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SPECTRAL REPRESENTATIONS FOR A CLASS OF BANDED JACOBI-TYPE MATRICES

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Abstract. We describe some spectral representations for a class of non-self-adjoint banded Jacobi-type matrices. Our results extend those obtained by P.B. Naïman for (two-sided infinite) periodic tridiagonal Jacobi matrices.

Keywords: spectral representation, Laurent operators, Jacobi matrices.

Mathematics Subject Classification: 47A67, 47A75.

1. INTRODUCTION

There is a wide literature dealing with classical spectral theory, which was initiated by von Neumann ([27, 28]), and its various extensions to non-normal operators. One of the most notable attempts to obtain an adequate spectral analysis of a number of non-self-adjoint operators was made by Dunford who introduced and investigated the notion of a spectral operator. The reader is urged to refer to [11] for a survey of the research on such operators as well as the historical background (see also [9, 10], and the work of Colojoară [7] for a different outlook on the subject). In [14], Fixman showed that the aforementioned class, although large, does not contain all non-self-adjoint operators (see also [17]). The theory of spectral operators has been extended in different directions. One of its developments was initiated in [16] by Foiaş who introduced a class of so-called decomposable operators (embracing those having rich spectral properties) which was then extensively investigated in his joint book [8] with Colojoară (see also the book [32] by Sz.-Nagy and Foiaş). Other extensions can be found, for example, in the works of Schaefer [29–31]. For an incisive treatment of spectral operators and spectral measures one should consult [12] (as well as [32]) and numerous references therein. The bibliography on diverse extensions of the spectral theory is steadily growing. One can see, for instance, [2] or more recent works [3, 4], where some integral representations for bounded operators were obtained.

In the present paper we deal with entirely different spectral representations of bounded operators acting on a Hilbert space. Our approach originates from that proposed by Schur for finite matrices (cf. [13, 23]). Schur's ideas were adopted in the work [21] by Ljubič and Macaev, where the concept of a separable spectrum was studied (see also [20, 22]). It should be remarked that the results of [21] can also be related to the problem of the existence of non-trivial invariant spaces for operators (cf. [15, 34]). The approach of Ljubič and Macaev was subsequently developed by Naïman in the paper [26], which is of most interest to us here, where some spectral representations for non-self-adjoint (two-sided infinite) periodic tridiagonal Jacobi matrices were given (see also [24, 25] or [18, Chapter 14]). This work inspired us to obtain reminiscent results for banded Jacobi-type matrices. In contrast to [26], we do not exploit computationally complex algebraic properties of matrices, but utilize the notion of the symbol of an operator which enables us to simplify and extend Naïman's arguments to an essentially wider class of operators. This is due to the fact that the spectral properties of banded Jacobi-type matrices can be easily derived from those of the corresponding symbol. A motivation for our considerations comes from [5] as well as [6].

The paper is organized as follows. In Section 2 we recall some key definitions and facts from [26] and clarify the terminology used therein. Our main results are found in Section 3, where we show that each operator A corresponding to a banded Jacobi-type matrix is unitarily equivalent to the multiplication operator by an appropriate matrix-valued function whose values are (depending on the case) of the triangular or diagonal form. Further, we prove that there exists a spectral resolution or a skew spectral resolution of A, respectively. Moreover, in the second case we obtain an integral representation of A. In Section 4 we show that these results apply to a particular class of two-sided infinite periodic Jacobi-type matrices (with finitely many diagonals). Finally, we point out that this class includes all d-periodic banded Jacobi matrices of order 1 which were considered by Naïman in [26].

2. SPECTRAL RESOLUTIONS (ABSTRACT FRAMEWORK). PRELIMINARY RESULTS

Let us consider a finite family of smooth non-intersecting curves on the complex plane. We enumerate them as $\Gamma_1, \ldots, \Gamma_d$, impose an orientation on each of them, and denote by α and β the beginning of Γ_1 and the end of Γ_d, respectively. We set $\Gamma^\circ = \Gamma_1 \cup \ldots \cup \Gamma_d$ and introduce the order relation $\prec$ on Γ° as follows: for $\lambda, \mu \in \Gamma^\circ$, we write $\lambda \prec \mu$ if λ and μ are on the same curve, whereas λ lies earlier than μ in accordance with the fixed orientation, or $\lambda \in \Gamma_i$ and $\mu \in \Gamma_j$ for some $i < j$ $(i, j = 1, \ldots, d)$. We denote by Γ the closure of Γ° and next in a natural way extend the order $\prec$ to Γ distinguishing, to avoid the ambiguity, the points which are simultaneously beginnings and ends of the corresponding curves.

Let $\mathcal{H}$ be a Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}$. We say that an operator-valued function $E : \Gamma \to \mathcal{B}(\mathcal{H})$ is a *resolution of the identity* if:

(i) for each $\lambda \in \Gamma$, $E(\lambda)$ is an orthogonal projection,
(ii) $E(\alpha) = 0$ and $E(\beta) = I$.

Take an operator $A \in \mathcal{B}(\mathcal{H})$ (which, in general, is non-self-adjoint). Assume henceforth that $\sigma(A) = \Gamma$, where $\sigma(A)$ stands for the spectrum of A. We provide the following definitions (cf. [26]). A resolution of the identity E is called a *spectral resolution* of the operator $A \in \mathcal{B}(\mathcal{H})$ if:

(i) for each $\lambda \in \Gamma$, the space $E(\lambda)\mathcal{H}$ is invariant for A,
(ii) for each $\lambda \in \Gamma$,

$$\sigma(AE(\lambda)) = \overline{\{\mu \in \Gamma : \mu \prec \lambda\}} \quad \text{and} \quad \sigma(A(I - E(\lambda))) = \overline{\{\mu \in \Gamma : \lambda \prec \mu\}}.$$

The latter can be regarded as the separability condition of the spectrum of A. It turns out that not all non-self-adjoint operators, contrary to self-adjoint ones, possess this property. It may even happen that a non-self-adjoint operator restricted to each non-trivial invariant subspace always has the same spectrum. An example of an such operator, arising from harmonic analysis, can be found in [21].

