DOI 10.2478/ama-2024-0014

# POSITIVE STATE CONTROLLABILITY OF DISCRETE LINEAR TIME-INVARIANT SYSTEMS

**Mourad OUYADRI*, Mohamed LAABISSI*, Mohammed Elarbi ACHHAB***

*Faculty of Sciences, Department of Mathematics, Chouaib Doukkali University, BP.20, 24000 El Jadida, Morocco

ouyadri.mourad@gmail.com, laabissi.m@ucd.ac.ma, elarbi.achhab@gmail.com

*received 26 April 2023, revised 2 August 2023, accepted 8 August 2023*

**Abstract:** Positive state controllability is the controllability of systems where the state is positive and the input remains in $\mathbb{R}^n$. Under some conditions, we established a relation between the reachability map of systems with only the positive state and the reachability map of a related positive system where the state and input are both positive. Using this connection, necessary and sufficient conditions are obtained for the positive state reachability of discrete linear time-invariant (LTI) systems, and then we deduced the positive state controllability. These conditions are evaluated over some numerical examples that support the theoretical results.

**Key words:** discrete LTI systems, controllability, reachability map, positive control, positive state controllability.

## 1. INTRODUCTION

In control theory, the study of controllability has received much attention. A system is controllable if any initial state can be transformed to any final state by a feasible control sequence. The concepts of reachability and null controllability are defined in the same manner as the controllability for the initial state equal to zero and the final state equal to zero, respectively. For linear discrete system, the controllability is equivalent to the reachability and implies the null controllability [16]. Kalman et al. [11] gave a formulation of this concept, the full rank of the reachability matrix.

The study of controllability considers neither any non-negativity of the states nor that of the inputs of a system. Then, the problem explained for many applications, where the state and/or input variables necessarily take non-negative values. A positive linear system is the system where both the state and the input are always non-negatives (or positives). We can find these systems in many areas, such as compartmental biological systems, pharmacology, manufacturing, economics, telecommunications, manpower planning and others [1, 2, 3, 6, 13, 15, 17, 18, 19, 20, 22]. The field study of the positive linear systems is restrictive compared with that of the linear systems, and this is due to the fact that the positive system is defined on the cone and not on the whole linear space. Thus, many important properties of the linear system are not applied to the positive linear system. Positive input controllability is the controllability of positive systems. This situation is treated in many textbooks and papers in the literature [4, 7, 9, 12, 16]. The study of Caccetta and Rumchev [4] demonstrates that the positive input controllability is equivalent to both positive input reachability and positive input null controllability at once (unlike the controllability of linear discrete systems). In the research of Valcher [21], it is proved that the positive input reachability is equivalent to the positive input reachability set contained in the positive orthant $\mathbb{R}^n_+$. A characterisation for positive input null controllability in finite (or infinite) time is given in Guiver et al. [10].

Positive state controllability is the controllability of linear systems where the state must be non-negative, and the input can take negative values. There are many papers where the system is suitable for describing the addition or removal of individuals from a population, and for a full description of these actions, we require that the control $u$ can take negative values [8, 14]. The framework of positive state controllability puts the non-negativity constraint of the state on the cone, and not on the whole linear space, and it is not obvious that the positive input controllability theory is applicable.

Guiver et al. [10] studied that concept for discrete LTI systems. Their results for the positive state controllability (including reachability and null controllability) are based on the main result shown in Theorem 2.6. Theorem 2.6 shows that the non-negative state trajectories of a system $(A, B)$ with possibly negative inputs are precisely the non-negative state trajectories of a related system with non-negative inputs (the states stay non-negative, and the possibly negative inputs are observed as non-negative inputs, i.e. positive system). Additionally, a characterisation of nonnegative inputs for the related system is given in Lemma 2.1. Using the Theorem 2.6, Guiver et al. [10] prove that under certain assumptions the positive state controllability of system $(A, B)$ is equivalent to the positive input controllability of a related positive system.

This paper is a generalisation of Guiver et al.'s study [10], where we enrich the research framework on the positive state controllability for discrete LTI systems. The main idea is to give a characterisation of positive state controllability based on the form of the reachability matrix for the system $(A, B)$. Accordingly, we demonstrate that the positive state controllability of a system $(A, B)$ with non-negative states and possibly negative inputs is equivalent to the positive input controllability of a related positive system. Besides, conditions for positive state controllability of the system $(A, B)$ are developed and proved. These results indicate that there is no need for the related positive system (unlike the results derived in the study of Guiver et al. [10]) to check the positive state reachability of the system $(A, B)$. Additionally, we demonstrated positive state controllability for systems where $A$ belongs to the class of Leslie matrices [5].

The remainder of this paper is structured as follows: Section 2 presents a mathematical notation of the positive matrix; additionally, we recall some definitions and proprieties about the theory of positive linear systems (controllability, reachability and null controllability). In Section 3, we begin with the problem formulation and present the class of systems studied in this paper. In Subsection 3.1, the positive state reachability problem is stated for discrete LTI systems. Under some conditions, we prove that the reachability matrix of the system $(A, B)$ is equivalent to the reachability matrix of a related positive system. Subsections 3.2 and 3.3, respectively, introduce necessary and sufficient conditions for positive state null controllability and positive state controllability of discrete LTI systems. The final section is devoted to illustrate the obtained results in some examples and an application from population dynamics.

## 2. PRELIMINARILY RESULTS

Notation: Denote by $\mathbb{N}$ the set of integer numbers. $\mathbb{R}_+^*$ is the set of nonzero positive real numbers. Given a matrix $A \in \mathbb{R}^{n\times n}$, the kernel and positive kernel of $A$, are, respectively, defined as $\ker A = \{x \in \mathbb{R}^n | Ax = 0$ and $\ker_+ A = \{x \in \mathbb{R}_+^n | Ax = 0\} = \mathbb{R}_+^n \cap \ker A$. The matrix $A$ is Schur if $|\lambda| < 1$, for all $\lambda \in \sigma(A)$. $e_i \in \mathbb{R}^n$ is the $i$th standard basis vector of $\mathbb{R}^n$. A matrix $A \in \mathbb{R}_+^{n\times n}$ ( $A \geq 0$ ) if for all $i, j \in \{1,2, \dots, n\}$, the components $A_{i,j} \in \mathbb{R}_+$ $(A_{i,j} \geq 0)$. A vector $v \in \mathbb{R}_+^n$ $(v \geq 0)$ if for all $j \in \{1,2, \dots, n\}$, the components $v_j \in \mathbb{R}_+$ $(v_j \geq 0)$. A positive vector (row or column) is said to be monomial if all of its components are zero except for a single one that is positive. If $v \in \mathbb{R}_+^n$ is a monomial column, then $v = ce_i$ for some $c > 0$ and $i = 1, \cdots, n$, and we can say that $v$ is i-monomial. A matrix $A \in \mathbb{R}_+^{n\times n}$ is said to be monomial if all its rows and columns are monomial. Further, a matrix $A$ is said to be nilpotent if there exist $p$ such that $A^p = 0_n$, $(A \in \mathbb{R}^{n\times n})$.

