmatrix} -2468.11 & 0 & 0 \\ 0 & -4.96 & 0 \\ 0 & 0 & -2.32 \end{bmatrix}$	$[-1608.3\ -66.21\ -0.06]$	$-10.00 + 10.21i$ $-10.00 - 10.21i$ -10.00
0.5 s	0.75	3.1	$\begin{bmatrix} -2469.19 & 0 & 0 \\ 0 & -4.98 & 0 \\ 0 & 0 & -2.32 \end{bmatrix}$	$[-1587.9\ -63.28\ -0.05]$	$-8.00 + 7.06i$ $-8.00 - 7.06i$ -8.00

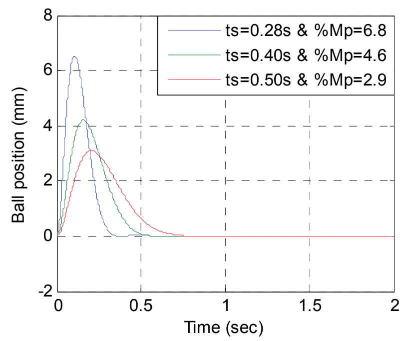

Fig. 5. Impulse response of ball position

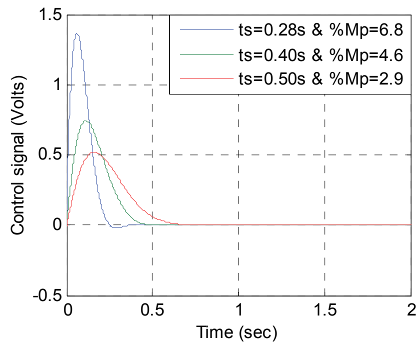

Fig. 6. Control signal

The impulse response of the system for three design requirements in time domain is shown in Figure 5. It is interesting to note that both the settling time and overshoot of the response exactly meet the design requirement. From Figure 6, which shows the amount of control input given to the plant, it is worth to note that the maximum value of coil voltage required to levitate the ball is contained well below 2 V and it does not reach the saturation level.

To assess the robustness of LQR design, the Bode plot of the system for three natural frequencies is plotted and shown in Figure 7. Positive gain margins of all the three cases suggest that the system is stable, and it can also accommodate the disturbances present in the system.

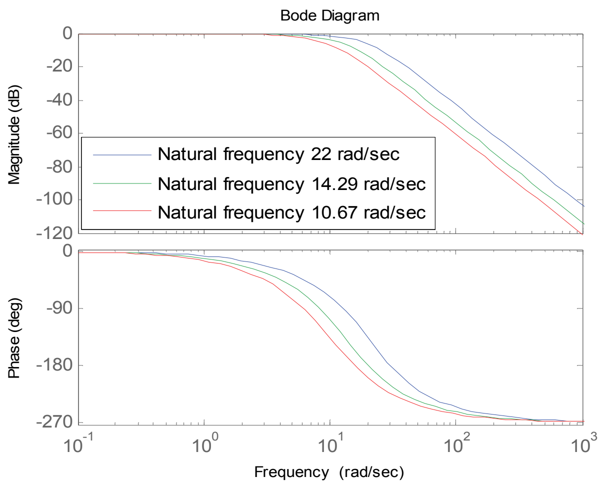

Fig. 7. Bode plot of the closed loop system

5.1. Trajectory tracking

The ability of the controller to track the reference signal is tested by providing a sinusoidal signal with a magnitude of ±1 mm. Figures 8-10 show the response of the control scheme for the three design requirements given in Table 2. From the tracking response, it is observed that the error between actual trajectory and reference trajectory decreases as the settling time decreases.