In the spectral theory of non-self-adjoint operators resolutions with orthogonal values play an important role, however those with non-orthogonal values are also widely considered. An operator-valued function $F : \Gamma \to \mathcal{B}(\mathcal{H})$ is said to be a *skew resolution of the identity* if:

(i) for each $\lambda \in \Gamma$, $F(\lambda)$ is a projection (i.e. $F^2(\lambda) = F(\lambda)$),
(ii) $F(\alpha) = \lim_{\lambda \downarrow \alpha} F(\lambda) = 0$ and $F(\beta) = \lim_{\lambda \uparrow \beta} F(\lambda) = I$,
(iii) $F(\lambda)\mathcal{H} \subset F(\mu)\mathcal{H}$ and $(I - F(\mu))\mathcal{H} \subset (I - F(\lambda))\mathcal{H}$ for all $\lambda, \mu \in \Gamma$ such that $\lambda \prec \mu$,
(iv) there exist positive real numbers m and M such that for each $g \in \mathcal{H}$ and each division $\{\Delta_1, \ldots, \Delta_s\}$ of Γ on open intervals $\Delta_k = \{\lambda \in \Gamma : \alpha_k \prec \lambda \prec \beta_k\}$ $(\alpha_k, \beta_k \in \Gamma)$, $k = 1, \ldots, s$, we have

$$m \sum_{k=1}^{s} \|F(\Delta_k)g\|^2 \leq \|g\|^2 \leq M \sum_{k=1}^{s} \|F(\Delta_k)g\|^2, \tag{2.1}$$

where $F(\Delta_k) := F(\beta_k) - F(\alpha_k)$.

Remark that projections in (i), in general, are not assumed to be orthogonal. (iii) can be regarded as the monotonicity condition for a family of subspaces of $\mathcal{H}$. Condition (iv), in turn, is a continuous version of the sufficient discrete condition for the existence of the Riesz basis in $\mathcal{H}$ (see [19, p. 320] or [35, p. 32]). By a division of Γ, here and throughout, we understand a finite family $\{\Delta_1, \ldots, \Delta_s\}$ whose members are pairwise disjoint and the closure of their sum gives the whole Γ. It is easily seen that

each $F(\Delta_k)$ is a projection (on the closed subspace $F(\beta_k)\mathcal{H} \cap (I - F(\alpha_k))\mathcal{H}$ of $\mathcal{H}$), and $F(\Delta_k)F(\Delta_j) = 0$ if $\Delta_k \cap \Delta_j = \emptyset$ $(k, j = 1, \ldots, s)$. (2.2)

A skew resolution of the identity F is called a *skew spectral resolution* of $A \in \mathcal{B}(\mathcal{H})$ if:

(i) for each $\lambda \in \Gamma$, the spaces $F(\lambda)\mathcal{H}$ and $(I - F(\lambda))\mathcal{H}$ are invariant for A,
(ii) for each $\lambda \in \Gamma$,

$$\sigma(AF(\lambda)) = \overline{\{\mu \in \Gamma : \mu \prec \lambda\}} \quad \text{and} \quad \sigma(A(I - F(\lambda))) = \overline{\{\mu \in \Gamma : \lambda \prec \mu\}},$$

(iii) for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each interval $\Delta \subset \Gamma$ of length less than δ, and all $f \in F(\Delta)\mathcal{H}$ and $\lambda \in \Delta$, we have

$$\|Af - \lambda f\| < \varepsilon. \tag{2.3}$$

Suppose now that A possesses a skew spectral resolution F. For a fixed $\varepsilon > 0$, we consider a division $\{\Delta_1, \ldots, \Delta_s\}$ of Γ satisfying the above condition (iii). Let λ_k be chosen from Δ_k ($k = 1, \ldots, s$). Take $f \in \mathcal{H}$. Then, since each space $F(\Delta_k)\mathcal{H}$ is invariant for A, it follows that $AF(\Delta_k)f - \lambda_k F(\Delta_k)f = F(\Delta_k)h_k$ for some $h_k \in \mathcal{H}$. After summing up both sides over k, we get $Af - \sum_{k=1}^{s} \lambda_k F(\Delta_k)f = \sum_{k=1}^{s} F(\Delta_k)h_k$. Next, we apply the right-hand side inequality of (2.1) to $g = \sum_{k=1}^{s} F(\Delta_k)h_k$, take into account the properties (2.2), use the inequality (2.3), and finally utilize the left-hand side inequality of (2.1) to $g = f$. As a consequence, we can deduce that

$$\Big\|Af - \sum_{k=1}^{s} \lambda_k F(\Delta_k)f\Big\| \leq \sqrt{\frac{M}{m}}\varepsilon\|f\|.$$

This argument shows that, given f, the Riemann-type sums $\sum_{k=1}^{s} \lambda_k F(\Delta_k)f$ are convergent to Af. Regarding the integral below as their limit, we can write

$$Af = \int_{\Gamma} \lambda \mathrm{d}F(\lambda)f, \quad f \in \mathcal{H}.$$

This explains how the integral representation of A proposed by Naïman in [26, Theorem 6] should be understood.

3. SPECTRAL RESOLUTIONS FOR LAURENT OPERATORS

Here and throughout, $\mathbb{C}^{d\times d}$ stands for the set of all $d \times d$ complex matrices ($d \in \mathbb{N}$). The usual norms in all spaces $\mathbb{C}^d$ and $\mathbb{C}^{d\times d}$ are denoted by the same symbol $|\cdot|$. By $l^2(\mathbb{Z}, \mathbb{C}^d)$ we mean the Hilbert space of all $\mathbb{C}^d$-valued sequences $u = (u_n)_{n\in\mathbb{Z}}$, $u_n \in \mathbb{C}^d$ ($n \in \mathbb{Z}$), such that

$$\|u\|_{l^2(\mathbb{Z},\mathbb{C}^d)} := \Big(\sum_{n\in\mathbb{Z}} |u_n|^2\Big)^{\frac{1}{2}} < \infty.$$

Let $A : l^2(\mathbb{Z}, \mathbb{C}^d) \to l^2(\mathbb{Z}, \mathbb{C}^d)$ be a Laurent operator defined by

$$(Au)_n = \sum_{j=-\infty}^{\infty} A_j u_{n-j}, \quad n \in \mathbb{Z}, \tag{3.1}$$

for $u = (u_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z}, \mathbb{C}^d)$, where $A_j \in \mathbb{C}^{d\times d}$ $(j \in \mathbb{Z})$ and

$$\sum_{j=-\infty}^{\infty} |A_j| < \infty. \tag{3.2}$$

Let $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. With regard to the terminology used in [1], the matrix-valued function

$$\mathcal{A}(\zeta) = \sum_{k=-\infty}^{\infty} A_k \zeta^k, \quad \zeta \in \mathbb{T}, \tag{3.3}$$

is called the *symbol* of the operator A.