Consider the discrete LTI system defined as follows:

$$x(k+1) = Ax(k) + Bu(k), \quad k \in \mathbb{N}, \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. The solution $x(k)$, for some $k \in \mathbb{N}$, of the system (1) is given by:

$$x(k) = A^k x_0 + [B \quad AB \quad \cdots \quad A^{k-1}B] \begin{bmatrix} u(k-1) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix}. \tag{2}$$

In this section we give some definitions and characterisations about null controllability, reachability and controllability of positive discrete LTI systems [4, 7, 16].

Definition 2.1 The system (1) is said to be positive if $x(k) \geq 0$ for any $x_0 \geq 0$ and all input sequences $u(k) \geq 0$, $k \in \mathbb{N}$.

Proposition 2.2 [9] The system (1) is positive if and only if $A \geq 0$ and $B \geq 0$.

Proposition 2.2 shows that, if $x_0 \in \mathbb{R}_+^n$ then $x(k) \in \mathbb{R}_+^n$ for all $k \in \mathbb{N}$.

In the rest of this section, we consider the system (1) as a positive system.

Definition 2.3 Given a positive system (1), we say that $x_f \in \mathbb{R}_+^n$ is reachable in finite time if there exist $k \in \mathbb{N}$ and a control sequence $[u(0) \quad u(1) \quad \cdots \quad u(k-1)] \in \mathbb{R}_+^m$ that steers the state $x \in \mathbb{R}_+^n$ of the system (1) from the origin to $x_f$ in $k$ steps.

The state trajectory $x(k)$, for some $k \in \mathbb{N}$, of the positive system (1) with $x_0 = 0$ is given by:

$$x(k) = R_k(A, B)U_k,$$

where

$$R_k(A, B) = [B \quad AB \quad \cdots \quad A^{k-1}B] \ and \ U_k = \begin{bmatrix} u(k-1) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix}.$$

We recall that $R_k(A, B)$ is the reachability matrix at time $k$ of the positive system (1). The set of all reachable states $x_f$ at time $k$ is defined as follows:

$$\mathfrak{R}_k(A, B) = \{x_f \in \mathbb{R}_+^n : \ x_f = R_k(A, B)U_k, \ U_k \in \mathbb{R}_+^{mk}\}.$$

We define also $\mathfrak{R}_\infty(A, B)$ as the set of all reachable positive states in finite time, with positive inputs, i.e.

$$\mathfrak{R}_\infty(A, B) = \bigcup_{k=1}^{\infty} \mathfrak{R}_k(A, B).$$

Thus, the positive system (1) is reachable if $\mathfrak{R}_\infty(A, B) = \mathbb{R}_+^n$. The following corollary is a characterisation of the reachability of the positive system (1) [21].

Corollary 2.4 [21] The positive system (1) is reachable if and only if for some $k \in \mathbb{N}$, $R_k(A, B)$ contains an $n \times n$ monomial sub-matrix.

Definition 2.5 Given a positive system (1), we say that $x_f \in \mathbb{R}_+^n$ is reachable in infinite time if for $x(0) = 0$, there exists an infinite control sequence $u \in \mathbb{R}_+^m$ such that the state $x(k)$ of the positive system (1) satisfies $x(k) \to x_f$ for $k \to +\infty$.

Definition 2.6 Given a positive system (1), we say that $x_0 \in \mathbb{R}_+^n$ is null controllable in finite time if there exist $k \in \mathbb{N}$ and a control sequence $[u(0) \quad u(1) \quad \cdots \quad u(k-1)] \in \mathbb{R}_+^m$ that steers the state $x \in \mathbb{R}_+^n$ of the system (1) from $x_0$ to the origin in $k$ steps.

If $x_f = 0$, then the state $x(k)$ of the positive system (1) becomes as follows:

$$-A^k x_0 = R_k(A, B)U_k.$$

Since $R_k(A, B)U_k \geq 0$ and $A^k x_0 \geq 0$, then this is possible if there is $k = 1,2, \cdots$, such that $A^k = 0_n$. Thus, the state can reach the origin.

Definition 2.7 Given a positive system (1), we say that $x_0 \in \mathbb{R}_+^n$ is null controllable in infinite time if there exists an infinite control sequence $u \in \mathbb{R}_+^m$ such that the state $x(k)$ of the positive system (1) can reach the origin for $k \to +\infty$.

The following corollary is a characterisation of the null controllability of the positive system (1).

Corollary 2.8 [10]

1. The positive system (1) is null controllable in finite time if $A$ is nilpotent.
2. The positive system (1) is null controllable in infinite time if $A$ is Schur.

Definition 2.9 We say that the positive system (1) is controllable in finite time if for all $x_0, x_f \in \mathbb{R}_+^n$, there exist $k \in \mathbb{N}$ and a control sequence $[u(0) \quad u(1) \quad \cdots \quad u(k-1)] \in \mathbb{R}_+^m$ that steers the state $x \in \mathbb{R}_+^n$ of the system (1) from $x_0$ to $x_f$ in $k$ steps.

DOI 10.2478/ama-2024-0014

Definition 2.10 Given a positive system (1), we say that $x_f \in \mathbb{R}_+^n$ is controllable in infinite time if for all $x(0) \in \mathbb{R}_+^n$, there exists an infinite control sequence $u \in \mathbb{R}_+^m$ such that the state $x(k)$ of the positive system (1) satisfies $x(k) \to x_f$ for $k \to +\infty$.

The following corollary is a characterisation of the controllability of the positive system (1).

Corollary 2.11 [4] The positive system (1) is controllable in infinite - or finite - time if and only if it is both null controllable and reachable in infinite - or finite - time.

## 3. PROBLEM FORMULATION AND MAIN RESULTS

Consider the discrete LTI system defined as follows:

$$x(k+1) = Ax(k) + Bu(k), \tag{3}$$

where $A \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times m}$ and the state $x \in \mathbb{R}_+^n$, with $u \in \mathbb{R}^m$.