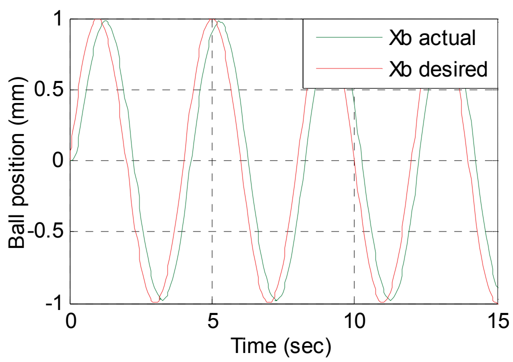

Fig. 8. Sine wave trajectory for $t_s = 0.5$ and $\delta = 0.75$

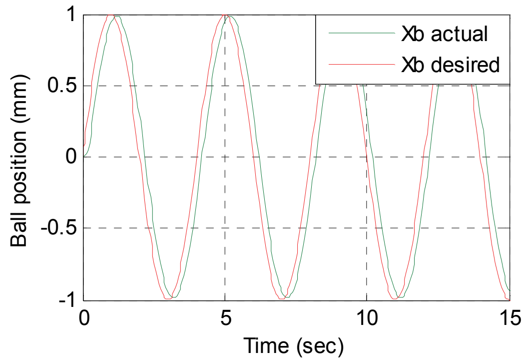

Fig. 9. Sine wave trajectory for $t_s = 0.4$ and $\delta = 0.7$

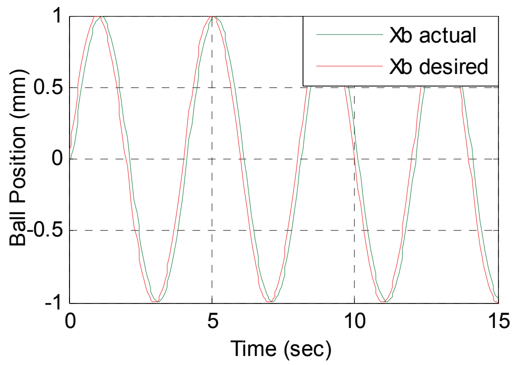

Fig. 10. Sine wave trajectory for $t_s = 0.28$ and $\delta = 0.65$

6. Experimental validation

The experimental set up, as shown in Figure 11, consists of a personal computer and a Quanser Magnetic levitation plant. The controller is implemented using a 32 bit personal computer. The proposed control algorithm is realized using QUARC, which is a real time algorithm similar to C like language. The Q8 USB hardware-in-loop (HIL) data acquisition board has 8 digital inputs and 8 pulse width modulated (PWM) digital outputs, and it is capable of reaching 4 kHz sampling rate. In addition, the system contains a Volt-PAQ power amplifier, which provides a regulated ±10 V at 1 A, to amplify the control signal given to the electromagnet. In order to attenuate the high frequency noise current, a derivative low pass filter with a cut off frequency of 80 Hz is added to the ball position sensor output.

Figure 12, which shows the trajectory tracking response of the plant for a sinusoidal test signal of ±1 mm amplitude at a frequency of 1 Hz, accentuates that the optimal controller could provide stable regulation over the entire operating point. Figure 13 illustrates the response of coil current, which tracks the specified command to make the ball follow the reference trajectory. Figure 14, which shows the amount of coil voltage given to the electromagnet, underscores that minimum control input is always required to levitate the ball. Figure 15 shows the tracking error, which is the difference between actual trajectory and reference trajectory. To assess the closeness of reference tracking, four performance indices namely, IAE, ISE, ITAE and ITSE, are computed and given in Table 3. The minimum value of integral errors accentuate that the controller yields good tracking ability while maintaining minimum control input.