In what follows, let us assume that

$$\sigma(A) = \Gamma, \text{ where } \Gamma \text{ is the closure of a finite sum of curves as in Section 2.} \tag{3.4}$$

We will show that a spectral representation of A can be derived from that of the symbol $\mathcal{A}(\zeta)$. We will consider two cases: first a more general one, when the symbol $\mathcal{A}(\zeta)$ is triangularizable, and then a particular one, when it is diagonalizable.

3.1. SPECTRAL RESOLUTION FOR A LAURENT OPERATOR

For a fixed $\zeta \in \mathbb{T}$, $\mathcal{A}(\zeta)$ is a $d \times d$ complex matrix (whose entries are well-defined owing to (3.2)), so we can employ the Schur theorem ([23, pp. 25–26], see also [33, pp. 124–125]) to construct a triangular matrix $T(\zeta)$ and a unitary matrix $U(\zeta)$ such that

$$\mathcal{A}(\zeta) = U(\zeta)T(\zeta)(U(\zeta))^*. \tag{3.5}$$

A careful look at the proof of the Schur theorem (in [23]) reveals that if $\lambda_1(\zeta), \ldots, \lambda_d(\zeta)$ are the eigenvalues of $\mathcal{A}(\zeta)$, then we can find orthonormal (column) vectors $\varphi^{(1)}(\zeta), \ldots, \varphi^{(d)}(\zeta) \in \mathbb{C}^d$ such that

$$\mathcal{A}(\zeta)\varphi^{(k)}(\zeta) = \lambda_k(\zeta)\varphi^{(k)}(\zeta) + \sum_{l=1}^{k-1} \mu_{kl}(\zeta)\varphi^{(l)}(\zeta), \quad k = 1, \ldots, d,$$

where $\mu_{kl}(\zeta) \in \mathbb{C}$ $(k = 2, \ldots, d$ and $l = 1, \ldots, d-1)$ and $\varphi^{(1)}(\zeta)$ is the eigenvector corresponding to $\lambda_1(\zeta)$. These vectors form the *Schur orthonormal basis* in $\mathbb{C}^d$. Moreover, we have

$$T(\zeta) = \begin{bmatrix} \lambda_1(\zeta) & \mu_{21}(\zeta) & \cdots & \mu_{d1}(\zeta) \\ 0 & \lambda_2(\zeta) & \cdots & \mu_{d2}(\zeta) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_d(\zeta) \end{bmatrix} \tag{3.6}$$

and

$$U(\zeta) = \begin{bmatrix} \varphi_1^{(1)}(\zeta) & \varphi_1^{(2)}(\zeta) & \cdots & \varphi_1^{(d)}(\zeta) \\ \varphi_2^{(1)}(\zeta) & \varphi_2^{(2)}(\zeta) & \cdots & \varphi_2^{(d)}(\zeta) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_d^{(1)}(\zeta) & \varphi_d^{(2)}(\zeta) & \cdots & \varphi_d^{(d)}(\zeta) \end{bmatrix}, \tag{3.7}$$

where $\varphi_r^{(k)}(\zeta)$ is the r-th coordinate of $\varphi^{(k)}(\zeta) \in \mathbb{C}^d$ ($\zeta \in \mathbb{T}$, $k, r = 1, \ldots, d$). Remark that all λ_k, μ_{kl}, and $\varphi^{(k)}$ above can be chosen to be continuous as functions of a variable $\zeta \in \mathbb{T}$.

Throughout what follows, we write ψ_r for the r-th coordinate ($r = 1, \ldots, d$) of the vector-valued function $\psi \in L^2(\mathbb{Z}, \mathbb{C}^d)$. Clearly, by $L^2(\mathbb{T}, \mathbb{C}^d)$ we mean the Hilbert space of all measurable $\mathbb{C}^d$-valued functions ψ on $\mathbb{T}$ such that

$$\|\psi\|_{L^2(\mathbb{T},\mathbb{C}^d)} := \Big(\int_{\mathbb{T}} |\psi(\zeta)|^2 \mathrm{d}m \Big)^{\frac{1}{2}} < \infty,$$

where m is the normalized Lebesgue measure on $\mathbb{T}$.

Next, we consider the operator $T : L^2(\mathbb{T}, \mathbb{C}^d) \to L^2(\mathbb{T}, \mathbb{C}^d)$ defined by

$$(T\psi)(\zeta) = T(\zeta)\psi(\zeta), \quad \psi \in L^2(\mathbb{T}, \mathbb{C}^d), \zeta \in \mathbb{T}. \tag{3.8}$$

We call T a *canonical operator* of A.

Our objective is to construct a spectral resolution E of T (compare with [26, p. 148]). Let $\Gamma_k := \{\lambda_k(\zeta) : \zeta \in \mathbb{T}\}$ ($k = 1, \ldots, d$) be a curve on the complex plane, where $\lambda_k : \mathbb{T} \ni \zeta \mapsto \lambda_k(\zeta) \in \mathbb{C}$ is a continuous function. Note that $\sigma(T)$ is the closure of $\Gamma_1 \cup \ldots \cup \Gamma_d$. Set $\zeta_0 = 1 \in \mathbb{T}$. In what follows, for $\eta \in \mathbb{T}$, by $(\zeta_0\eta)$ we mean the subset $\{e^{it} : 0 < t < t_\eta\}$ of $\mathbb{T}$, where $t_\eta \in (0, 2\pi)$ is such that $\eta = e^{it_\eta}$, and by $[\zeta_0\eta]$ we denote the closure of $(\zeta_0\eta)$ in $\mathbb{C}$. Now let us consider a few cases.

Case 1. First we suppose that for each $\zeta \in \mathbb{T}$ all values $\lambda_k(\zeta)$ ($k = 1, \ldots, d$) on the main diagonal of the matrix $T(\zeta)$ are distinct.