This section aims to study the controllability of the system (3), called *positive state controllability*. The framework of *positive state controllability* puts the non-negativity constraint of the states on the cone and not on the whole linear space, and it is not clear that the *positive input controllability* theory is applicable. Therefore, Guiver et al. [10] proves that the *positive state controllability* of system (3) is equivalent to the *positive input controllability* of a related positive system. This connection is based on the main result arrived at in Guiver et al.'s study [10] (see Theorem 2.6). Theorem 2.6 proves that the non-negative state trajectories of system (3) with possibly negative inputs are precisely the non-negative state trajectories of a related system with non-negative inputs (i.e., positive system). To demonstrate this relation, Guiver et al. [10] provide the following assumption: Given $A \in \mathbb{R}_+^{n\times n}$ and $B \in \mathbb{R}_+^{n\times m}$, there is $F \in \mathbb{R}^{m\times n}$ such that $\widehat{A} = A - BF \geq 0$ and if $v \in \mathbb{R}_+^n$ and $w \in \mathbb{R}^m$ satisfy $\widehat{A}v + Bw \geq 0$, then $w \geq 0$. This assumption shows the transformation from the system (3) with non-negative state trajectory $x$ and possibly negative control $u$ to a related positive system $(\widehat{A}, b)$ with a non-negative control $w = Fx + u$. Besides, Guiver et al. [10] provide a constructive characterisation of the positive input control $w$ for the related system (see Lemma 2.1). This lemma takes a monomial sub-matrix from $B$ and the associate sub-matrix from $A$ to formulate the matrix (vector) $F$ and then the related positive system. Consequently, *positive state controllability* is tied necessarily to the related positive system. That becomes a problem when the matrix $B$ contains no monomial sub-matrix (i.e. the above assumption does not hold), and in this instance, it is not in any unique sense that the choice of $F$ is carried out (see Remark 2.13 in Guiver et al. [10]). We aim to find a characterisation for the *positive state controllability* of the system (3) without using its related positive system (i.e. using $A$ and $B$ of the system (3) itself).

We consider in the sequel the discrete LTI system with a single input defined as follows:

$$x(k+1) = Ax(k) + bu(k), \tag{4}$$

where $A \in \mathbb{R}_+^{n\times n}$, $b \in \mathbb{R}_+^n$ and the state $x \in \mathbb{R}_+^n$, with $u \in \mathbb{R}$.

First, we state the important following theorem:

Theorem 3.1 [10] The non-negative state trajectories of system (4) from $x_0 \in \mathbb{R}_+^n$ are exactly the non-negative state trajectories of the relative system $(\widehat{A}, b)$ from $x_0 \in \mathbb{R}_+^n$ with non-negative control.

Remark 3.2 We can deduce from Theorem 3.1 that to verify the *positive state controllability* of the system (4) it is sufficient to verify the *positive input controllability* of the related positive system $(\widehat{A}, b)$.

We assume in the sequel that Guiver's assumption is verified for the system (4).

### 3.1. Positive state reachability (PSR)

In this section, we give a definition and characterisation of *positive state reachability* of the system (4).

Definition 3.3 We say that $x_f \in \mathbb{R}_+^n$ is *PSR* in finite time if there exists $k \in \mathbb{N}$ and a control $u \in \mathbb{R}$ that steers the state $x \in \mathbb{R}_+^n$ of the system (4) from the origin to $x_f$ in $k$ steps.

Using Corollary 2.4 and Definition 3.3, the following corollary gives a characterisation for PSR of the system (4).

Corollary 3.4 [10] The system (4) is *PSR* in finite time if and only if for some $k \geq n$, the matrix $R_k(\widehat{A}, b)$ is monomial.

Remark 3.5 For the system (4), we describe the positive state reachability set $\mathfrak{R}_\infty(\widehat{A}, b)$ as the set of all reachable states $x_f \in \mathbb{R}_+^n$ of the system (4). We say that the system (4) is PSR in finite time if $\mathfrak{R}_\infty(\widehat{A}, b) = \mathbb{R}_+^n$.

Our principal result is Theorem 3.6, which is a new characterisation for *PSR* of the system (4). For ascertaining the characterisation for reachability, we return to the well-known method followed in the research of Kalman et al. [11], which is based on the span of the reachability matrix of such a system. In the same way, the idea expressed in Theorem 3.6 is to demonstrate a characterisation of *PSR* based on the form of the reachability matrix of the system (4).Thus, we prove that there is a deep connection between the reachability matrix of the system (4) and the reachability matrix of the related positive system $(\widehat{A}, b)$, and provide a new algorithm for ascertaining the validity of Guiver's assumption.

Theorem 3.6 The following assertions are equivalents:

1. The reachability matrix of the system (4) is an upper triangular matrix, such as:

$$R_n(A,b) = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ 0 & \alpha_{2,2} & \cdots & \alpha_{2,n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{n,n} \end{bmatrix},$$

with components verifying the following conditions:

$$\alpha_{j,j} > 0, \ \ \alpha_{1,1}\alpha_{j,k+1} = \alpha_{1,k-j+2}\alpha_{j,j} \quad \forall\, 1 \leq j \leq k+1. \tag{5}$$

2. There exist $f \in \mathbb{R}^{1\times n}$, of the following form:

$$\begin{cases} f_0 = 1 \\ f_1 = \frac{\alpha_{1,2}}{\alpha_{1,1}^2} \\ \vdots \\ f_{k+1} = \frac{\alpha_{1,k+2} - \alpha_{1,1}\sum_{j=1}^{k}\alpha_{j,k+1}f_j}{\alpha_{1,1}\alpha_{k+1,k+1}} \quad \forall\, 1 \leq k \leq n-2, \end{cases} \tag{6}$$

such that the matrix $R_n(\widehat{A}, b)$ is monomial.

Proof. Assume that,

$$R_n(A,b) = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ 0 & \alpha_{2,2} & \cdots & \alpha_{2,n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{n,n} \end{bmatrix},$$

with $\alpha_{j,j} > 0$ and $\alpha_{1,1}\alpha_{j,k+1} = \alpha_{1,k-j+2}\alpha_{j,j}$, $\forall\, 1 \leq j \leq k+1$.

Since

$$A^k b = \sum_{j=1}^{k+1} \alpha_{j,k+1} e_j, \tag{7}$$

then we have

$$\begin{aligned} A^{k+1}b = AA^k b &= \sum_{j=1}^{k+1} \alpha_{j,k+1} A e_j, \\ &= \sum_{j=1}^{k} \alpha_{j,k+1} A e_j + \alpha_{k+1,k+1} A e_{k+1}. \end{aligned} \tag{8}$$

Now, by using the recurrence method, let us prove that $\widehat{\mathrm{A}}^{\mathrm{k}} b = \alpha_{k+1,k+1} e_{k+1}$:

for $k = 0$,

$$\hat{A}^0 b = b = \alpha_{1,1} e_1 = \alpha_{1,1} f_0 e_1,$$

and then $f_0 = 1$.

for $k = 1$,

$$\hat{A} b = (A - bf) b = Ab - bfb,$$

and we have:

$$\begin{cases} b = \alpha_{1,1} e_1, \\ Ab = \alpha_{1,2} e_1 + \alpha_{2,2} e_2 \\ fb = \alpha_{1,1} f e_1 = \alpha_{1,1} f_1, \end{cases} \quad \text{(by assumption)}$$

then

$$\begin{aligned} \hat{A} b &= (\alpha_{1,2} - \alpha_{1,1}^2 f_1) e_1 + \alpha_{2,2} e_2, \\ &= \alpha_{2,2} e_2, \end{aligned}$$

if $f_1 = \frac{\alpha_{1,2}}{\alpha_{1,1}^2}$.