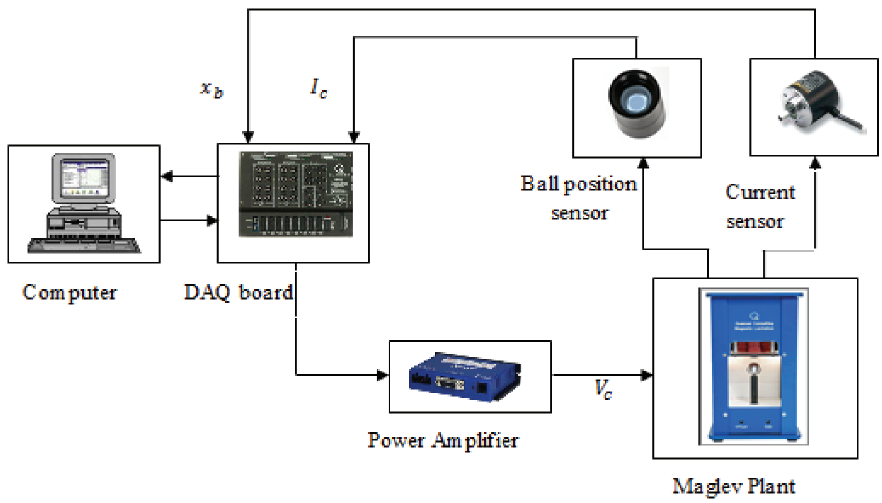

Fig. 11. Connection diagram of maglev system

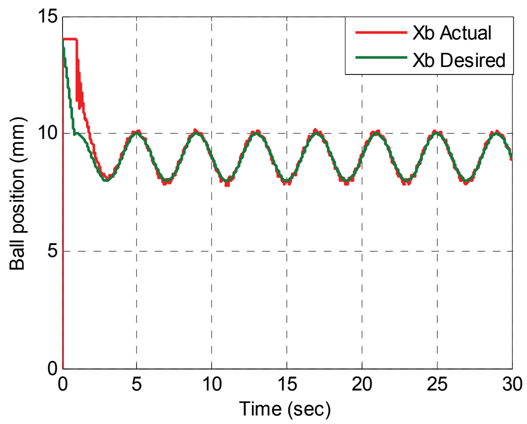

Fig. 12. Sinusoidal trajectory

Table 3. Integral performance indices of tracking error

IAE	ITAE	ISE	ITSE
0.0039	0.116	0.000016	0.000017

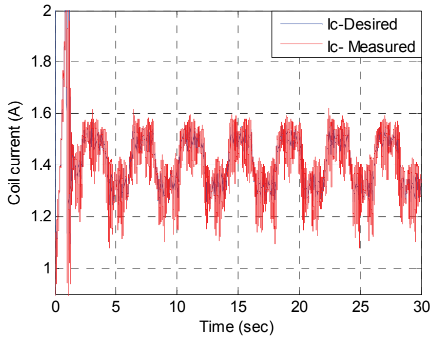

Fig. 13. Coil current

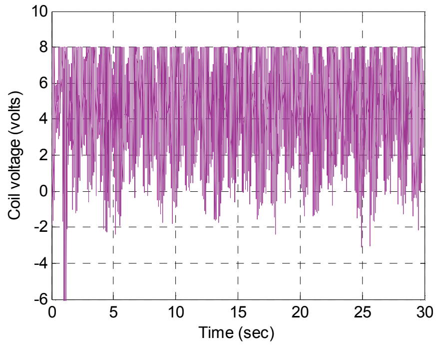

Fig. 14. Coil voltage

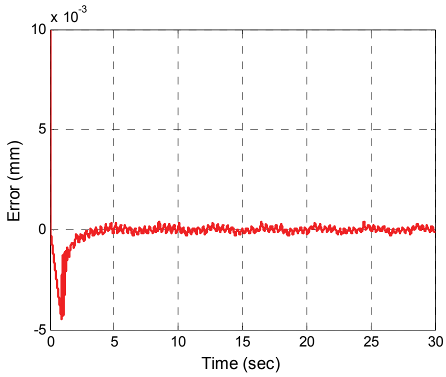

Fig. 15. Trajectory tracking error

7. Conclusions

In this paper, an analytical approach for selecting the weighting matrices of LQR for a third order system based on the time domain specification has been proposed and implemented for controlling the ball position of a magnetic levitation system. Normally, the Q and

R matrices of LQR are chosen based on iterative approach which further increases the complexity of the system to implement the LQR in real time. Therefore, to avoid the tedious manual tuning of LQR, an analytical methodology, by making use of ARE and the Lagrangian optimization technique, has been formulated. The novelty of the methodology is that it exploits the relation between the solution of ARE and the state model of the system with the design requirement in time domain specifications and provides a systematic way of selecting the elements of Q and R matrices. The effectiveness of the methodology has been tested on the benchmark Quanser magnetic levitation system for reference following. Three different test cases of damping ratio and settling time have been assessed and the performance of the algorithm in meeting the design requirement has been validated. In our future work, we also intend to extend the methodology to a multi input and multi output system and assess the robustness of the algebraically tuned LQR control scheme against external disturbances and model uncertainty.

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