Case 1.1. Assume that all λ_k's are injective as functions. Then we obtain exactly d disjoint non-self-intersecting curves Γ_k ($k = 1, \ldots, d$). For a fixed $\lambda \in \bigcup_{k=1}^d \Gamma_k$, there exist $l \in \{1, \ldots, d\}$ and $\xi \in \mathbb{T}$ such that $\lambda = \lambda_l(\xi)$. Then, for $\psi(\zeta) = (\psi_k(\zeta))_{k=1}^d \in L^2(\mathbb{T}, \mathbb{C}^d)$, we set

$$E(\lambda)\psi(\zeta) = \begin{bmatrix} \psi_1(\zeta) \\ \ldots \\ \psi_{l-1}(\zeta) \\ \psi_l(\zeta)\chi_{(\zeta_0\xi)}(\zeta) \\ 0 \\ \ldots \\ 0 \end{bmatrix}.$$

Case 1.2. Assume that at least one function λ_k is not injective. For simplicity, assume that the only such function is λ_1. We can choose $\eta \in \mathbb{T}$ such that λ_1 is injective on $[\zeta_0\eta]$ and $\Gamma_1 = \{\lambda_1(\zeta) : \zeta \in [\zeta_0\eta]\}$. Then, for a fixed λ and each ψ as in Case 1.1, we set

$$E(\lambda)\psi(\zeta) = \begin{bmatrix} \psi_1(\zeta)\chi_{(\zeta_0\eta)}(\zeta) \\ \psi_2(\zeta) \\ \dots \\ \psi_{l-1}(\zeta) \\ \psi_l(\zeta)\chi_{(\zeta_0\xi)}(\zeta) \\ 0 \\ \dots \\ 0 \end{bmatrix}.$$

Case 2. Suppose that the matrix $T(\zeta)$ has multiple eigenvalues for some $\zeta \in \mathbb{T}$. To simplify the writing, we confine our attention to the case of $d = 2$ only. Let

$$T(\zeta) = \begin{bmatrix} \lambda_1(\zeta) & \mu(\zeta) \\ 0 & \lambda_1(\zeta) \end{bmatrix}.$$

Case 2.1. Assume that the function λ_1 is injective. Then, for a fixed $\lambda \in \Gamma_1$, there exists $\xi \in \mathbb{T}$ such that $\lambda = \lambda_1(\xi)$. For $\psi = (\psi_k(\zeta))_{k=1}^2 \in L^2(\mathbb{T}, \mathbb{C}^2)$, we set

$$E(\lambda)\psi(\zeta) = \begin{bmatrix} \psi_1(\zeta)\chi_{(\zeta_0\xi)}(\zeta) \\ \psi_2(\zeta)\chi_{(\zeta_0\xi)}(\zeta) \end{bmatrix}.$$

Case 2.2. If λ_1 is not injective, then we take η as in Case 1.2. For a fixed $\lambda \in \Gamma_1$, there exists $\xi \in [\zeta_0\eta]$ such that $\lambda = \lambda_1(\xi)$. Then E can be defined by the same formula as in Case 2.1.

We are in a position to conduct a construction of E in full generality. Namely, we can define the values of $E(\lambda)$ on the elements of $L^2(\mathbb{T}, \mathbb{C}^d)$ as column vectors whose entries are chosen according to the following rules. If some λ_k's coincide ($k = 1, \dots, d$), then the corresponding entries are given as in Case 2.1 (resp. Case 2.2) whenever such repeated functions are (resp. are not) injective. Next, we call upon Case 2.2 to get the entries corresponding to pairwise distinct functions λ_k which are not injective. Finally, we derive the reminder entries from Case 1.1. The construction is complete.

We now prove a generalization of Theorem 7 of [26] for the operator A. The arguments are in a sense similar to those given by Naïman.

Theorem 3.1. *Let $A : l^2(\mathbb{Z}, \mathbb{C}^d) \to l^2(\mathbb{Z}, \mathbb{C}^d)$ be a Laurent operator defined by* (3.1)*, $\mathcal{A}(\zeta)$ its symbol, and $T(\zeta)$ a Schur triangularization* (3.6) *of $\mathcal{A}(\zeta)$ ($\zeta \in \mathbb{T}$). Then the operator A is unitarily equivalent to the canonical operator $T : L^2(\mathbb{T}, \mathbb{C}^d) \to L^2(\mathbb{T}, \mathbb{C}^d)$ defined by* (3.8)*.*

Proof. We first show that the operator U defined on $L^2(\mathbb{T}, \mathbb{C}^d)$ by

$$(U\psi)(\zeta) = U(\zeta)\psi(\zeta), \quad \psi \in L^2(\mathbb{T}, \mathbb{C}^d), \zeta \in \mathbb{T}, \tag{3.9}$$

is unitary. In view of (3.7), for $\zeta \in \mathbb{T}$, we obtain

$$U(\zeta)\psi(\zeta) = \left(\sum_{k=1}^{d} \varphi_r^{(k)}(\zeta)\psi_k(\zeta) \right)_{r=1}^{d}, \quad \psi \in L^2(\mathbb{T}, \mathbb{C}^d).$$

By this and the fact that the vectors $\varphi^{(k)}(\zeta)$ $(k = 1, 2, \ldots, d)$ form the orthonormal basis in $\mathbb{C}^d$, we can perform the following calculations for $\psi \in L^2(\mathbb{T}, \mathbb{C}^d)$:

$$\begin{aligned}\sum_{r=1}^{d} \int_{\mathbb{T}} \left|\big(U(\zeta)\psi(\zeta)\big)_r\right|^2 \mathrm{d}m &= \sum_{r=1}^{d} \int_{\mathbb{T}} \left|\sum_{k=1}^{d} \varphi_r^{(k)}(\zeta)\psi_k(\zeta)\right|^2 \mathrm{d}m \\ &= \sum_{r=1}^{d} \int_{\mathbb{T}} \sum_{k,l=1}^{d} \left(\psi_k(\zeta)\,\overline{\psi_l(\zeta)}\,\varphi_r^{(k)}(\zeta)\,\overline{\varphi_r^{(l)}(\zeta)}\right) \mathrm{d}m \\ &= \int_{\mathbb{T}} \sum_{k,l=1}^{d} \left(\psi_k(\zeta)\,\overline{\psi_l(\zeta)}\right) \left(\sum_{r=1}^{d} \varphi_r^{(k)}(\zeta)\,\overline{\varphi_r^{(l)}(\zeta)}\right) \mathrm{d}m \\ &= \int_{\mathbb{T}} \sum_{k,l=1}^{d} \psi_k(\zeta)\,\overline{\psi_l(\zeta)} \cdot \langle \varphi^{(k)}(\zeta), \varphi^{(l)}(\zeta)\rangle_{\mathbb{C}^d}\, \mathrm{d}m \\ &= \sum_{k=1}^{d} \int_{\mathbb{T}} |\psi_k(\zeta)|^2\, \mathrm{d}m \;=\; \|\psi\|^2_{L^2(\mathbb{T},\mathbb{C}^d)} < \infty.\end{aligned}$$