Finally, we assume that $\hat{A}^k b = \alpha_{k+1,k+1} e_{k+1}$, and prove that $\hat{A}^{k+1} b = \alpha_{k+2,k+2} e_{k+2}$.

Let

$$\begin{aligned} \hat{A}^{k+1} b = \hat{A}\hat{A}^k b &= \hat{A}(\alpha_{k+1,k+1} e_{k+1}), \\ &= \alpha_{k+1,k+1} A e_{k+1} - \alpha_{k+1,k+1} bf e_{k+1}, \\ &\overset{(8)}{=} A^{k+1} b - \sum_{j=1}^{k} \alpha_{j,k+1} A e_j \\ &\quad - \alpha_{k+1,k+1} bf e_{k+1}, \\ &\overset{(7)}{=} \sum_{j=1}^{k+2} \alpha_{j,k+2} e_j - \sum_{j=1}^{k} \alpha_{j,k+1} A e_j \\ &\quad - \alpha_{k+1,k+1} bf e_{k+1}, \end{aligned}$$

And by hypotheses (7) and (8), we have for every $1 \le \mathrm{j} \le \mathrm{k}$,

$$\hat{A}^{j-1} b = \alpha_{j,j} e_j \quad \Rightarrow \widehat{A}^j b = \widehat{A}\widehat{A}^{j-1} b = \alpha_{j,j} A e_j - \alpha_{j,j} bf e_j$$

and $\widehat{A}^j b = \alpha_{j+1,j+1} e_{j+1}$,

$$\Rightarrow \alpha_{j,j} A e_j - \alpha_{j,j} bf e_j = \alpha_{j+1,j+1} e_{j+1}$$

$$and \quad \alpha_{j,j} > 0 \quad \text{(by assumption)}$$

$$\Rightarrow A e_j = bf e_j + \frac{\alpha_{j+1,j+1}}{\alpha_{j,j}} e_{j+1}.$$

Then $\hat{A}^{k+1} b$ becomes:

$$\begin{aligned} \hat{A}^{k+1} b &= \sum_{j=1}^{k+2} \alpha_{j,k+2} e_j - \sum_{j=1}^{k} \alpha_{j,k+1} \left( bf e_j + \frac{\alpha_{j+1,j+1}}{\alpha_{j,j}} e_{j+1} \right) \\ &\quad - \alpha_{k+1,k+1} bf e_{k+1}, \\ &= \alpha_{k+2,k+2} e_{k+2} + \alpha_{1,k+2} e_1 + \sum_{j=2}^{k+1} \alpha_{j,k+2} e_j \\ &\quad - \sum_{j=1}^{k} \frac{\alpha_{j,k+1}\alpha_{j+1,j+1}}{\alpha_{j,j}} e_{j+1} - \alpha_{1,1} \sum_{j=1}^{k} \alpha_{j,k+1} f_j e_1 \\ &\quad - \alpha_{1,1}\alpha_{k+1,k+1} f_{k+1} e_1, \\ &= \alpha_{k+2,k+2} e_{k+2} \\ &\quad + \sum_{j=1}^{k} \left( \alpha_{j+1,k+2} - \frac{\alpha_{j,k+1}\alpha_{j+1,j+1}}{\alpha_{j,j}} \right) e_{j+1} \\ &\quad + \left( \alpha_{1,k+2} - \alpha_{1,1} \sum_{j=1}^{k} \alpha_{j,k+1} f_j - \alpha_{1,1}\alpha_{k+1,k+1} f_{k+1} \right) e_1. \end{aligned}$$

Besides, by the hypothesis $\alpha_{1,1}\alpha_{j,k+1} = \alpha_{1,k-j+2}\alpha_{j,j}$, we find that:

$$\begin{aligned} \alpha_{1,1}\alpha_{j+1,k+2} &= \alpha_{1,k-j+2}\alpha_{j+1,j+1}, \\ &= \alpha_{1,k-j+2}\alpha_{j,j} \frac{\alpha_{j+1,j+1}}{\alpha_{j,j}}, \\ &= \alpha_{1,1}\alpha_{j,k+1} \frac{\alpha_{j+1,j+1}}{\alpha_{j,j}}, \end{aligned}$$

which gives $\alpha_{j+1,k+2} - \frac{\alpha_{j,k+1}\alpha_{j+1,j+1}}{\alpha_{j,j}} = 0$.

Then

$$\hat{A}^{k+1} b = \alpha_{k+2,k+2} e_{k+2},$$

if

$$f_{k+1} = \frac{\alpha_{1,k+2} - \alpha_{1,1} \sum_{j=1}^{k} \alpha_{j,k+1} f_j}{\alpha_{1,1}\alpha_{k+1,k+1}}, \quad \forall\, 1 \le k \le n-2.$$

Now, we can deduce that there is a vector $f \in \mathbb{R}^{1\times n}$, of the following form:

$$\begin{cases} f_0 = 1 \\ f_1 = \dfrac{\alpha_{1,2}}{\alpha_{1,1}^2} \\ \vdots \\ f_{k+1} = \dfrac{\alpha_{1,k+2} - \alpha_{1,1} \sum_{j=1}^{k} \alpha_{j,k+1} f_j}{\alpha_{1,1}\alpha_{k+1,k+1}} \quad \forall\, 1 \le k \le n-2, \end{cases}$$

such that the matrix $[b \quad \hat{A}b \quad \hat{A}^2 b \quad \cdots \quad \hat{A}^{n-1} b]$ is monomial.

Conversely, we assume that $\mathrm{R}_\infty(\widehat{\mathrm{A}}, \mathrm{b})$ is monomial, i.e. $\widehat{\mathrm{A}}^{\mathrm{k}} \mathrm{b} = \alpha_{\mathrm{k}+1,\mathrm{k}+1} \mathrm{e}_{\mathrm{i},\mathrm{k}+1}$ and by recurrence, we prove that $\mathrm{R}_\infty(\mathrm{A}, \mathrm{b})$ is upper triangular with its components verifying the conditions $(5)$, i.e. we prove that

$$A^k b = \sum_{j=1}^{k+1} \alpha_{j,k+1} e_{i,j}, \quad \forall\, 0 \le k \le n-1,$$

DOI 10.2478/ama-2024-0014

such that $\alpha_{k+1,k+1} > 0$, $\alpha_{j,k+1} \geq 0$ and $\alpha_{1,1}\alpha_{j,k+1} = \alpha_{1,k-j+2}\alpha_{j,j}$ for every $1 \leq j \leq k+1$.

For all $0 \leq k \leq n-2$, we assume that

$$\begin{cases} A^k b = \sum_{j=1}^{k+1} \alpha_{j,k+1} e_{i,j}, \quad \forall\, 0 \leq k \leq n-2, \\ \alpha_{k+1,k+1} > 0, \alpha_{j,k+1} \geq 0 \quad and \\ \alpha_{1,1}\alpha_{j,k+1} = \alpha_{1,k-j+2}\alpha_{j,j}, \quad \forall\, 1 \leq j \leq k+1, \end{cases}$$

and by recurrence, we prove that

$$\begin{cases} A^{k+1} b = \sum_{j=1}^{k+2} \alpha_{j,k+2} e_{i,j}, \quad \forall\; 0 \leq k \leq n-2, \\ \alpha_{k+2,k+2} > 0, \; \alpha_{j,k+2} \geq 0 \quad and \\ \alpha_{1,1}\alpha_{j,k+2} = \alpha_{1,k-j+3}\alpha_{j,j}, \quad \forall\; 1 \leq j \leq k+2. \end{cases}$$

Then,

$$A^{k+1}b = AA^k b = \left(\hat{A} + bf\right) \sum_{j=1}^{k+1} \alpha_{j,k+1} e_{i,j}$$

$$= \sum_{j=1}^{k+1} \alpha_{j,k+1} \hat{A} e_{i,j} + \sum_{j=1}^{k+1} \alpha_{j,k+1} bf e_{i,j}.$$

Moreover, through the hypothesis $\hat{A}^{j-1}b = \alpha_{j,j}e_{i,j}$, we have

$$\hat{A}^j b = \hat{A}\hat{A}^{j-1}b = \alpha_{j,j}\hat{A}e_{i,j}$$

and by hypothesis, we also have

$$\hat{A}^j b = \alpha_{j+1,j+1} e_{i,j+1}$$

Comparing the last two equalities, we find that

$$\hat{A}e_{i,j} = \frac{\alpha_{j+1,j+1}}{\alpha_{j,j}} e_{i,j+1}.$$

Thus, $A^{k+1}b$ becomes

$$\begin{aligned} A^{k+1}b \quad &= \sum_{j=1}^{k+1} \alpha_{j,k+1} \frac{\alpha_{j+1,j+1}}{\alpha_{j,j}} e_{i,j+1} + \sum_{j=1}^{k+1} \alpha_{j,k+1} bf e_{i,j}, \\ &= \sum_{j=1}^{k+1} \alpha_{j,k+1} \frac{\alpha_{j+1,j+1}}{\alpha_{j,j}} e_{i,j+1} \\ &\quad + \left( \alpha_{1,1} \sum_{j=1}^{k+1} \alpha_{j,k+1} f_j \right) e_{i,1}, \\ &= \sum_{j=2}^{k+2} \alpha_{j-1,k+1} \frac{\alpha_{j,j}}{\alpha_{j-1,j-1}} e_{i,j} \\ &\quad + \left( \alpha_{1,1} \sum_{j=1}^{k+1} \alpha_{j,k+1} f_j \right) e_{i,1}, \\ &= \sum_{j=1}^{k+2} \alpha_{j,k+2} e_{i,j}, \end{aligned}$$

such that

$$\begin{cases} \alpha_{1,k+2} = \alpha_{1,1} \sum_{j=1}^{k+1} \alpha_{j,k+1} f_j, \\ \alpha_{j,k+2} = \alpha_{j-1,k+1} \dfrac{\alpha_{j,j}}{\alpha_{j-1,j-1}} \quad \forall\; 2 \leq j \leq k+2. \end{cases}$$

The first equality $\alpha_{1,k+2} = \alpha_{1,1} \sum_{j=1}^{k+1} \alpha_{j,k+1} f_j$ came from the following hypothesis:

$$f_{k+1} = \frac{\alpha_{1,k+2} - \alpha_{1,1} \sum_{j=1}^{k} \alpha_{j,k+1} f_j}{\alpha_{1,1}\alpha_{k+1,k+1}}, \quad \forall\; 1 \leq j \leq k-2.$$

Secondary, since

$$\alpha_{j,k+2} = \alpha_{j-1,k+1} \frac{\alpha_{j,j}}{\alpha_{j-1,j-1}}, \quad \forall\; 2 \leq j \leq k+2, \tag{9}$$

and by hypothesis $(5)$, the following equality

$$\alpha_{j,k+1} = \alpha_{1,k-j+2} \frac{\alpha_{j,j}}{\alpha_{1,1}}, \quad \forall 1 \leq j \leq k+1,$$

gives

$$\alpha_{j-1,k+1} = \alpha_{1,k-j+3} \frac{\alpha_{j-1,j-1}}{\alpha_{1,1}}, \quad \forall\, 2 \leq j \leq k+2. \tag{10}$$

Substituting (10) in (9) we find that:

$$\alpha_{j,k+2} = \alpha_{1,k-j+3} \frac{\alpha_{j,j}}{\alpha_{1,1}}, \quad \forall\, 2 \leq j \leq k+2,$$

with $\alpha_{j,k+2} \geq 0$, for all $2 \leq j \leq k+2$ (hypothesis).

Theorem 3.6 shows that the choice of a vector $\mathrm{f}$, which makes $\mathrm{R_n}(\widehat{\mathrm{A}}, \mathrm{b})$ monomial, is well characterised by the form (6), under the assumption that $\mathrm{R_n}(\mathrm{A}, \mathrm{b})$ is an upper triangular matrix and its components verify the conditions (5).

Corollary 3.7. The system $(4)$ is $\mathrm{PSR}$ if and only if $\mathrm{R_n}(\mathrm{A}, \mathrm{b})$ is an upper triangular matrix and its components verify the conditions (5).

Proof. Assume that $\mathrm{R_n}(\mathrm{A}, \mathrm{b})$ is an upper triangular matrix and its components verify the conditions (5). Then, by theorem 3.6, there is $\mathrm{f} \in \mathbb{R}^{1\times n}$ such that $\mathrm{R_n}(\widehat{\mathrm{A}}, \mathrm{b})$ is $\mathrm{n} \times \mathrm{n}$ monomial. Thus, according to corollary 3.4, the system (4) is *PSR*. Conversely, if the system (4) is *PSR*, then by Corollary 3.4, $\mathrm{R_n}(\widehat{\mathrm{A}}, \mathrm{b})$ is $\mathrm{n} \times \mathrm{n}$ monomial. Then by Theorem 3.6, $\mathrm{R_n}(\mathrm{A}, \mathrm{b})$ is an upper triangular matrix and its components verify the conditions (5).

Proposition 3.8 If $\mathrm{A}$ and $\mathrm{b}$ are given by

$$A = \begin{bmatrix} b_1 & b_1 & \cdots & b_1 & a_n \\ a_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n-1} & 0 \end{bmatrix} and\; b = \begin{bmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{11}$$

where $\mathrm{a_i}, \mathrm{b_1} \in \mathbb{R}_+^*$, $\forall \mathrm{i} \in \{1,2, \ldots, \mathrm{n}\}$ and $\mathrm{a_n} \geq \mathrm{b_1}$, then the system $(4)$ is *PSR*.