As a consequence, we deduce that $U\psi \in L^2(\mathbb{T}, \mathbb{C}^d)$ and

$$\|U\psi\|_{L^2(\mathbb{T},\mathbb{C}^d)} = \|\psi\|_{L^2(\mathbb{T},\mathbb{C}^d)}.$$

Since, for each $\zeta \in \mathbb{T}$, the matrix $U(\zeta)$ is invertible, a direct calculation shows that the operator U^{-1} on $L^2(\mathbb{T}, \mathbb{C}^d)$ given by

$$(U^{-1}\psi)(\zeta) = (U(\zeta))^{-1}\psi(\zeta), \quad \psi \in L^2(\mathbb{T}, \mathbb{C}^d), \zeta \in \mathbb{T}, \tag{3.10}$$

is the inverse of U. What is more, since $(U(\zeta))^{-1} = (U(\zeta))^*$ for each $\zeta \in \mathbb{T}$, the reader will have no trouble verifying that $U^{-1} = U^*$.

Next, consider the operator $\widehat{A} : L^2(\mathbb{T}, \mathbb{C}^d) \to L^2(\mathbb{T}, \mathbb{C}^d)$ defined by

$$(\widehat{A}\psi)(\zeta) = \mathcal{A}(\zeta)\psi(\zeta), \quad \psi \in L^2(\mathbb{T}, \mathbb{C}^d), \zeta \in \mathbb{T}. \tag{3.11}$$

In view of (3.11), (3.10), (3.8), (3.9) and (3.5), we get

$$\widehat{A} = UTU^*. \tag{3.12}$$

In accordance with the theory of Laurent operators ([1, pp. 48–49]), the operators A and $\widehat{A}$ are unitarily equivalent. Namely, we have

$$A = G\widehat{A}G^*, \tag{3.13}$$

where $G : L^2(\mathbb{T}, \mathbb{C}^d) \to l^2(\mathbb{Z}, \mathbb{C}^d)$ is a unitary operator given by

$$(G\psi)(\zeta) = \left(\frac{1}{2\pi} \int_{\mathbb{T}} \psi(\zeta)\zeta^{-n}\, \mathrm{d}m\right)_{n\in\mathbb{Z}}, \quad \psi \in L^2(\mathbb{T}, \mathbb{C}^d), \zeta \in \mathbb{T}, \tag{3.14}$$

and $G^*: l^2(\mathbb{Z},\mathbb{C}^d) \to L^2(\mathbb{T},\mathbb{C}^d)$ is of the form

$$G^*(u) = \sum_{n=-\infty}^{\infty} u_n \zeta^n, \quad u = (u_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z},\mathbb{C}^d), \zeta \in \mathbb{T}.$$

As a conclusion, by (3.12) and (3.13), we get

$$A = \big(GU\big)T\big(GU\big)^*, \tag{3.15}$$

which completes the proof. □

Suppose now that the assertion of Theorem 3.1 holds. Let E be a spectral resolution of T (whose construction was described before Theorem 3.1). From (3.15) and the fact that the operators U and G are unitary (so, in particular, $\sigma(A) = \sigma(T)$) we infer that the operator-valued function $\mathcal{E}$ given by

$$\mathcal{E}(\lambda) = \big(GU\big)E(\lambda)\big(GU\big)^*, \quad \lambda \in \sigma(A), \tag{3.16}$$

is a spectral resolution of A. In this way, we arrived at the following generalization of Theorem 8 of [26] for A.

Theorem 3.2. *Let A and T be as in Theorem 3.1. Then the operator-valued function $\mathcal{E}$ defined by* (3.16) *is a spectral resolution of A.*

It should be noted that not all Laurent operators A satisfy condition (3.4), and hence do not fall within the scope of Theorem 3.2 even in the case when all matrices A_j ($j \in \mathbb{Z}$) in (3.1) are non-zero. An example of such an operator is given below.

Example 3.3. Consider a Laurent operator A acting on the space $l^2(\mathbb{Z},\mathbb{C}^2)$ corresponding to the symbol (3.3), where

$$A_0 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad A_k = \begin{bmatrix} 0 & 2^{-|k|} \\ 0 & 0 \end{bmatrix}, \; k \in \mathbb{Z}\setminus\{0\}.$$

Obviously, the symbol of A can be expressed explicitly as

$$\mathcal{A}(\zeta) = \begin{bmatrix} 2 & 3\zeta(2-\zeta)^{-1}(2\zeta-1)^{-1} \\ 0 & 2 \end{bmatrix}, \quad \zeta \in \mathbb{T}.$$

Clearly, its spectrum consists of only one point, and so A does not have a spectral resolution.

3.2. SKEW SPECTRAL RESOLUTION FOR A LAURENT OPERATOR

We now move on to considering a particular situation when, for each $\zeta \in \mathbb{T}$, $\mathcal{A}(\zeta)$ has all simple eigenvalues $\lambda_k(\zeta)$ ($k = 1,\dots,d$). Then, owing to Theorem 4.15.11 of [33], each matrix $\mathcal{A}(\zeta)$ is diagonalizable. More precisely, for $\zeta \in \mathbb{T}$, we can find a diagonal matrix $D(\zeta)$ and an invertible matrix $V(\zeta)$ such that

$$\mathcal{A}(\zeta) = V(\zeta)D(\zeta)(V(\zeta))^{-1}. \tag{3.17}$$

It is worth noting that the matrix $V(\zeta)$ may not be unitary. An analysis of the proof of this result (in [33]) shows that

$$D(\zeta) = \begin{bmatrix} \lambda_1(\zeta) & 0 & \dots & 0 \\ 0 & \lambda_2(\zeta) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_d(\zeta) \end{bmatrix} \tag{3.18}$$

and

$$V(\zeta) = \begin{bmatrix} \phi_1^{(1)}(\zeta) & \phi_1^{(2)}(\zeta) & \cdots & \phi_1^{(d)}(\zeta) \\ \phi_2^{(1)}(\zeta) & \phi_2^{(2)}(\zeta) & \cdots & \phi_2^{(d)}(\zeta) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_d^{(1)}(\zeta) & \phi_d^{(2)}(\zeta) & \cdots & \phi_d^{(d)}(\zeta) \end{bmatrix}, \tag{3.19}$$