Proof. First, we take $\mathrm{A}$ and $\mathrm{b}$ with $\mathrm{n} = 2$ (dimension of $\mathrm{A}$), such that:

$$A = \begin{bmatrix} b_1 & a_2 \\ a_1 & 0 \end{bmatrix} and\; b = \begin{bmatrix} b_1 \\ 0 \end{bmatrix}.$$

The reachability matrix of the pair $(A,b)$ given by

$$R_2(A,b)=\begin{bmatrix} b_1 & b_1^2 \\ 0 & a_1b_1 \end{bmatrix}$$

is upper triangular and the conditions (5) have been verified. The form (6) gives $f=[1 \quad 1]$ and

$$\hat{A}=A-bf=\begin{bmatrix} 0 & a_2-b_1 \\ a_1 & 0 \end{bmatrix}.$$

Then the reachability matrix of the pair $(\hat{A},b)$ given by

$$R_2(\hat{A},b)=\begin{bmatrix} b_1 & 0 \\ 0 & a_1b_1 \end{bmatrix}$$

is monomial. Thus, by corollary 3.4 the system is *PSR*.

Now if $A$ and $b$ are with $n=3$, then

$$A=\begin{bmatrix} b_1 & b_1 & a_3 \\ a_1 & 0 & 0 \\ 0 & a_2 & 0 \end{bmatrix} \ and \ b=\begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix}.$$

The reachability matrix of the pair $(A,b)$ given by

$$R_3(A,b)=\begin{bmatrix} b_1 & b_1^2 & b_1^3+a_1b_1^2 \\ 0 & a_1b_1 & a_1b_1^2 \\ 0 & 0 & a_1a_2b_1 \end{bmatrix}$$

is upper triangular and the conditions (5) have been verified. The form (6) gives $f=[1 \quad 1 \quad 1]$ and

$$\hat{A}=A-bf=\begin{bmatrix} 0 & 0 & a_3-b_1 \\ a_1 & 0 & 0 \\ 0 & a_2 & 0 \end{bmatrix}.$$

Then the reachability matrix of the pair $(\hat{A},b)$ given by

$$R_3(\hat{A},b)=\begin{bmatrix} b_1 & 0 & 0 \\ 0 & a_1b_1 & 0 \\ 0 & 0 & a_1a_2b_1 \end{bmatrix}$$

is monomial. Thus, by corollary 3.4 the system is *PSR*.

If $A$ and $b$ are with $n=4$, then

$$A=\begin{bmatrix} b_1 & b_1 & b_1 & a_4 \\ a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \end{bmatrix} \ and \ b=\begin{bmatrix} b_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The reachability matrix of the pair $(A,b)$ given by

$$R_4(A,b)=$$

$$\begin{bmatrix} b_1 & b_1^2 & b_1^3+a_1b_1^2 & b_1^4+2a_1b_1^3+a_1a_2b_1^2 \\ 0 & a_1b_1 & a_1b_1^2 & a_1b_1^3+a_1^2b_1^2 \\ 0 & 0 & a_1a_2b_1 & a_1a_2b_1^2 \\ 0 & 0 & 0 & a_1a_2a_3b_1 \end{bmatrix}$$

is upper triangular and the conditions (5) have been verified. The form (6) gives $f=[1 \quad 1 \quad 1 \quad 1]$ and

$$\hat{A}=A-bf=\begin{bmatrix} 0 & 0 & 0 & a_4-b_1 \\ a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \end{bmatrix}.$$

Then the reachability matrix of the pair $(\hat{A},b)$ given by

$$R_4(\hat{A},b)=\begin{bmatrix} b_1 & 0 & 0 & 0 \\ 0 & a_1b_1 & 0 & 0 \\ 0 & 0 & a_1a_2b_1 & 0 \\ 0 & 0 & 0 & a_1a_2a_3b_1 \end{bmatrix}$$

is monomial. Thus, by corollary 3.4 the system is *PSR*.

Then, without loss of generality, for all $\mathrm{A}$ and $\mathrm{b}$ given in (11), we can conclude that the reachability matrix $\mathrm{R_n(A,b)}$ is upper triangular and the conditions (5) are verified. Using the form (6), we ascertain that:
$f=[1 \quad \cdots \quad 1]$.

Then the reachability matrix of the pair $(\hat{A},b)$ is given by

$$R_n(\hat{A},b)=\begin{bmatrix} b_1 & 0 & 0 & \dots & 0 \\ 0 & a_1b_1 & 0 & \dots & 0 \\ 0 & 0 & a_1a_2b_1 & \dots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & \prod_{i=1}^{n} a_ib_1 \end{bmatrix}$$

and it is a monomial matrix. Consequently, by corollary 3.4, the system (11) is *PSR*.

Remark 3.9. In general, the use of a base change produces a similar system. The system (4) is similar to another system $(\widetilde{\mathrm{A}},\widetilde{\mathrm{b}})$ if there is a non-singular matrix $\mathrm{P}$ such that $\widetilde{\mathrm{A}}=\mathrm{P^{-1}AP}$ and $\widetilde{\mathrm{b}}=\mathrm{P^{-1}b}$. Thus, if there is a permutation matrix $\mathrm{P}$, which makes the reachability matrix $\mathrm{R_n(A,b)}$ upper triangular, and if its components verify the conditions (5), then by corollary 3.4, we deduce that the system is *PSR*.

Corollary 3.10 The system (4) is *PSR* if there is a permutation matrix $\mathrm{P}$ that makes the reachability matrix of the system (4) upper triangular with its components verifying the conditions (5).

Proof. It is similar to that applying for corollary 3.7.

## 3.2. Positive state null controllability (PSNC)

Definition 3.11 We say that $\mathrm{x_0} \in \mathbb{R}_+^{\mathrm{n}}$ is *PSNC* in finite time if there exist $\mathrm{k} \in \mathbb{N}$ and a control $\mathrm{u} \in \mathbb{R}$ that steers the state $\mathrm{x} \in \mathbb{R}_+^{\mathrm{n}}$ of the system (4) from $\mathrm{x_0}$ to the origin in $\mathrm{k}$ steps.

Remark 3.12. For the system (4), we describe the *positive state null controllability* set $\ker_+(\widehat{\mathrm{A}}^{\mathrm{n}})$ as the set of all null controllable initial states $\mathrm{x_0} \in \mathbb{R}_+^{\mathrm{n}}$ of the system (4). We say that the system (4) is *PSNC* in finite time if $\ker_+(\widehat{\mathrm{A}}^{\mathrm{n}})=\mathbb{R}_+^{\mathrm{n}}$.

Based on theorem 3.1 and corollary 2.8, a characterisation of *PSNC* in finite and infinite time is the following:

Corollary 3.13 [10]

The system (4) is *PSNC* in finite time if and only if $\widehat{\mathrm{A}}$ is nilpotent.

The system (4) is *PSNC* in infinite time if and only if $\widehat{\mathrm{A}}$ is Schur.

Proposition 3.14. Let the system (4) be as given in (4)
If $\mathrm{a_n}=\mathrm{b}$, then the system $(4)$ is *PSNC* in finite time.
If $\mathrm{a_n}>\mathrm{b}$ and $\prod_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{a_ib}<1$, then the system (4) is *PSNC* in infinite time.