where $\phi^{(k)}(\zeta) = \big(\phi_r^{(k)}(\zeta)\big)_{r=1}^{d}$ $(k = 1,\dots,d)$ are the eigenvectors corresponding to $\lambda_k(\zeta)$ $(k = 1,\dots,d)$. Next, denote by $\vartheta^{(k)}(\zeta)$ $(k = 1,\dots,d)$ the eigenvectors of the matrix $(\mathcal{A}(\zeta))^*$ corresponding to its eigenvalues $\overline{\lambda_k(\zeta)}$. As noticed in [26, pp. 148–149], we may assume that these eigenvectors are chosen in such a way that

$$(V(\zeta))^{-1} = \begin{bmatrix} \overline{\vartheta_1^{(1)}(\zeta)} & \overline{\vartheta_2^{(1)}(\zeta)} & \cdots & \overline{\vartheta_d^{(1)}(\zeta)} \\ \overline{\vartheta_1^{(2)}(\zeta)} & \overline{\vartheta_2^{(2)}(\zeta)} & \cdots & \overline{\vartheta_d^{(2)}(\zeta)} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\vartheta_1^{(d)}(\zeta)} & \overline{\vartheta_2^{(d)}(\zeta)} & \cdots & \overline{\vartheta_d^{(d)}(\zeta)} \end{bmatrix}. \tag{3.20}$$

We show that the operator V defined on $L^2(\mathbb{T},\mathbb{C}^d)$ by

$$(V\psi)(\zeta) = V(\zeta)\psi(\zeta), \quad \psi \in L^2(\mathbb{T},\mathbb{C}^d), \zeta \in \mathbb{T}, \tag{3.21}$$

is bounded and invertible. Indeed, since $V(\zeta)$ is of the form (3.19) for $\zeta \in \mathbb{T}$, it follows that

$$V(\zeta)\psi(\zeta) = \bigg(\sum_{k=1}^{d} \phi_r^{(k)}(\zeta)\psi_k(\zeta)\bigg)_{r=1}^{d}, \quad \psi \in L^2(\mathbb{T},\mathbb{C}^d).$$

Then, by using the inequality

$$\Big|\sum_{r=1}^{d} x_r\Big|^2 \le d \sum_{r=1}^{d} |x_r|^2, \quad x_1,\dots,x_d \in \mathbb{C},$$

for $\psi \in L^2(\mathbb{T}, \mathbb{C}^d)$, we get

$$\begin{aligned}\sum_{r=1}^{d} \int_{\mathbb{T}} \left|\big(V(\zeta)\psi(\zeta)\big)_r\right|^2 \mathrm{d}m &= \sum_{r=1}^{d} \int_{\mathbb{T}} \left|\sum_{k=1}^{d} \phi_r^{(k)}(\zeta)\psi_k(\zeta)\right|^2 \mathrm{d}m \\ &\le d \sum_{r=1}^{d} \int_{\mathbb{T}} \sum_{k=1}^{d} |\psi_k(\zeta)|^2 \, |\phi_r^{(k)}(\zeta)|^2 \, \mathrm{d}m \\ &= d^2 \sum_{k=1}^{d} \int_{\mathbb{T}} |\psi_k(\zeta)|^2 \, \mathrm{d}m \\ &= d^2 \|\psi\|^2_{L^2(\mathbb{T},\mathbb{C}^d)} < \infty.\end{aligned}$$

Hence $V\psi \in L^2(\mathbb{T}, \mathbb{C}^d)$ and

$$\|V\psi\|_{L^2(\mathbb{T},\mathbb{C}^d)} \le d\|\psi\|_{L^2(\mathbb{T},\mathbb{C}^d)},$$

so V is a bounded operator. Since, for each $\zeta \in \mathbb{T}$, $(V(\zeta))^{-1}$ is the inverse of $V(\zeta)$, from a direct calculation we immediately conclude that the operator V^{-1} given by

$$(V^{-1}\psi)(\zeta) = V^{-1}(\zeta)\psi(\zeta), \quad \psi \in L^2(\mathbb{T}, \mathbb{C}^d), \zeta \in \mathbb{T}, \tag{3.22}$$

is the inverse of V. As $(V(\zeta))^{-1}$ is of the form (3.20) for $\zeta \in \mathbb{T}$ and $\vartheta^{(k)}(\zeta) = (\vartheta_r^{(k)}(\zeta))_{r=1}^d$, $k = 1, \dots, d$, we have

$$(V(\zeta))^{-1}\psi(\zeta) = \left(\sum_{r=1}^{d} \overline{\vartheta_r^{(k)}(\zeta)}\psi_r(\zeta)\right)_{k=1}^{d}, \quad \psi \in L^2(\mathbb{T}, \mathbb{C}^d).$$

Since $|\vartheta^{(k)}| \le c$ $(k = 1, \dots, d)$ for some constant c, arguing in a similar fashion as above, for $\psi \in L^2(\mathbb{T}, \mathbb{C}^d)$, we get

$$\begin{aligned}\sum_{k=1}^{d} \int_{\mathbb{T}} \left|\big((V(\zeta))^{-1}\psi(\zeta)\big)_k\right|^2 \mathrm{d}m &= \sum_{k=1}^{d} \int_{\mathbb{T}} \left|\sum_{r=1}^{d} \overline{\vartheta_r^{(k)}(\zeta)}\psi_r(\zeta)\right|^2 \mathrm{d}m \\ &\le d \sum_{k=1}^{d} \int_{\mathbb{T}} \sum_{r=1}^{d} |\psi_r(\zeta)|^2 \left|\overline{\vartheta_r^{(k)}(\zeta)}\right|^2 \mathrm{d}m \\ &\le d^2 c^2 \sum_{r=1}^{d} \int_{\mathbb{T}} |\psi_r(\zeta)|^2 \, \mathrm{d}m \\ &= (cd)^2 \|\psi\|^2_{L^2(\mathbb{T},\mathbb{C}^d)} < \infty.\end{aligned}$$

Hence $V^{-1}\psi \in L^2(\mathbb{T}, \mathbb{C}^d)$ and

$$\|V^{-1}\psi\|_{L^2(\mathbb{T},\mathbb{C}^d)} \leq cd\|\psi\|_{L^2(\mathbb{T},\mathbb{C}^d)},$$

so V^{-1} is a bounded operator.

Furthermore, let $D : L^2(\mathbb{T}, \mathbb{C}^d) \to L^2(\mathbb{T}, \mathbb{C}^d)$ be the operator given by

$$(D\psi)(\zeta) = D(\zeta)\psi(\zeta) = \big(\lambda_k(\zeta)\psi_k(\zeta)\big)_{k=1}^{d}, \quad \zeta \in \mathbb{T}, \tag{3.23}$$

where $\psi(\zeta) = \big(\psi_k(\zeta)\big)_{k=1}^{d} \in L^2(\mathbb{T}, \mathbb{C}^d)$, and $\widehat{A} : L^2(\mathbb{T}, \mathbb{C}^d) \to L^2(\mathbb{T}, \mathbb{C}^d)$ be the operator given by (3.11). A combination of (3.11), (3.22), (3.23), (3.21) and (3.17) reveals that

$$\widehat{A} = VDV^{-1}. \tag{3.24}$$

Recall that A stands for the Laurent operator with the corresponding symbol $\mathcal{A}(\zeta)$ ($\zeta \in \mathbb{T}$). Then, in view of (3.24) and (3.13), we obtain

$$A = WDW^{-1},$$

where

$$W := GV \tag{3.25}$$

and $G : L^2(\mathbb{T}, \mathbb{C}^d) \to l^2(\mathbb{Z}, \mathbb{C}^d)$ is the operator from the proof of Theorem 3.1 given by (3.14). Then, clearly, $\sigma(A) = \sigma(D)$. Next, we define the operator-valued function $\mathcal{F}$ by

$$\mathcal{F}(\lambda) = WF(\lambda)W^{-1}, \quad \lambda \in \sigma(A), \tag{3.26}$$

where F is a spectral resolution of D. (Remark that F can be defined in exactly the same way as E in Subsection 3.1.) It easily checked that $\mathcal{F}$ is a skew spectral resolution of A. By this and the discussion made at the end of Section 2, A possesses an integral representation with respect to $\mathcal{F}$.

The above discussion enables us to formulate the following generalization of Theorems 9 and 10 of [26] for the operator A.

Theorem 3.4. *Let $A : l^2(\mathbb{Z}, \mathbb{C}^d) \to l^2(\mathbb{Z}, \mathbb{C}^d)$ be a Laurent operator defined by* (3.1) *and $\mathcal{A}(\zeta)$ ($\zeta \in \mathbb{T}$) its symbol. Suppose that for each $\zeta \in \mathbb{T}$ the matrix $\mathcal{A}(\zeta)$ has simple eigenvalues only. Then:*

- (i) *A is similar to the operator $D : L^2(\mathbb{T}, \mathbb{C}^d) \to L^2(\mathbb{T}, \mathbb{C}^d)$ defined by* (3.23) (*more precisely, $A = WDW^{-1}$, where $W : L^2(\mathbb{T}, \mathbb{C}^d) \to l^2(\mathbb{Z}, \mathbb{C}^d)$ is an invertible bounded operator given by* (3.25)),
- (ii) *the operator-valued function $\mathcal{F}$ defined by* (3.26) *is a skew spectral resolution of A,*
- (iii) *A has an integral representation*

$$Af = \int_{\Gamma} \lambda \, \mathrm{d}\mathcal{F}(\lambda) f, \quad f \in l^2(\mathbb{Z}, \mathbb{C}^d).$$

4. APPLICATIONS TO PERIODIC JACOBI-TYPE MATRICES

Our next aim is to show that the results of Section 3 apply to operators associated with some periodic infinite matrices, which are defined below.

Let us consider a Jacobi-type matrix $J = [a_{p\,q}]_{p,q\in\mathbb{Z}}$, that is, a complex matrix whose entries satisfy the following two conditions:

1) there exists $k \in \mathbb{N}$ such that for all $p, q \in \mathbb{Z}$, we have $a_{p\,q} = 0$ if $|p-q| > k$,
2) there exists $d \in \mathbb{N}$ such that for all $p, q \in \mathbb{Z}$, we have $a_{p+d\,q+d} = a_{p\,q}$.

Remark that if additionally $a_{p\,q} \neq 0$ for all $p, q \in \mathbb{Z}$ such that $|p-q| = k$, then J is called a *d-periodic banded Jacobi matrix of order k*. Such a class of matrices for $k = 1$ was considered by Naïman in [26]. Here, however, we do not confine ourselves to this particular case. For $l \in \mathbb{Z}$, we denote by A_l a complex $d\times d$-matrix with entries $(A_l)_{p\,q} = a_{p\,q}$ ($p = 0, \ldots, d-1$ and $q = -dl, \ldots, d(1-l)-1$). Note that $A_l = 0$ for $|l| > m$, where $m = \lceil \frac{k}{d} \rceil$ (i.e. m is the smallest integer not less than $\frac{k}{d}$). Under the above notation, J can be converted to the block matrix $\widetilde{J}$ of the form

$$\begin{bmatrix} \ddots & \ddots & & \ddots & \ddots & \ddots & & \ddots & \ddots & & & & \\ \ldots & 0 & A_m & \ldots & A_1 & A_0 & A_{-1} & \ldots & A_{-m} & 0 & \ldots & & \\ & \ldots & 0 & A_m & \ldots & A_1 & A_0 & A_{-1} & \ldots & A_{-m} & 0 & \ldots & \\ & & \ldots & 0 & A_m & \ldots & A_1 & A_0 & A_{-1} & \ldots & A_{-m} & 0 & \ldots \\ & & & & \ddots & \ddots & & \ddots & \ddots & \ddots & & \ddots & \ddots \end{bmatrix}. \tag{4.1}$$