Proof. Let the system (4) be as given in (11).

First, if $\mathrm{a_n}=\mathrm{b}$ we obtain

DOI 10.2478/ama-2024-0014

$$\hat{A} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ a_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n-1} & 0 \end{bmatrix},$$

which is a nilpotent matrix; then, by corollary 3.13, the system (4) is *PSNC* in finite time. Secondarily, if $a_n > b$, we obtain

$$\hat{A} = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_n - b \\ a_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n-1} & 0 \end{bmatrix},$$

and

$$\hat{A}^{n\times k} =$$

$$\begin{bmatrix} \left(\prod_{i=1}^{n} a_i b\right)^k & 0 & 0 & \dots & 0 \\ 0 & \left(\prod_{i=1}^{n} a_i b\right)^k & 0 & \dots & 0 \\ 0 & 0 & \left(\prod_{i=1}^{n} a_i b\right)^k & \dots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & \left(\prod_{i=1}^{n} a_i b\right)^k \end{bmatrix},$$

then $\hat{A}^{n\times k} = 0_n$ if $\prod_{i=1}^{n} a_i b < 1$ and $k \to \infty$. Therefore, by corollary 3.13, the system (4) is *PSNC* in infinite time.

### 3.3. Positive state controllability (PSC)

Definition 3.15 We say that the system $(4)$ is PSC in finite time if for all $x_0, x_f \in \mathbb{R}_+^n$, there exist $k \in \mathbb{N}$ and a control $u \in \mathbb{R}$ that steers the state $x \in \mathbb{R}_+^n$ of the system (4) from $x_0$ to $x_f$ in $k$ steps.

Proposition 3.16 [10]. The system (4) is *PSC* in finite - or infinite - time if and only if it is PSR and PSNC in finite - or infinite - time.

According to proposition 3.16, as well as Corollaries 3.7 and 3.13, we deduce the following propositions.

Proposition 3.17. The system (4) is *PSC* in finite - or infinite - time if and only if $R_n(A, b)$ is an upper triangular matrix with its components verifying the conditions (5) and $\widehat{A}$ is nilpotent, - or Schur -.

Proof. It is clear according to proposition 3.16 and corollaries 3.7 and 3.13.

Corollary 3.18. If the system (4) is *PSC* in $k$ steps, then $x_0 \in \ker\widehat{A}^k$ and $x_f \in R_k(\widehat{A}, b)$, for all $k \geq n$.

Proposition 3.19. Let the system (4) be as given in (11).
If $a_n = b$, then the system (4) is PSC in finite time.
If $a_n > b$ and $\prod_{i=1}^{n} a_i b < 1$, then the system $(4)$ is *PSC* in infinite time.

Proof. It is clear from Propositions $3.8$ and $3.14$.

Using Propositions $3.8$ and $3.19$ we obtain the following result:

Corollary 3.20 The System $(11)$ is *PSC* if it is *PSNC*.

Proof. According to proposition 3.8, the system (11) is always *PSR*. Then, by proposition 3.19, this system is *PSC* if it is *PSNC*.

## 4. EXAMPLES

**Example 4.1** *(Formulation test)*

Let us consider the system $(4)$ given by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \; and \; b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The reachability matrix of the system $(A, b)$

$$R_3(A, b) = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

is upper triangular and its components verify the conditions (5). Using the form (6), there exist $f = [1 \quad 1 \quad 1]$ such that $\hat{A} = A - bf = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Then the reachability matrix of the system $(\hat{A}, b)$ given by

$$R_3(\hat{A}, b) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

is monomial. Then, by corollary 3.7, the system $(A, b)$ is *PSR*. Additionally, the matrix $\widehat{A}^3 = 0_3$. Therefore, by Corollary 3.13, the system $(A, b)$ is *PSNC*. According to Proposition 3.17, we deduce that the system $(A, b)$ is *PSC*.

Example 4.2. (Matrix of Leslie in $\mathbb{R}^{6\times 6}$).

Consider the discrete *LTI* system (4) defined by the following matrices:

$$A = \begin{bmatrix} 3 & 3 & 3 & 3 & 3 & 6 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \; and \; b = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The reachability matrix of the system $(A, b)$ is as follows

$$R_6(A, b) = \begin{bmatrix} 3 & 9 & 45 & 279 & 1467 & 7875 \\ 0 & 6 & 18 & 90 & 558 & 2934 \\ 0 & 0 & 30 & 90 & 450 & 2790 \\ 0 & 0 & 0 & 30 & 90 & 450 \\ 0 & 0 & 0 & 0 & 60 & 180 \\ 0 & 0 & 0 & 0 & 0 & 60 \end{bmatrix}.$$

$R_6(A, b)$ is upper triangular and its components verify the conditions (5). Then, by corollary 3.7, the system $(A, b)$ is PSR. We can check this by using the form (6). Then, we obtain

$$f = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$$

and

$$\hat{A} = A - bf = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The reachability matrix of the system $(\hat{A}, b)$ is given by

$$R_6(\hat{A}, b) = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 30 & 0 & 0 & 0 \\ 0 & 0 & 0 & 30 & 0 & 0 \\ 0 & 0 & 0 & 0 & 60 & 0 \\ 0 & 0 & 0 & 0 & 0 & 60 \end{bmatrix}.$$

Since $R_6(\widehat{A}, b)$ is monomial, then by corollary 3.7, this system $(A, b)$ is *PSR*. Furthermore, according to the second assertion in proposition 3.14, we have $a_6 = 6 > b = 3$ but $\prod_{i=1}^{n} a_i b = 120 > 1$, which implies that this system is not *PSNC*. Thus, this system is not *PSC*.

With a little change, if we take $a_1 = 10^{-5}$, we obtain $\prod_{i=1}^{n} a_i b = 6 \times 10^{-4} < 1$. Therefore, the matrix $\widehat{A}^k \to 0_6$ when $k \to \infty$. Then, by proposition 3.19 and corollary 3.20, this system is *PSC*.

Example 4.3. (Population dynamics).