It should be noted that this elegant matrix reduction procedure was borrowed from [5, pp. 139–140]. Next, we consider the operators acting on the spaces $l^2(\mathbb{Z}, \mathbb{C})$ and $l^2(\mathbb{Z}, \mathbb{C}^d)$ corresponding to the matrices J and $\widetilde{J}$, respectively, defined as follows:

$$(Jv)_n = \sum_{j=n-k}^{n+k} a_{n\,j} v_j, \quad n \in \mathbb{Z}, \tag{4.2}$$

for $v = (v_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z}, \mathbb{C})$ and

$$(\widetilde{J}u)_n = \sum_{j=-m}^{m} A_j u_{n-j}, \quad n \in \mathbb{Z}, \tag{4.3}$$

for $u = (u_n) \in l^2(\mathbb{Z}, \mathbb{C}^d)$. Note that, as it does not lead to the ambiguity, these operators are denoted by the same symbols as the respective matrices. It is plain that

$$J = I_d \widetilde{J} I_d^*, \tag{4.4}$$

where I_d stands for the unitary operator from $l^2(\mathbb{Z}, \mathbb{C}^d)$ to $l^2(\mathbb{Z}, \mathbb{C})$ given by

$$I_d u = (\ldots, v_{-1}^1, \ldots, v_{-1}^d, v_0^1, \ldots, v_0^d, \ldots)$$

for $u = (u_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z},\mathbb{C}^d)$, $u_n = (v_n^1,\dots,v_n^d)$, $n \in \mathbb{Z}$. Suppose that $\sigma(\widetilde{J}) = \Gamma$, where Γ is as in Section 2. Consider the symbol of the operator $\widetilde{J}$, which is defined as

$$\widetilde{\mathcal{J}}(\zeta) = \sum_{k=-m}^{m} A_k \zeta^k, \quad \zeta \in \mathbb{T},$$

a Schur triangularization $T_{\widetilde{J}}(\zeta)$ of $\widetilde{\mathcal{J}}(\zeta)$ ($\zeta \in \mathbb{T}$) (see (3.6)), and the canonical operator $T_{\widetilde{J}} : L^2(\mathbb{T},\mathbb{C}^d) \to L^2(\mathbb{T},\mathbb{C}^d)$ of $\widetilde{J}$ given by

$$(T_{\widetilde{J}}\psi)(\zeta) = T_{\widetilde{J}}(\zeta)\psi(\zeta), \quad \psi \in L^2(\mathbb{T},\mathbb{C}^d), \zeta \in \mathbb{T}. \tag{4.5}$$

Now one can repeat the argument of Section 3.1 (putting $\widetilde{J}$ and $T_{\widetilde{J}}$ in place of A and T, respectively) to derive a particular version of combined Theorems 3.1 and 3.2.

Theorem 4.1. *Let $\widetilde{J} : l^2(\mathbb{Z},\mathbb{C}^d) \to l^2(\mathbb{Z},\mathbb{C}^d)$ be a Laurent operator defined by* (4.3), *$\widetilde{\mathcal{J}}(\zeta)$ its symbol, and $T_{\widetilde{J}}(\zeta)$ a Schur triangularization of $\widetilde{\mathcal{J}}(\zeta)$ ($\zeta \in \mathbb{T}$). Then the operator $\widetilde{J}$ is unitarily equivalent to the canonical operator $T_{\widetilde{J}} : L^2(\mathbb{T},\mathbb{C}^d) \to L^2(\mathbb{T},\mathbb{C}^d)$ defined by* (4.5) *and it possesses a spectral resolution.*

Moreover, if for each $\zeta \in \mathbb{T}$ the matrix $\widetilde{\mathcal{J}}(\zeta)$ is diagonalizable, then denoting by $D_{\widetilde{J}}(\zeta)$ its diagonalization (see (3.18)) we can define the operator $D_{\widetilde{J}} : L^2(\mathbb{T},\mathbb{C}^d) \to L^2(\mathbb{T},\mathbb{C}^d)$ by

$$(D_{\widetilde{J}}\psi)(\zeta) = D_{\widetilde{J}}(\zeta)\psi(\zeta), \quad \psi \in L^2(\mathbb{T},\mathbb{C}^d), \zeta \in \mathbb{T}. \tag{4.6}$$

Finally, mimicking the argument of Section 3.2 (replace $\widetilde{J}$ and $D_{\widetilde{J}}$ by A and D, respectively) we get the following particular version of Theorem 3.4.

Theorem 4.2. *Let $\widetilde{J} : l^2(\mathbb{Z},\mathbb{C}^d) \to l^2(\mathbb{Z},\mathbb{C}^d)$ be a Laurent operator defined by* (4.3) *and $\widetilde{\mathcal{J}}(\zeta)$ its symbol. Suppose that for each $\zeta \in \mathbb{T}$ the matrix $\widetilde{\mathcal{J}}(\zeta)$ has simple eigenvalues only. Then $\widetilde{J}$ is similar to the operator $D_{\widetilde{J}} : L^2(\mathbb{T},\mathbb{C}^d) \to L^2(\mathbb{T},\mathbb{C}^d)$ defined by* (4.6) *and it possesses a skew spectral resolution $\mathcal{F}_{\widetilde{J}}$ as well as a spectral representation*

$$\widetilde{J}f = \int_\Gamma \lambda \, \mathrm{d}\mathcal{F}_{\widetilde{J}}(\lambda) f, \quad f \in l^2(\mathbb{Z},\mathbb{C}^d).$$

It should be emphasized that in the above theorems the explicit formulae, which are skipped here, for a spectral resolution and a skew spectral resolution of the operator $\widetilde{J}$ can be obtained on the basis of Section 3 (consult (3.16) and (3.26), respectively).

It is clear that, owing to (4.4) and the unitarity of the operator I_d, the assertions of Theorems 4.1 and 4.2 hold also for the operator J given by (4.2).

Let now $J = [a_{p\,q}]_{p,q\in\mathbb{Z}}$ be a d-periodic banded Jacobi matrix of order 1. If we set $a_{p\,p-1} = b_r$, $a_{p\,p} = a_r$, and $a_{p\,p+1} = c_r$ ($p = nd + r$, $r = 1, \ldots, d$, $n \in \mathbb{Z}$), then

$$J = \begin{bmatrix} \ddots & \ddots & \ddots & \vdots & & & & \vdots & & & \\ & b_d & a_d & \vdots & c_d & & & \vdots & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & b_1 & \vdots & a_1 & c_1 & & \vdots & & & \\ & & & \vdots & \ddots & \ddots & \ddots & \vdots & & & \\ & & & \vdots & & b_d & a_d & \vdots & c_d & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & \vdots & & & b_1 & \vdots & a_1 & \ddots & \\ & & & \vdots & & & & \vdots & \ddots & \ddots & \end{bmatrix}.$$

The assumption that all entries b_r and c_r are non-zero was essential in [26] for the spectrum of the Laurent operator J to consist of curves irreducible to single points (which is not the case of Example 3.3). Clearly, this special tridiagonal case of J investigated by Naïman is embraced by our reasoning above for general Jacobi-type matrices.

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