We consider a population structured in equal lengths age classes. This length is also used to measure time discretely. We divide animals into three subgroups: aged 0 to 10 years (size measured by the sequence $(x_n)$), 10 to 20 years (size measured by the sequence $(y_n)$) and 20 to 30 years (size measured by the sequence $(z_n)$). Here the initial conditions (i.e. the data of $x_0$, $y_0$ and $z_0$) correspond to the year 2000. Then, $x_n$ is the size of the subgroup 0 to 10 years in 2000+10n… etc. All informations about the population are contained in the following vector (i.e. a matrix of size 3.1):

$$X_n = \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}.$$

In these groups of animals, it is assumed that group 1 is too young to reproduce, group 2 reproduces with a fertility rate equal to 1 and group 3 reproduces with a fertility rate equal to 5. It is assumed that the probability that an individual from group 1 survives so as to be comprised in group 2 is 0,2 and the probability that an individual from group 2 survives so as to be comprised in group 3 is 0,5. We can then make a life cycle diagram and write the linear system as follows:

$$\begin{cases} x_{n+1} = y_n + 5z_n \\ y_{n+1} = 0.2x_n \\ z_{n+1} = 0.5y_n \end{cases}$$

which can be rewritten in the following matrix form:

$$X_{n+1} = AX_n = \begin{bmatrix} 0 & 1 & 5 \\ 0.2 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} X_n.$$

The matrix $A$ is a Leslie matrix. The dynamics of this system shows that, since the fertility rate of $z_n$ is equal to $5$, we may predict that the size of $x_n$ will increase, and so will the population. Then, given the prevalence of a favourable condition characterised by an absence of threats and the unlimited availability of nutrition (abundant eating and drinking, absence of epidemic, absence of predators, etc.), this population will destroy the ecological system. So we added the control term to control (minimise) the size of the first subgroup $x_n$ (the new births), and thus the system becomes as follows:

$$X_{n+1} = AX_n + bu_n = \begin{bmatrix} 0 & 1 & 5 \\ 0.2 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} X_n + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_n, \qquad (12)$$

where $u_n \in \mathbb{R}$. Now, we check the positive state controllability of this system. The reachability matrix of the system (12) is as follows:

$$R_2(A, b) = \begin{bmatrix} 1 & 0 & 0.2 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},$$

which is an upper triangular matrix, and its components verify the conditions (5) Then, by corollary 3.7, this system is *PSR*. Additionally, using the form (6), there exist $f = [1 \quad 0 \quad 1]$ such that

$$\hat{A} = A - bf = \begin{bmatrix} -1 & 1 & 4 \\ 0.2 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}.$$

The eigenvalues $\lambda_1 \approx 0.572$, $\lambda_2 \approx -0.786 + i(0.285)$ and $\lambda_3 \approx -0.786 - i(0.285)$ of $\widehat{A}$ are strictly less than (1) Then, by corollary 3.13 this system is *PSNC*. According to Proposition 3.17 this system is *PSC*. Then, we choose to take $u_n = -3z_n$ to make the rate of fertility of $z_n$ equal to 2 in order to minimise $x_n$ (which represents the new births).

The study of Cáceres and Cáceres-Saez [5] presents a similar application. They consider a single-sex population model, with three age compartments (adults, juveniles and calves). The study is on the female reproductive success in bottlenose dolphins (with more information being available from the cited study [5]). Therefore, the 3x3 population dynamics model is subject to a Leslie matrix characterised as we mentioned in our application (example 4.3). Consequently, the results obtained in example 4.3 can be projected to the application from the study of Cáceres and Cáceres-Saez [5].

## 5. CONCLUSION

In this paper, we investigate the controllability of discrete *LTI* systems (4) where the state is positive, and the input can take a negative value, called *positive state controllability*. Our idea was to demonstrate the connection between the *positive input controllability* and *positive state controllability*. In section 3, we demonstrated the equivalence between the *positive state reachability* of the system (4) and the *positive input reachability* of the related positive system $(\widehat{A}, b)$ under some conditions. Moreover, we proved that, if the reachability matrix of the system (4) is upper triangular and satisfies some conditions (see conditions (5), then the system is *positive state reachable*. The subject of further research will be the development of the result of *the positive state controllability* of discrete *LTI* systems with multiple inputs. Non-linear and singular systems engage our attention, providing us with the impetus to proceed with a new study.

**REFERENCES**

1. Bartholomew DJ, Forbes AF, McLean SI. Statistical techniques for manpower planning. 2nd ed, Wiley, New York. 1979.
2. Berman A, Neumann M, Stern RJ. Nonnegative matrices in dynamic systems. Vol. 3. Wiley-Interscience. 1989.
3. Brown RF. Biomedical systems analysis via compartmental concept. CRC Press; 1985.
4. Caccetta L, Rumchev VG. A survey of reachability and controllability for positive linear systems. Annals of Operations Research. 2000:101-22.

DOI 10.2478/ama-2024-0014

5. Cáceres MO, Cáceres-Saez I. Random Leslie matrices in population dynamics. Journal of Mathematical Biology. 2011;63:519-56.
6. Chase RB, Aquilano NJ. Production and Operations Management, Richard D. Irwin, Chicago, IL. 1992.
7. Coxson PG, Shapiro H. Positive input reachability and controllability of positive systems. Linear Algebra and its Applications. 1987:1;94:35-53.
8. Doak D, Kareiva P, Klepetka B. Modeling population viability for the desert tortoise in the western Mojave Desert. Ecological applications. 1994 Aug;4(3):446-60.
9. Farina L, Rinaldi S. Positive linear systems: theory and applications. John Wiley & Sons; 2000 Jul 3.
10. Guiver C, Hodgson D, Townley S. Positive state controllability of positive linear systems. Systems & Control Letters. 2014:1;65: 23-9.
11. Kalman RE, Ho YC, Narendra KS. Controllability of linear dynamical systems. In Contributions to Differential Equations. 1962;1: 189–213.
12. Kaczorek T. Positive 1D and 2D systems. Springer Science & Business Media; 2012 Dec 6.
13. Krasnoselskii MA, Lifshitz EA, Sobolev AV. Positive Linear Systems, Nauka, Moscow, in Russian. 1985.
14. Lubben J, Tenhumberg B, Tyre A, Rebarber R. Management recommendations based on matrix projection models: the importance of considering biological limits. Biological Conservation. 2008; 1;141(2):517-23.
15. Luenberger DG. Theory, models and applications. Stanford University. John Wiley & Sons Inc; 1979.
16. Ouyadri M, Laabissi M, Achhab ME. Positive output controllability of linear discrete–time invariant systems. Control and Cybernetics. 2021 Oct 1;50(4):521-39.
17. Rantzer A, Valcher ME. A tutorial on positive systems and large scale control. In 2018 IEEE Conference on Decision and Control (CDC). 2018:3686-3697.
18. Rumchev VG, Konin AL. Decision support systems for manpower planning: Mathematical Models. Radio and Communication Press, Moscow. 1984.
19. Sethi SP, Thompson GL. Optimal Control Theory: Applications to Management Sciences. Martinus Nijhoff, Boston. 1981.
20. Szidarovszky F. Linear systems theory. Routledge; 2018.
21. Valcher ME. Controllability and reachability criteria for discrete time positive systems. International Journal of Control. 1996:1;65(3): 511-36.
22. Vollmann TE, Berry WL, Whybark DC. Manufacturing planning and control systems. Irwin/McGraw-Hill. 1997.

Mourad Ouyadri: https://orcid.org/0009-0009-3297-8576

Mohamed Laabissi: https://orcid.org/0000-0001-8287-1034

Mohammed Elarbi Achhab: https://orcid.org/0000-0003-4361-4347